Probability and Measure 1

1.1 Let $E$ be a set and let $S$ be a set of $\sigma$-algebras on $E$. Define

$$E^* = \{A \subseteq E : A \in \mathcal{E} \text{ for all } \mathcal{E} \in S\}.$$ 

Show that $E^*$ is a $\sigma$-algebra on $E$. Show, on the other hand, by example, that the union of two $\sigma$-algebras on the same set need not be a $\sigma$-algebra.

1.2 Show that the following sets of subsets of $\mathbb{R}$ all generate the same $\sigma$-algebra:

(a) $\{(a, b) : a < b\}$,  
(b) $\{(a, b] : a < b\}$,  
(c) $\{(-\infty, b] : b \in \mathbb{R}\}$.

1.3 Show that a countably additive set function on a ring is additive, increasing and countably subadditive.

1.4 Show that a $\pi$-system which is also a $\mathcal{D}$-system is a $\sigma$-algebra.

1.5 Let $\mu$ be a finite-valued additive set function on a ring $\mathcal{A}$. Show that $\mu$ is countably additive if and only if the following condition holds: for any decreasing sequence $(A_n : n \in \mathbb{N})$ of sets in $\mathcal{A}$, with $\bigcap_n A_n = \emptyset$, we have $\mu(A_n) \to 0$.

1.6 Let $(E, \mathcal{E}, \mu)$ be a finite measure space. Show that, for any sequence of sets $(A_n : n \in \mathbb{N})$ in $\mathcal{E}$,

$$\mu(\lim \inf A_n) \leq \lim \inf \mu(A_n) \leq \lim \sup \mu(A_n) \leq \mu(\lim \sup A_n).$$

Show that the first inequality remains true without the assumption that $\mu(E) < \infty$, but that the last inequality may then be false.

1.7 Let $(A_n : n \in \mathbb{N})$ be a sequence of events in a probability space. Show that the events $A_n$ are independent if and only if the $\sigma$-algebras $\sigma(A_n) = \{\emptyset, A_n, A_n^c, \Omega\}$ are independent.

1.8 Let $B$ be a Borel subset of the interval $[0, 1]$. Show that for every $\varepsilon > 0$, there exists a finite union of disjoint intervals $A = (a_1, b_1] \cup \cdots \cup (a_n, b_n]$ such that the Lebesgue measure of $A \Delta B = (A^c \cap B) \cup (A \cap B^c)$ is less than $\varepsilon$. Show further that this remains true for every Borel set in $\mathbb{R}$ of finite Lebesgue measure.

1.9 Let $(E, \mathcal{E}, \mu)$ be a measure space. Call a subset $N \subseteq E$ null if $N \subseteq B$ for some $B \in \mathcal{E}$ with $\mu(B) = 0$. Write $\mathcal{N}$ for the set of all null sets. Prove that the set of subsets $\mathcal{E}^\mu = \{A \cup N : A \in \mathcal{E}, N \in \mathcal{N}\}$ is a $\sigma$-algebra and show that $\mu$ has a well-defined and countably additive extension to $\mathcal{E}^\mu$ given by $\mu(A \cup N) = \mu(A)$. We call $\mathcal{E}^\mu$ the completion of $\mathcal{E}$ with respect to $\mu$. Suppose now that $E$ is $\sigma$-finite and write $\mu^*$ for the outer measure associated to $\mu$, as in the proof of Carathéodory’s Extension Theorem. Show that $\mathcal{E}^\mu$ is exactly the set of $\mu^*$-measurable sets.
2.1 Let \((f_n : n \in \mathbb{N})\) be a sequence of measurable functions on a measurable space \((E, \mathcal{E})\). Show that the following functions are also measurable: \(f_1 + f_2, f_1 f_2, \inf_n f_n, \sup_n f_n, \liminf_n f_n, \limsup_n f_n\). Show also that \(\{x \in E : f_n(x)\text{ converges as } n \to \infty\} \in \mathcal{E}\).

2.2 Let \((E, \mathcal{E})\) and \((G, \mathcal{G})\) be measurable spaces, let \(\mu\) be a measure on \(\mathcal{E}\), and let \(f : E \to G\) be a measurable function. Show that we can define a measure \(\nu\) on \(\mathcal{G}\) by setting \(\nu(A) = \mu(f^{-1}(A))\) for each \(A \in \mathcal{G}\).

2.3 Show that the following condition implies that random variables \(X\) and \(Y\) are independent: \(\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)\) for all \(x, y \in \mathbb{R}\).

2.4 Let \((A_n : n \in \mathbb{N})\) be a sequence of events, with \(\mathbb{P}(A_n) = 1/n^2\) for all \(n\). Set \(X_n = n^2 1_{A_n} - 1\) and set \(\bar{X}_n = (X_1 + \cdots + X_n)/n\). Show that \(\mathbb{E}(\bar{X}_n) = 0\) for all \(n\), but that \(\bar{X}_n \to -1\) almost surely as \(n \to \infty\).

2.5 The zeta function is defined for \(s > 1\) by \(\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\). Let \(X\) and \(Y\) be independent random variables with
\[
\mathbb{P}(X = n) = \mathbb{P}(Y = n) = n^{-s}/\zeta(s).
\]
Write \(A_n\) for the event that \(n\) divides \(X\). Show that the events \((A_p : p \text{ prime})\) are independent and deduce Euler’s formula
\[
\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).
\]
Show also that \(\mathbb{P}(X \text{ is square-free}) = 1/\zeta(2s)\). Write \(H\) for the highest common factor of \(X\) and \(Y\). Show finally that \(\mathbb{P}(H = n) = n^{-2s}/\zeta(2s)\).

2.6 Let \((X_n : n \in \mathbb{N})\) be independent \(N(0, 1)\) random variables. Prove that
\[
\limsup_n \frac{X_n}{\sqrt{2 \log n}} = 1 \text{ a.s.}
\]

2.7 Let \(C_n\) denote the \(n\)th approximation to the Cantor set \(C\); thus \(C_0 = [0, 1], C_1 = [0, 1/2] \cup [1/3, 1], C_2 = [0, 1/4] \cup [1/3, 1/2] \cup [2/3, 2/4] \cup [3/4, 1], \text{ etc. and } C_n \downarrow C\) as \(n \to \infty\). Denote by \(F_n\) the distribution function of a random variable uniformly distributed on \(C_n\). Show that
(a) \(C\) is uncountable and has Lebesgue measure 0,
(b) for all \(x \in [0, 1]\), the limit \(F(x) = \lim_{n \to \infty} F_n(x)\) exists,
(c) the function \(F\) is continuous on \([0, 1]\), with \(F(0) = 0\) and \(F(1) = 1\),
(d) for almost all \(x \in [0, 1]\), \(F\) is differentiable at \(x\) with \(F'(x) = 0\).

Hint: express \(F_{n+1}\) recursively in terms of \(F_n\) and use this relation to obtain a uniform estimate on \(F_{n+1} - F_n\).