Advanced Probability (M24)

I. Bailleul

This course aims to cover some advanced topics at the heart of research in probability theory, with an emphasis on the tools needed for the analysis of stochastic processes like Brownian motion.

It will be assumed that students have some familiarity with the measure theoretic formulation of probability theory, at the level of the Part II(B) course Probability and Measure, or part A of D. Williams' book.

1. 'Static' theory of Stochastic Processes

- Measure theoretic tools for construction of probabilities.

  a) Construction of measures.

  b) Good modifications. Application: construction of Wiener measure and Brownian motion.

- Weak convergence in separable Banach spaces:

  a) Finite dimensional theory (characteristic functions, Lévy's continuity theorem).

  b) Infinite dimensional theory: Prohorov's theorem, couplings.

  c) Application: Donsker's invariance principle.

2. Dynamic theory of Stochastic Processes

- Conditional expectation (application to sufficient statistics).
- Dynamics and filtrations, stopping times.
- Martingales: *a)* Discrete time theory, *b)* Applications: Radon-Nikodym theorem, changes of measure, a glimpse at concentration of measure phenomenon, *c)* Continuous time theory.

3. Brownian motion and Lévy processes

- Distributional and sample paths properties of Brownian motion.
- Martingales and strong Markov property (application to Dirichlet problem).
- Lévy processes: integral with respect to a random Poisson measure, Lévy-Khinchin structure theorem.

Appropriate books • L.C.G. Rogers and D. Williams, *Diffusions, Markov processes and Martingales, Vol. 1 (2nd edition).* Chapters I and II. Wiley 1994

• K.L. Chung, *Green, Brown and Probability & Brownian motion on the line.* World Scientific 2002.

• D. Williams, *Probability with Martingales.* C.U.P. 1991

• O. Kallenberg, *Foundations of Modern Probability.* Chapters 1-3, 6, 7, 16. Springer 1997

(...) and many others which I will mention along the course.