10 Linear systems with non-negative quadratic costs

The general theory of dynamic optimization for non-negative costs specializes in a computationally explicit way in the case of linear systems with quadratic costs.

Consider the linear controllable dynamical system

$$f(x,a) = Ax + Ba, \quad x \in \mathbb{R}^d, \quad a \in \mathbb{R}^m,$$

with non-negative quadratic cost function

$$c(x,a) = x^T Rx + x^T S^T a + a^T Sx + a^T Qa,$$

where R is a $d \times d$ symmetric matrix, S is am $m \times d$ matrix and Q is an $m \times m$ symmetric matrix. We assume throughout that Q is positive-definite. We begin with some calculations regarding partial minimization of quadratic forms. Note that

$$\inf_a c(x,a) = c(x, Kx) = x^T(R - S^T Q^{-1} S)x,$$

where $K = -Q^{-1}S$. Thus the requirement that c be non-negative imposes the constraint that $R - S^T Q^{-1} S$ is non-negative definite. For a non-negative definite matrix Π, we can write

$$c(x,a) + f(x,a)^T \Pi f(x,a) = \tilde{c}(x,a) = x^T \tilde{R} x + x^T \tilde{S}^T a + a^T \tilde{S} x + a^T \tilde{Q} a,$$

where $\tilde{R} = R + A^T \Pi A$, $\tilde{S} = S + B^T \Pi A$ and $\tilde{Q} = Q + B^T \Pi B$. Since $B^T \Pi B$ is non-negative definite, $\tilde{Q}$ is positive-definite. Hence

$$\inf_a \{c(x,a) + f(x,a)^T \Pi f(x,a)\} = \tilde{c}(x, K(\Pi)x) = x^T r(\Pi) x, \tag{3}$$

where

$$K(\Pi) = -\tilde{Q}^{-1}\tilde{S}, \quad r(\Pi) = \tilde{R} - \tilde{S}^T \tilde{Q}^{-1} \tilde{S}.$$

Since the left-hand side of equation (3) is non-negative, $r(\Pi)$ must be non-negative definite. Fix now a non-negative definite matrix Π_0 and consider the n-horizon problem with final cost $c(x) = x^T \Pi_0 x$. Define, as usual, for $n \geqslant 0$,

$$V_n^u(x) = \sum_{k=0}^{n-1} c(x_k, u_k) + c(x_n), \quad V_n(x) = \inf_u V_n^u(x),$$

where $x_0 = x$ and $x_{k+1} = Ax_k + Bu_k$, $k \geqslant 0$. Then (see footnote 12) $V_0 = c$ and

$$V_{n+1}(x) = \inf_a \{c(x,a) + V_n(Ax + Ba)\}, \quad n \geqslant 0.$$

Hence we obtain the following result by using equation (3) and an induction on $n \geqslant 0$.

Proposition 10.1. *Define* $(\Pi_n)_{n\geqslant 0}$ *by the* Riccati recursion

$$\Pi_{n+1} = r(\Pi_n), \quad n \geqslant 0.$$

Then,

$$V_n(x) = x^T \Pi_n x$$

and the optimal sequence $(x_0, \dots, x_n)$ *is given by*

$$x_k = \Gamma_{n-k} \dots \Gamma_{n-1} x_0, \quad k = 0, 1, \dots, n,$$

where $\Gamma_n = A + BK(\Pi_n)$ *is the* gain matrix.

We turn now to the infinite-horizon case. Define, as usual,

$$V^u(x) = \sum_{k=0}^{\infty} c(x_k, u_k), \quad V(x) = \inf_u V^u(x).$$

Note that, if f is fully controllable, we can choose u so that $x_k = 0$ and $u_k = 0$ for all $k \geqslant d$, so $V(x) < \infty$ for all $x \in \mathbb{R}^d$.

A matrix A is a (discrete-time) *stability matrix* if $A^n \to 0$ as $n \to \infty$. We call f *stabilizable* if $A + BK$ is a stability matrix for some K. We use the matrix norm $|A| = \sup\{|Ax| : |x| = 1\}$, for which $|Ax| \leqslant |A||x|$ for all $x \in \mathbb{R}^d$, $|A| = |A^T|$ and $|AB| \leqslant |A||B|$. Then A is a stability matrix if and only if $|A|^n \leqslant C\alpha^n$ for all $n \geqslant 0$, for some constants $C < \infty$ and $\alpha \in [0, 1)$.

Example. Suppose

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then $f(x, a) = Ax + Ba$ is stabilized by $K = \begin{pmatrix} -2 & 0 \end{pmatrix}$, but f is not fully controllable.

Note that, if f is stabilized by K, and we set $u_n = Kx_n$, then $x_n = \Gamma^n x_0$, where $\Gamma = A + BK$. Choose $C < \infty$ and $\alpha < 1$ such that $|\Gamma^n| \leqslant C\alpha^n$ for all $n \geqslant 0$. Then, for all $x \in \mathbb{R}^d$,

$$V(x) \leqslant V^u(x) = x^T \sum_{n=0}^{\infty} (\Gamma^n)^T Q_K \Gamma^n x \leqslant C^2 |Q_K| |x|^2 / (1 - \alpha^2) < \infty,$$

where

$$Q_K = \begin{pmatrix} I \\ K \end{pmatrix}^T \begin{pmatrix} R & S^T \\ S & Q \end{pmatrix} \begin{pmatrix} I \\ K \end{pmatrix}.$$

Proposition 10.2. *Assume that f is fully controllable or stabilizable. Then the infimal cost function is given by*

$$V(x) = x^T \Pi x, \quad x \in \mathbb{R}^d,$$

where Π is the minimal non-negative definite solution to the equilibrium Riccati equation

$$\Pi = r(\Pi),$$

and, for $K = K(\Pi)$, $u(x) = Kx$ defines an optimal control. Moreover, if Q_K is positive-definite, in particular, if c is positive-definite, then $\Gamma = A + BK$ is a stability matrix, Π is the only non-negative definite solution to $\Pi = r(\Pi)$, and, for any non-negative definite matrix Π_0, if we define $\Pi_{n+1} = r(\Pi_n)$, for $n \geqslant 0$, then $\Pi_n \to \Pi$ as $n \to \infty$.

Proof. By Proposition 2.1,

$$V(x) = \inf_a \{c(x, a) + V(Ax + Ba)\}, \quad x \in \mathbb{R}^d.$$

Take $\Pi_0 = 0$ in the preceding proposition to obtain for the infimal cost function of the n-horizon problem with no final cost,

$$x^T \Pi_n x = V_n(x) \uparrow V_\infty(x) \leqslant V(x), \quad x \in \mathbb{R}^d.$$

Since f is fully controllable or stabilizable, $V(x) < \infty$ for all $x \in \mathbb{R}^d$. Hence[24] there is a non-negative definite matrix Π such that $V_\infty(x) = x^T \Pi x$ for all x. Since r is continuous, we can let $n \to \infty$ in $\Pi_{n+1} = r(\Pi_n)$ to obtain $\Pi = r(\Pi)$. Then

$$V_\infty(x) = \min_a \{c(x,a) + V_\infty(Ax + Ba)\}, \quad x \in \mathbb{R}^d,$$

with minimum at $a = u(x) = K(\Pi)x$. Then $V_\infty \geqslant V^u \geqslant V$ by the argument of Proposition 6.1, so $V(x) = x^T \Pi x$ and u is optimal. For $\Gamma = A + BK$, we have

$$\sum_{n=0}^\infty (\Gamma^n)^T Q_K \Gamma^n = \Pi < \infty,$$

so, if Q_K is positive-definite, then Γ is a stability matrix.

Consider the n-horizon problem with final cost $x^T \tilde{\Pi}_0 x$, where $\tilde{\Pi}_0$ is any non-negative definite matrix. The infimal cost function is $\tilde{V}_n(x) = x^T \tilde{\Pi}_n x$, where $\tilde{\Pi}_{n+1} = r(\tilde{\Pi}_n)$ for $n \geqslant 0$. Then

$$V_n(x) \leqslant \tilde{V}_n(x) \leqslant V_n^u(x) + x^T (\Gamma^n)^T \tilde{\Pi}_0 \Gamma^n x.$$

If $r(\tilde{\Pi}_0) = \tilde{\Pi}_0$, then we obtain $\Pi \leqslant \tilde{\Pi}_0$, so Π is the minimal non-negative solution. In the case where Q_K is positive-definite, for general $\tilde{\Pi}_0$, as $n \to \infty$, the final term tends to 0, so we obtain

$$x^T \Pi x \leqslant \lim_{n\to\infty} x^T \tilde{\Pi}_n x \leqslant x^T \Pi x, \quad x \in \mathbb{R}^d,$$

so $\tilde{\Pi}_n \to \Pi$. In particular Π is the only solution to $r(\Pi) = \Pi$. □

[24] Write $e_1, \dots, e_d$ for the standard basis in $\mathbb{R}^d$, then $V_n(e_i \pm e_j)$ converges to a finite limit for all i, j, and so, by polarization, does $(\Pi_n)_{ij} = e_i^T \Pi_n e_j$. Denote the limit by Π_{ij}. Then $\Pi = (\Pi_{ij})$ is symmetric and $x^T \Pi_n x \to x^T \Pi x$ for all $x \in \mathbb{R}^d$.