13 Observability

We introduce the notion of observability for deterministic linear systems.

Consider the system

$$x_{n+1} = Ax_n, \quad y_{n+1} = Cx_n, \quad n \geqslant 0.$$

Here the state x takes values in $\mathbb{R}^d$, the observation y takes values in $\mathbb{R}^p$, and A and C are matrices of appropriate dimensions. We say that the system is *observable in n-steps* if $y_1, \dots, y_n$ determine uniquely the initial state x_0, for all $x_0 \in \mathbb{R}^d$. It is *observable* if it is observable in n-steps for some $n \geqslant 1$. Note that

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = N_n x_0, \quad N_n = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix},$$

so the system is observable in n-steps if and only if $\operatorname{rank}(N_n) = d$. Hence, by the Cayley–Hamilton theorem, the system is observable if and only if $\operatorname{rank}(N_d) = d$.

The continuous-time system

$$\dot{x}_t = Ax_t, \quad \dot{y}_t = Cx_t, \quad y_0 = 0, \quad t \geqslant 0$$

is *observable in time t* if $(y_s)_{0 \leqslant s \leqslant t}$ determines uniquely the initial state x_0, for all $x_0 \in \mathbb{R}^d$. Since

$$\left(\frac{d}{dt}\right)^n \bigg|_{t=0} y_t = CA^{n-1}x_0, \quad n \geqslant 1,$$

it is clear that, for any $t > 0$, the condition $\operatorname{rank}(N_d) = d$ is sufficient for observability in time t. On the other hand, if $\operatorname{rank}(N_d) \leqslant d - 1$, then there exists $x_0 \in \mathbb{R}^d \setminus \{0\}$ such that $CA^n x_0 = 0$ for $n = 0, 1, \dots, d-1$, and hence for all n by Cayley–Hamilton. Hence

$$y_t = \int_0^t Ce^{sA}x_0 ds = 0$$

for all $t \geqslant 0$ and we cannot distinguish x_0 from 0. The condition $\operatorname{rank}(N_d) = d$ is thus equivalent to observability (in any time $t > 0$).

Example (The sum of two populations). Suppose $\dot{x}_t = \lambda x_t$, $\dot{z}_t = \mu z_t$ and we observe $y_t = x_t + z_t$. Can we determine x_0 and z_0?. In this case

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & 1 \\ \lambda & \mu \end{pmatrix}.$$

So we can determine x_0 and z_0, provided $\lambda \neq \mu$. Even though we are provided with the extra information $x_0 + z_0$, if $\lambda = \mu$, we will have $y_t = (x_0 + z_0)e^{\lambda t}$, so we can never recover x_0 alone.

Example (Radioactive decay). Suppose atoms of element 1 can decay in two ways, to atoms of element 2 at rate α and to atoms of element 3 at rate β. Suppose that atoms of element 2 also decay to atoms of element 3, at rate γ. We observe the number of atoms of element 3. Can we determine the initial numbers of atoms of elements 1 and 2?

Here we have the system

$$\dot{x}^1_t = -(\alpha+\beta)x^1_t, \quad x^2_t = \alpha x^1_t - \gamma x^2_t, \quad \dot{x}^3_t = \beta x^1_t + \gamma x^2_t,$$

so

$$A = \begin{pmatrix} -\alpha-\beta & 0 & 0 \\ \alpha & -\gamma & 0 \\ \beta & \gamma & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, N_3 = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \beta & \gamma & 0 \\ \alpha\gamma - \beta(\alpha+\beta) & -\gamma^2 & 0 \end{pmatrix}.$$

Note that $\det N_3 = \gamma(\alpha+\beta)(\gamma-\beta)$, so the system is observable if $\gamma > 0, \alpha+\beta > 0$ and $\beta \neq \gamma$. It is easy to see that it is not so otherwise.