4.1 Let \((f_n : n \in \mathbb{N})\) be a sequence of integrable functions and suppose that \(f_n \to f\) a.e. for some integrable function \(f\). Show that, if \(\|f_n\|_1 \to \|f\|_1\), then \(\|f_n - f\|_1 \to 0\).

4.2 Let \(X\) be a random variable and let \(1 \leq p < \infty\). Show that, if \(X \in L^p(\mathbb{P})\), then \(\mathbb{P}(\{|X| \geq \lambda\}) = O(\lambda^{-p})\) as \(\lambda \to \infty\). Prove the identity
\[
\mathbb{E}(|X|^p) = \int_0^\infty p\lambda^{p-1}\mathbb{P}(|X| \geq \lambda)d\lambda
\]
and deduce that, for all \(q > p\), if \(\mathbb{P}(\{|X| \geq \lambda\}) = O(\lambda^{-q})\) as \(\lambda \to \infty\), then \(X \in L^p(\mathbb{P})\).

4.3 Give a simple proof of Schwarz’ inequality \(\|fg\|_1 \leq \|f\|_2\|g\|_2\) for measurable functions \(f\) and \(g\).

4.4 Show that \(\|XY\|_1 = \|X\|_1\|Y\|_1\) for independent random variables \(X\) and \(Y\). Show further that, if \(X\) and \(Y\) are also integrable, then \(\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)\).

4.5 A stepfunction \(f : \mathbb{R} \to \mathbb{R}\) is any finite linear combination of indicator functions of finite intervals. Show that the set of stepfunctions \(I\) is dense in \(L^p(\mathbb{R})\) for all \(p \in [1, \infty)\): that is, for all \(f \in L^p(\mathbb{R})\) and all \(\varepsilon > 0\) there exists \(g \in I\) such that \(\|f - g\|_p < \varepsilon\). Deduce that the set of continuous functions of compact support is also dense in \(L^p(\mathbb{R})\) for all \(p \in [1, \infty)\).

4.6 Let \((X_n : n \in \mathbb{N})\) be an identically distributed sequence in \(L^2(\mathbb{P})\). Show that \(n\mathbb{P}(|X_1| > \varepsilon \sqrt{n}) \to 0\) as \(n \to \infty\), for all \(\varepsilon > 0\). Deduce that \(n^{-1/2}\max_{k \leq n} |X_k| \to 0\) in probability.

5.1 Let \((E, \mathcal{E}, \mu)\) be a measure space and let \(V_1 \leq V_2 \leq \ldots\) be an increasing sequence of closed subspaces of \(L^2 = L^2(E, \mathcal{E}, \mu)\) for \(f \in L^2\), denote by \(f_n\) the orthogonal projection of \(f\) on \(V_n\). Show that \(f_n\) converges in \(L^2\).

5.2 Let \(X = (X_1, \ldots, X_n)\) be a random variable, with all components in \(L^2(\mathbb{P})\). The covariance matrix \(\text{var}(X) = (c_{ij} : 1 \leq i, j \leq n)\) of \(X\) is defined by \(c_{ij} = \text{cov}(X_i, X_j)\). Show that \(\text{var}(X)\) is a non-negative definite matrix.

6.1 Find a uniformly integrable sequence of random variables \((X_n : n \in \mathbb{N})\) such that both \(X_n \to 0\) a.s. and \(\mathbb{E}(\sup_n |X_n|) = \infty\).

6.2 Let \((X_n : n \in \mathbb{N})\) be an identically distributed sequence in \(L^2(\mathbb{P})\). Show that
\[
\mathbb{E}(\max_{k \leq n} |X_k|)/\sqrt{n} \to 0 \quad \text{as} \quad n \to \infty.
\]
7.1 Let \( u,v \in L^1(\mathbb{R}^d) \) and define \( f : \mathbb{R}^d \to \mathbb{C} \) by \( f(x) = u(x) + iv(x) \). Set
\[
\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} u(x)dx + i \int_{\mathbb{R}^d} v(x)dx.
\]
Show that, for all \( y \in \mathbb{R}^d \), we have
\[
\int_{\mathbb{R}^d} f(x-y)dx = \int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} f(-x)dx
\]
and show that
\[
\left| \int_{\mathbb{R}^d} f(x)dx \right| \leq \int_{\mathbb{R}^d} |f(x)|dx.
\]

7.2 Show that the Fourier transform of a finite Borel measure on \( \mathbb{R}^d \) is a bounded continuous function.

7.3 Determine which of the following distributions on \( \mathbb{R} \) have an integrable characteristic function: \( N(\mu, \sigma^2) \), \( \text{Bin}(N, p) \), \( \text{Poisson}(\lambda) \), \( U[0, 1] \).

7.4 For a finite Borel measure \( \mu \) on the line show that, if \( \int |x|^k d\mu(x) < \infty \), then the Fourier transform \( \hat{\mu} \) of \( \mu \) has a \( k \)th continuous derivative, which at 0 is given by
\[
\hat{\mu}^{(k)}(0) = i^k \int x^k d\mu(x).
\]

7.5 Define a function \( \psi \) on \( \mathbb{R} \) by setting \( \psi(x) = C \exp\{-1 - x^2\} \) for \( |x| < 1 \) and \( \psi(x) = 0 \) otherwise, where \( C \) is a constant chosen so that \( \int_{\mathbb{R}} \psi(x)dx = 1 \). For \( f \in L^1(\mathbb{R}) \) of compact support, show that \( f * \psi \) is \( C^\infty \) and of compact support.

7.6 (i) Show that for any real numbers \( a,b \) one has \( \int_a^b e^{ix}dx \to 0 \) as \( |t| \to \infty \).
(ii) Show that, for any \( f \in L^1(\mathbb{R}) \), the Fourier transform
\[
\hat{f}(t) = \int_{-\infty}^{\infty} e^{ix}f(x)dx
\]
tends to 0 as \( |t| \to \infty \). This is the Riemann–Lebesgue Lemma.

7.7 Say that \( f \in L^2(\mathbb{R}) \) is \( L^2 \)-differentiable with \( L^2 \)-derivative \( Df \) if
\[
\|\tau_h f - f - hDf\|_2/h \to 0 \quad \text{as} \quad h \to 0,
\]
where \( \tau_h f(x) = f(x+h) \). Show that the function \( f(x) = \max(1-|x|,0) \) is \( L^2 \)-differentiable and find its \( L^2 \)-derivative.

Suppose that \( f \in L^1 \cap L^2 \) is \( L^2 \)-differentiable. Show that \( u\hat{f}(u) \in L^2 \). Deduce that \( f \) has a continuous version and that \( \|f\|_\infty \leq C\|\langle 1+|u|\rangle \hat{f}(u)\|_2 \) for some absolute constant \( C < \infty \), to be determined. This is a simple example of a Sobolev inequality.