Advanced Probability 3

5.1 Assuming Prohorov's theorem, prove that if $(\mu_n : n \in \mathbb{N})$ is a tight sequence of finite measures on $\mathbb{R}$ and if

$$\sup_n \mu_n(\mathbb{R}) < \infty$$

then there is a subsequence (n_k) and a finite measure μ on $\mathbb{R}$ such that $\mu_{n_k} \to \mu$ weakly.

5.2 *Weak law of large numbers.* Let $(X_n : n \in \mathbb{N})$ be a sequence of independent, identically distributed, integrable random variables. Set $S_n = X_1 + \cdots + X_n$. Use characteristic functions to show that $S_n/n \to \mathbb{E}(X_1)$ weakly.

5.3 Let $(X_n : n \in \mathbb{N})$ be a sequence of random variables and suppose that $X_n \to X$ weakly. Show that, if X is almost surely constant, then also X_n converges to X in probability. Is the condition that X is almost surely constant necessary?

6.1 Let X be an integrable random variable in $\mathbb{R}$ which is not almost surely constant. Show that, for all $\lambda \geq 0$,

$$\mathbb{E}(e^{\lambda X} | X \leq K) \uparrow \mathbb{E}(e^{\lambda X}) \quad \text{as } K \to \infty.$$

Suppose that $M(\lambda) = \mathbb{E}(e^{\lambda X}) < \infty$ for all $\lambda \geq 0$. Show that M has a continuous derivative on $[0, \infty)$ and has derivatives of all orders on $(0, \infty)$. Set $\psi(\lambda) = \log M(\lambda)$ and define a new probability measure $\mathbb{P}_\lambda$ by

$$\mathbb{P}_\lambda(A) = \mathbb{E}(e^{\lambda X - \psi(\lambda)} 1_A)$$

Show that X has mean $\psi'(\lambda)$ and variance $\psi''(\lambda)$ under $\mathbb{P}_\lambda$. Hence show that the derivative ψ' define a homeomorphism on $[0, \infty)$ and determine its range.

6.2 Let $(X_t)_{t\geq 0}$ be a Poisson process of rate 1. Show that, for all $a \geq 1$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(X_t \geq at) = -a \log a + a - 1.$$

7.1 Let $(X_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d$ starting from 0. Let $\sigma \in (0, \infty)$ and let U be an orthogonal $d \times d$-matrix. Show that

(a) $(\sigma X_{\sigma^{-2}t})_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$ starting from 0,
(b) $(UX_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$ starting from 0.

7.2 Let $(X_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ starting from 0. Show that the processes $(X_t)_{t>0}$ and $(tX_{1/t})_{t>0}$ have the same distribution on $C((0, \infty), \mathbb{R})$. Deduce that $X_t/t \to 0$ almost surely as $t \to \infty$.

7.3 Let $(X_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion in $\mathbb{R}^d$ and let F be a bounded measurable function on $C([0, \infty), \mathbb{R}^d)$. Set $\tilde{X}_t = X_t - X_0$. Show that $\tilde{X}$ is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion starting from 0. Hence show, first when F has the form $F(w) = F_0(w_0)F_1(w - w_0)$, and then in general, that, almost surely,

$$\mathbb{E}(F(X) | \mathcal{F}_0) = f(X_0)$$

where $f(x)$ is given by integration of F with respect to Wiener measure starting from x.

7.4 Let $(X_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ starting from 0. Set $Q_t = X_t^2 - t$. Show that $(X_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ are continuous martingales. Define for $a \in \mathbb{R}$

$$T_a = \inf\{t \geq 0 : X_t = a\}.$$

Show that T_a is a stopping time. For $a, b > 0$, show that $\mathbb{P}(T_{-a} < T_b) = b/(a+b)$ and find $\mathbb{E}(T_{-a} \wedge T_b)$.

7.5 Let $(X_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ starting from 0. Show that, for $a > 0$ and $t \geq 0$,

$$\mathbb{P}(T_a \leq t) = 2\mathbb{P}(X_t \geq a).$$

Hence show that $T_a < \infty$ almost surely and find a density function for T_a.

7.6 Let $(X_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ starting from 0. Show that, almost surely,

$$\limsup_{t\to 0} X_t/t = -\liminf_{t\to 0} X_t/t = \infty.$$

For $a > 0$, set

$$L = \sup\{t > 0 : X_t = at\}.$$

Show that L has the same distribution as T_a^{-1}. Define

$$S = \sup\{t \leq 1 : X_t = 0\}, \quad T = \inf\{t \geq 1 : X_t = 0\}.$$

Show that S and T^{-1} have the same distribution. Hence show that

$$\mathbb{P}(S \leq t) = \frac{2}{\pi} \arcsin \sqrt{t}.$$

7.7 Let $(X_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ starting from 0. Find the joint distribution of $(X_t, \max_{s\leq t} X_s)$.

7.8 Let $(X_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^3$. You may assume that $B_t \neq 0$ for all $t > 0$ almost surely. Set $R_t = 1/|X_t|$. Show that

(i) $(R_t : t \geq 1)$ is bounded in L^2,
(ii) $\mathbb{E}(R_t) \to 0$ as $t \to \infty$,
(iii) R_t is a supermartingale.

Deduce that $|X_t| \to \infty$ almost surely as $t \to \infty$.

7.9 Let A be a non-empty open subset of the unit sphere in $\mathbb{R}^d$ and consider the cone

$$C = \{ty : t \in (0,1), y \in A\}.$$

Let $(X_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^d$ starting from 0 and set

$$T = \inf\{t \geq 0 : X_t \in C\}.$$

Use the Brownian scaling property to show that $\mathbb{P}(T < t)$ is non-increasing in $t > 0$. Deduce that $T = 0$ almost surely.