Advanced Probability 1

1.1 Let $X, Y \in L^1(\mathbb{P})$ and let $\mathcal{G}$ be a σ-algebra. Show that

$$\mathbb{E}(X+Y|\mathcal{G}) = \mathbb{E}(X|\mathcal{G}) + \mathbb{E}(Y|\mathcal{G}) \quad \text{almost surely.}$$

1.2 Let X be a non-negative random variable and let Y be a version of $\mathbb{E}(X|\mathcal{G})$. Show that $\{X > 0\} \subseteq \{Y > 0\}$ almost surely, that is, $1_{\{X>0\}} \leq 1_{\{Y>0\}}$ almost surely. Show further that, for all $A \in \mathcal{G}$, if $\{X > 0\} \subseteq A$ almost surely then $\{Y > 0\} \subseteq A$ almost surely.

1.3 Let $X, Y \in L^2(\mathbb{P})$. Show that if

$$\mathbb{E}(X|Y) = Y \quad \text{and} \quad \mathbb{E}(Y|X) = X \quad \text{almost surely}$$

then $X = Y$ almost surely. Show that this holds also for $X, Y \in L^1(\mathbb{P})$.

1.4 Let X be an integrable random variable and let $x \in \mathbb{R}$. Show that

$$\mathbb{E}(X|X \leq x) \leq \mathbb{E}(X).$$

Here and below, it is to be assumed that the conditioning event has positive probability. Let Y be another random variable, independent of X, and let f be a non-decreasing function such that $f(X+Y)$ is integrable. Show that

$$\mathbb{E}(f(X+Y)|X \leq x) \leq \mathbb{E}(f(X+Y)).$$

Let $S_n = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are independent, and let $x_1, \ldots, x_n \in \mathbb{R}$. Show that

$$\mathbb{P}(S_n \geq x | X_1 \leq x_1, \ldots, X_n \leq x_n) \leq \mathbb{P}(S_n \geq x).$$

1.5 Show that, for any sequence of non-negative random variables $(X_n : n \in \mathbb{N})$ and any σ-algebra $\mathcal{G}$,

$$\mathbb{E}(\liminf X_n|\mathcal{G}) \leq \liminf \mathbb{E}(X_n|\mathcal{G}) \quad \text{almost surely.}$$

1.6 Let X and Y be random variables and let $\lambda \in (0, \infty)$. Show that, if X and $Y - X$ are independent exponential random variables of parameter λ, then Y has density $\lambda^2 y e^{-\lambda y}$ on $(0, \infty)$ and, for all $x \geq 0$, almost surely,

$$\mathbb{P}(X \leq x|Y) = (x/Y) \wedge 1.$$

Show that the converse also holds.

2.1 Let $(X_n)_{n\geq 0}$ be an integrable process, taking values in a countable set $E \subseteq \mathbb{R}$. Show that $(X_n)_{n\geq 0}$ is a martingale in its natural filtration if and only if, for all n and for all $x_0, \ldots, x_n \in E$, whenever the conditioning event has positive probability, we have

$$\mathbb{E}(X_{n+1}|X_0 = x_0, \ldots, X_n = x_n) = x_n.$$

2.2 Let $(X_n)_{n\geq 0}$ be a martingale and let f be a convex function on $\mathbb{R}$ such that $f(X_n)$ is integrable for all n. Show that $(f(X_n)_{n\geq 0}$ is a submartingale.

2.3 Let $(X_n)_{n\geq 0}$ be a martingale and let T be a stopping time. Consider the following conditions

(a) $T \leq n$ for some $n \geq 0$,
(b) there is a constant $C < \infty$ such that $|X_n| \leq C$ for all $n \leq T$ almost surely,
(c) $\mathbb{E}(T) < \infty$ and there is a constant $C < \infty$ such that $|X_{n+1} - X_n| \leq C$ for all $n < T$ almost surely.

Show that, under each one of these conditions, X_T is integrable and $\mathbb{E}(X_T) = \mathbb{E}(X_0)$.

2.4 Let $X \in L^2(\mathbb{P})$ and set

$$X_n = \mathbb{E}(X|\mathcal{F}_n)$$

where $(\mathcal{F}_n)_{n\geq 0}$ is a given filtration. Show that, for all $m \leq n$,

$$\|X_m\|_2^2 + \|X_m - X_n\|_2^2 = \|X_n\|_2^2.$$

Hence show there exists $Y \in L^2(\mathbb{P})$ such that $X_n \to Y$ in L^2. Show further that $Y = X$ almost surely if and only if X is $\mathcal{F}_\infty$-measurable, where $\mathcal{F}_\infty = \sigma(\mathcal{F}_n : n \geq 0)$.

2.5 Let $(X_n)_{n\geq 0}$ be a martingale, starting from 0. Show that $(X_n)_{n\geq 0}$ is bounded in L^2 if and only if $\sum_n \|X_{n+1} - X_n\|_2^2 < \infty$.

3.1 *Pólya's urn.* At time 0, an urn contains two balls, one black, the other white. Suppose we repeatedly choose a ball at random from the urn and replace it together with a new ball of the same colour. Then, after n steps, there are $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of steps in which a black ball was chosen. Let $M_n = (B_n + 1)/(n + 2)$ the proportion of black balls in the urn after n steps. Show that $(M_n)_{n\geq 0}$ is a martingale, relative to a filtration which you should specify. Show also that

$$\mathbb{P}(B_n = k) = (n + 1)^{-1}, \quad k = 0, 1, \ldots, n.$$

Deduce that there is a random variable Θ such that $M_n \to \Theta$ almost surely and find the distribution of Θ.

For $\theta \in [0, 1]$, set

$$N_n^\theta = \frac{(n + 1)!}{B_n!(n - B_n)!}\,\theta^{B_n}(1 - \theta)^{n-B_n}.$$

Show that $(N_n^\theta)_{n\geq 0}$ is a martingale.

3.2 *Bayes' urn.* A random number Θ is chosen uniformly in $[0, 1]$, and a coin with probability Θ of heads is minted. The coin is tossed repeatedly. Let B_n be the number of heads in n tosses. Show that the process $(B_n)_{n\geq 0}$ has the same distribution as the process $(B_n)_{n\geq 0}$ in Example 3.1. Show that N_n^θ is a conditional density function of Θ given $B_1, \ldots, B_n$.

3.3 Let $X_1, X_2, \ldots$ be independent with $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$ for all n. Show that the series $\sum_n X_n/n$ converges almost surely.

3.4 Let $X_1, X_2, \ldots$ be independent with $\mathbb{P}(X_n = -1/p_n) = p_n$ and $\mathbb{P}(X_n = 1/q_n) = q_n$, where $p_n = 1/n^2$ and $p_n + q_n = 1$. Set $S_n = X_1 + \cdots + X_n$. Show that $(S_n)_{n\geq 0}$ is a martingale and that S_n/n converges almost surely as $n \to \infty$. Deduce that $S_n \to \infty$ almost surely as $n \to \infty$.