ADVANCED PROBABILITY
SOLUTIONS FOR EXAMPLE SHEET 3

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1. Set $\widetilde{B}_t = tB_{1/t}$ for $t > 0$ and $\widetilde{B}_0 = 0$. We know from proposition 94 that $\widetilde{B}$ is a Brownian motion. Also, as $\mathcal{F}^{\widetilde{B}}_{0+} = \mathcal{T}$, Blumenthal's 0-1 law applied to $\widetilde{B}$ shows that $\mathcal{T}$ is made up of trivial events *for* $\mathbb{P}$. (They might be non-trivial for a different probability!)

2. We proceed as in the proof of proposition 92, denoting by $\mathcal{C}$ the cone. As the event $\{\tau_U = 0\} \in \mathcal{F}_{0+}$, it suffices to prove that $\mathbb{P}(\tau_U = 0) \geqslant c$ for some positive constant c to prove that it has probability 1, by Blumenthal's 0-1 law. Let $\epsilon > 0$ be given. As $\mathbb{P}(\tau_U \leqslant \epsilon) \geqslant \mathbb{P}(B_\epsilon \in \mathcal{C})$ and the law of B is invariant by rotations, we have $\mathbb{P}(B_\epsilon \in \mathcal{C}) = |A|\Big(\int \frac{e^{-r^2/2\epsilon}}{(2\pi\epsilon)^{d/2}} \mathbf{1}_{r\leqslant a}\, r^{d-1} dr\Big) = |A|\Big(\int \mathbf{1}_{u\leqslant a\epsilon^{-1/2}} \frac{2^{-u^2/2}}{(2\pi)^{d/2}}\, u^{d-1} du\Big) \geqslant c$, where $|A|$ is the surface of $A \subset \mathbb{S}^{d-1}$. Sending ϵ to 0 gives the conclusion.

3. We make the same reasonning as in exercise 10 in example sheet 2. Set $T = \min\{H_{-a}, H_b\}$ and write p for $\mathbb{P}(H_{-a} < H_b)$. As the *stopped processes* $(B_t)_{t\geqslant T}$ is a bounded martingale, the optionnal stopping theorem gives us: $0 = p(-a) + (1-p)b$, hence the value of p. Use the martingale $B_t^2 - t$ to compute $\mathbb{E}[T]$.

4. b) Given $0 \leqslant s < t$ and $A \in \mathcal{F}_s$, we have $\mathbb{E}\big[e^{\sigma B_t - \frac{\sigma^2}{2}t}\mathbf{1}_A\big] = \mathbb{E}\big[e^{\sigma B_s - \frac{\sigma^2}{2}s}\mathbf{1}_A\big]$ for all σ. Expanding the exponential on both sides in power series of σ, use the fact that $\mathbb{E}[B^{2k}] = \prod_{p=0}^{k-1}(2k-2p-1)$ (induction) to justify the interchange of $\mathbb{E}$ and $\sum_k$. The term $\mathbb{E}\big[(B_t^2 - t)\mathbf{1}_A\big]$ appears as the coefficient of σ on the lhs and the term $\mathbb{E}\big[(B_s^2 - s)\mathbf{1}_A\big]$ as the coefficient of σ on the rhs. Their identification gives the martingale property of the first process as we can take any $0 \leqslant s < t$ and $A \in \mathcal{F}_s$. Look at the coefficients of σ^2 and σ^3 to obtain the martingale property of the two other processes.[1]

5. a) We need θ to satisfy $\lambda - \theta c = \frac{\theta^2}{2}$, that is $\theta = \sqrt{c^2 + 2\lambda} - c$, since it is positive.

b) As the *stopped martingale* $\Big(e^{\theta B^c_t - \lambda t}\Big)_{0\leqslant t\leqslant T}$ is bounded, the optionnal stopping theorem implies: $1 = \mathbb{E}\big[e^{\theta x - \lambda H^c_x}\big]$, hence the formula. We check that this function of $\lambda > 0$ coincides with the Laplace transform of the given density. As the Laplace transform characterizes the distribution H^c_x has the mentionned density/

c) It suffices to let λ decrease to 0.

6. b) Recall the strong Markov property: *Given any finite stopping time T, the process $(B_{T+t} - B_T)_{t\geqslant 0}$ is a Brownian motion independent of $\mathcal{F}_T$.* Apply it to T_a for some $a \geqslant 0$. The fact that it is a Brownian motion says that if $b > a$ then $T_b - T_a$ is distributed as

[1] Note that I have not tried to work directly with the conditionnal expectation identity $\mathbb{E}\big[e^{\sigma B_t - \frac{\sigma^2}{2}t}\big|\mathcal{F}_s\big] = e^{\sigma B_s - \frac{\sigma^2}{2}s}$ as this identity involves random variables defined only almost-surely, so it is not obvious how to differentiate with respect to σ in a mathematically neat way.

T_{b-a}, giving the stationnarity of the process $(T_a)_{a\geqslant 0}$. The independence of the increments comes from the second piece of information provided by the strong Markov property: the independence of $(B_{T+t}-B_T)_{t\geqslant 0}$ with respect to $\mathcal{F}_T$. Given $a_1<a_2<\cdots<a_n$, a straightforward induction enables to prove that the increments $T_{a_2}-T_{a_1},\dots,T_{a_n}-T_{a_{n-1}}$ are independent. It is not a Lévy process though, as it is not càdlàg but continuous on the *left* with right limits. Prove it!

7. a) We have $T_a\leqslant S_a$ and $S_a=T_a+\inf\{t\geqslant 0\,;\,B_{t+T_a}-B_{T_a}>0\}$. The strong Markov property gives $S_a=T_a$, almost-surely.

c) Take for L the time in $[0,1]$ where B_t is maximum. Prove that it is almost-surely <1. It follows that we have almost-surely $S_L\geqslant 1$ and $S_L\neq T_L$.

8. a) Set $T_0=0$ and define inductively $S_n=\inf\{t\geqslant T_{n-1}\,;\,B_t\in D\}$ and $T_n=\inf\{t\geqslant S_n\,;\,B_t\notin B(0,2r)\}$. By the strong Markov property and the invariance of the law of Brownian motion by rotations, the random variables $\int_{S_k}^{T_k}\mathbf{1}_D(B_s)\,ds$ are iid. As they have positive mean, the strong law of large numbers gives $\int_0^\infty\mathbf{1}_D(B_s)\,ds\geqslant\sum_{n=0}^\infty\int_{S_k}^{T_k}\mathbf{1}_D(B_s)\,ds=\infty$, almost-surely.

b) Denote by $p_t(x,y)$ the transition kernel of Brownian motion. By Fubini's theorem, we have $\mathbb{E}_x\big[\int_0^\infty f(B_t)\,dt\big]=\int\Big(\int_0^\infty p_t(x,y)\Big)f(y)\,dy$, for any non-negative function f. The time integral equals $|y-x|^{2-d}$ up to a multiplicative constant C. (Do the computation! We see why we need $d\geqslant 3$.) This function of y is locally integrable with respect to y.[2]

9. We know from exercise 7 the distribution of T. As it is independent of B^1, we have

$$\mathbb{E}\big[f(B^1_T)\big]=\int_0^\infty\frac{2^{-\frac{1}{2t}}}{\sqrt{2\pi t^3}}f(B^1_t)\,dt=\int f(x)\Big(\int_0^\infty\frac{2^{-\frac{1}{2t}}}{\sqrt{2\pi t^3}}\frac{2^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}\,dt\Big)\,dx=\int f(x)\frac{dx}{\pi(1+x^2)},$$

for any bounded measurable function $f:\mathbb{R}\to\mathbb{R}$. We read the distribution of B^1_T above: it is a Cauchy random variable.

10. The process $M_t=|B_t|^2-t\,d\;\big(=\sum_{i=1}^d|B^i_t|^2-t,\text{sum of independent martingales}\big)$ is a martingale. We would like to use the optionnal stopping theorem to the stopped martingale $(M_t)_{t\leqslant T}$; yet this process is not bounded, so it is convenient ot replace first T by $T\wedge n$ (rather than proving for instance that $(M_t)_{t\leqslant T}$ is uniformly integrable, which can be done). The new stopped martingale is bounded. So we have $|x|^2=\mathbb{E}\big[|B_{T\wedge n}|^2-d(T\wedge n)\big]$, that is $\mathbb{E}\big[T\wedge n\big]=\frac{\mathbb{E}\big[|B_{T\wedge n}|^2\big]-|x|^2}{d}$. Use monotone convergence on the lhs, and dominated convergence on the rhs, to conclude by sending n to infinity.

11. Suppose g has a maximum M at a point x_0 inside O. As it has the mean value property, g needs to be equal to M near x_0; this shows that the closed set where g attains its maximum is also open. As O is connected, g is constant, equal to its maximum, on the whole of O.

Would a given Dirichlet problem have two solutions, their difference would be a solution to the Dirichlet problem with null boundary condition, so would have a null maximum. As the opposite of this difference is also a solution, it would also have a null maximum, leading to the equality of the two functions.

[2] If x is not in the domain of integration, no problem; otherwise, use polar coordinates near x.

13. Let denote by $(N_t)_{t\geqslant 0}$ a Poisson process of intensity λ and jump measure J. Can you see why it suffices to consider the case where $J(\cdot)=\delta_1(\cdot)$? In that case, we need to prove that given any $n\geqslant 1$, any times $t_1<\cdots<t_n$ and any integers $i_1,\dots,i_n$, we have

$$\mathbb{P}\big(N_{t_2}-N_{t_1}=i_1,\dots,N_{t_n}-N_{t_{n-1}}=i_{n-1}\big)=\prod_{k=1}^{n-1}\frac{\big(\lambda(t_k-t_{k-1})\big)^{i_k}}{i_k!}e^{-\lambda(t_k-t_{k-1})}.$$

We proceed by induction on $n\geqslant 1$. The case $n=1$ is treated in exercise 12. To make the induction step, it suffices to prove that

$$\mathbb{P}\big(N_{t_{n+1}}-N_{t_n}=i_n\big|N_{t_k}-N_{t_{k-1}}=i_{k-1},\ \text{for } k=1..n\big)=\frac{\big(\lambda(t_n-t_{n-1})\big)^{i_n}}{i_n!}e^{-\lambda(t_{n+1}-t_n)} \tag{0.1}$$

Set $i=i_1+\cdots+i_{n-1}$, and denote by H_i the hitting time of $\{i\}$ by the process N. Then, conditionally on the event $\{H_i<t_{n-1}<H_i+S_i\}$, the time $H_i+S_i-t_{n-1}$ to wait after t_{n-1} before the next jump is exponentially distributed, with parameter λ, by the memoryless property of S_i. Identity (0.1) follows as $\mathbb{P}\big(N_{t_{n+1}}-N_{t_n}=i_n\big|N_{t_k}-N_{t_{k-1}}=i_{k-1},\ \text{for } k=1..n\big)=\mathbb{P}\big(N_{t_{n+1}}-N_{t_n}=i_n\big|H_i<t_{n-1}<H_i+S_i\big)$, by the strong Markov property of the Markov chain $(N_t)_{t\geqslant 0}$.

14. Denote by S_1 the first holding time. The obvserver is proved wrong if at some time t he observes that $\{N_t=N_{t-S_1}\}$. Given $s>0$, let define the stopping time $T_s=\inf\{t\geqslant s\,;\,N_t=N_{t-s}\}$ – with respect to which filtration? Then, conditionning on the first jump, the strong Markov property gives

$$\mathbb{E}\big[T_s\big]=se^{-\lambda s}+\int_0^s\big(a+\mathbb{E}[T_s]\big)\lambda e^{-\lambda a}\,da,$$

so $\mathbb{E}\big[T_s\big]=\frac{e^{\lambda S_1}-1}{\lambda}$. The mean time until one sees a holding time bigger than S_1 is thus

$$\int_0^\infty\big(s+\mathbb{E}[T_s]\big)\,\lambda e^{-\lambda s}\,ds=\infty.$$

15. a) It suffices to prove, for all $n\geqslant 1$, that $S_1+\cdots+S_n$ is almost-surely different from t. (Can you see why?) This follows from the fact that the random variable $S_1+\cdots+S_n$ has a density with respect to Lebesgue measure on $\mathbb{R}_+$.

b) Note that

$$\begin{aligned}\mathbb{P}(T_t>t+s)&=\sum_{k=1}^\infty\mathbb{P}(N_t=k,N_{t+s}=k)=\sum_{k=1}^\infty\mathbb{P}(N_t=k,N_{t+s}-N_t=0)\\&=\sum_{k=1}^\infty\mathbb{P}(N_t=k)\,\mathbb{P}(,N_{t+s}-N_t=0)=\sum_{k=1}^\infty\mathbb{P}(N_t=k)\,e^{-\lambda s}=e^{-\lambda s}.\end{aligned}\tag{0.2}$$

So T_t-t is exponentially distributed, with parameter λ.

16. It's even worse! The sum of two Brownian motions can be non-Brownian! To see that, let us work on the subset Ω of $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^2)$ made up of paths starting from 0, equipped with its Borel σ-algebra. Let X be the coordinate process $X_t(\omega)=\omega(t)=(\omega_1(t),\omega_2(t))\in\mathbb{R}^2$, for $\omega\in\Omega$, and let $\mathbb{P}$ be Wiener measure. Let $\mathbb{P}'$ be the measure on $(\Omega,\mathcal{F})$ under which X is a Wiener measure with correlation -1. Let $\mathbb{Q}=\frac{\mathbb{P}+\mathbb{P}'}{2}$. I let you prove that the processes ω_1 and ω_2 are Wiener processes under $\mathbb{Q}$. Can you prove by a simple calculation that the continuous process $\omega_1 + \omega_2$ is not Gaussian? As Brownian motion with dift (a Gaussian process!) is the only continuous Lévy process (see exercise 21), this proves the claim.

17. Let[3] Ω be an arbitrary space and $\mathcal{F}$ be the trivial σ-algebra over it. (We work with deterministic processes!). Let also $(x_\alpha)_\alpha$ be a Hamel basis of Haar over the rational numbers. For every $t \geqslant 0$, let X_t be the sum of the coordinates of t in the Hamel basis. As $X_{t+s} = X_t + X_s$, the process X has stationnary independent increments. As X is highly discontinuous (it takes values in $\mathbb{Q}$!), it does not have a modification which is càdlàg .

18. For $s < t$, we have $\mathbb{E}\left[e^{i\lambda(X_t - X_s)}\right] = e^{(t-s)g(\lambda)}$. Senging $s \uparrow t$ we conclude that $\mathbb{E}\left[e^{i\lambda(\Delta X_t)}\right] = 1$, so ΔX_t has the same fistribution as the constant 0, that is ΔX_t is almost-surely null.

19. **a)** We know, from the general construction of Lévy processes given in the course, that X has the same law as the sum of a difted Brownian motion, an independent Poisson process with finite intensity, and a infinite sum of independent compensated Poisson processes. (This sum takes care of the fact that the jump measure can have an infinite mass.) In the case of a finite jump measure, only the first two termsare needed; as Poisson processes have almost-surely finitely many jumps in any finite time interval, we are done.

b) We can forget the continuous part (drifted Brownian motion) and work only with the Poisson process. Let S_i, J_i be the successive holding and jump times of the process; they are all independent. By construction, the process Y is constructed out of the sequence of jump times $\left((S_1 + \cdots + S_i)\mathbf{1}_{\epsilon_i = 1}\right)_{i \geqslant 1}$ and the corresponding jumps. The time between two jumps will have the same law as $S_1 + \cdots + S_N$, where N is a geometrical random variable with parameter p. A straightforward computation shows that this random sum with exponentially distributed, with parameter $p\lambda$. So Y is a Lévy process with jump measure $p\Lambda_X$.

20 - 21. Copy word by word what has been done previously elsewhere.

22. See for instance theorem (28.12), p. 76, in Rogers and Williams' book.

References

[Med07] P. Medvegyev. *Stochastic integration theory*, volume 14 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2007.

[3] This solution is taken from the excellent book [Med07] by P. Medvegyev.