ADVANCED PROBABILITY
SOLUTIONS FOR EXAMPLE SHEET 2

BATI SENGUL

1. a) We need to prove that we have $\mathbb{E}\big[h(V)\mathbf{1}_A\big] = \mathbb{E}[g(U)\mathbf{1}_A]$, for each $A \in \sigma(U)$; any such event is by definition of the form $\mathbf{1}_B(U)$, for some measurable subset B of $\mathbb{R}$. Using Fubini's theorem, we have

$$\begin{aligned}\mathbb{E}\big[h(V)\mathbf{1}_B(U)\big] &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_{U,V}(u,v)h(v)\mathbf{1}_B(u)\,dudv = \int_{\mathbb{R}} \mathbf{1}_B(u)\int_{\mathbb{R}} f_{U,V}(u,v)h(v)\,dvdu \\ &= \int_{\mathbb{R}} \mathbf{1}_B(u)g(u)f_U(u)\,du = \mathbb{E}\big[g(U)\mathbf{1}_B(u)\big].\end{aligned}$$

b) Consider

$$X := V - \frac{Cov(U,V)}{Var(U)}U$$

then X is a centred Gaussian random variable, moreover

$$\mathbb{E}[XU] = \mathbb{E}[UV] - \frac{Cov(U,V)}{Var(U)}\mathbb{E}\big[U^2\big] = \mathbb{E}[UV] - Cov(U,V) = 0$$

hence X is independent of U, so $\mathbb{E}[X|\sigma(U)] = 0$. Now

$$\begin{aligned}\mathbb{E}\big[V|\sigma(U)\big] &= \mathbb{E}\Big[\frac{Cov(U,V)}{Var(U)}U + X\Big|\sigma(U)\Big] \\ &= \frac{Cov(U,V)}{Var(U)}U.\end{aligned}$$

c) Let us prove more generally that any $\sigma(U)$-measurable almost-surely finite random variable X is of the form $f(U)$ for some measurable function $f : \mathbb{R} \to \mathbb{R}$.

Suppose first that X takes only finitely many values $x_1, \ldots, x_n$. As each set $A_i = X^{-1}\big(\{x_i\}\big)$ belongs to $\sigma(U)$, it is of the form $U^{-1}(B_i)$ for some measurable $B_i \subset \mathbb{R}$; the B_i's are disjoint. Set $f(x) = x_i$ if $x \in B_i$ for some i, and $f(x) = 0$ elsewhere. We check directly that $f(U) = X$.

For $X \geqslant 0$, we define a $\sigma(U)$-measurable random variable setting

$$X_n = \sum_{j=0}^{n2^n} \frac{j}{2^n}\mathbf{1}_{X\in(j2^{-n},(j+1)2^{-n}]}.$$

As it takes only finitely many values, it is of the form $f_n(U)$. Note that $X_n \uparrow X$ almost-surely. Set $\overline{f} = \overline{\lim} f_n$ and $f = \overline{f}\mathbf{1}_{\overline{f}<\infty}$ and check that $f(U) = X$ as X is almost-surely finite.

2. The trivial case $k = 1$ is obvious. So suppose that the statement holds for k, i.e.

$$\mathbb{P}(T > kN) \leqslant (1-\epsilon)^k.$$

Then by using $\mathbb{P}(T > n + N|\mathcal{F}_n) \leqslant 1 - \epsilon$ and the fact $\{T > (k+1)N\} \subset \{T > kN\}$ we have that[1]

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{T>(k+1)N}] &= \mathbb{E}[\mathbb{E}[\mathbf{1}_{T>(k+1)N}|\mathcal{F}_{kN}]] \\
&= \mathbb{E}[\mathbb{E}[\mathbf{1}_{T>(k+1)N}\mathbf{1}_{T>kN}|\mathcal{F}_{kN}]] \\
&= \mathbb{E}[\mathbf{1}_{T>kN}\mathbb{E}[\mathbf{1}_{T>(k+1)N}|\mathcal{F}_{kN}]] \\
&\leqslant \mathbb{E}[\mathbf{1}_{T>kN}(1-\epsilon)] \leqslant (1-\epsilon)^k(1-\epsilon).
\end{aligned}$$

3. a) Work with $\Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, the coordinate process and its filtration $(\mathcal{F}_t)_{t \geqslant 0}$, and set $\mathcal{G}_t = \{\emptyset, \Omega\}$ for $t \leqslant 1$, and $\mathcal{G}_t = \mathcal{F}_{t-1}$ for $t \geqslant 1$. Look at the hitting time of some level.

b) For any $b \leqslant a$ we have by the continuity of ω

$$\{\gamma_a \leqslant b\} = \{\omega \in \Omega\,;\, \omega_t > 0 \text{ for all } t \in (b, a]\} = \bigcap_{t \in (b,a] \cap \mathbb{Q}} \{\omega \in \Omega\,;\, \omega_t > 0\} \in \mathcal{F}_a.$$

Next we show that $\{\gamma_a < t\} \notin \mathcal{F}_t$ for $t < a$. Intuitively this fails ultimately because at time $t < a$ we cannot deduce if γ_a has happened or not, given the path up to time t. More rigorously

$$\{\gamma_a < t\} = \{\omega_s \neq 0\, \forall s \in [t, a]\} = \{\omega_s \neq 0\, \forall s \in [t, a] \cap \mathbb{Q}\} = \bigcap_{s \in [t,a] \cap \mathbb{Q}} \{\omega_s \neq 0\}$$

where we have used the continuity in the second equality. Now the last part is not in $\mathcal{F}_t$ and hence $\{\gamma_a < t\} \notin \mathcal{F}_t$.

4. b) Obviously we have that $\mathcal{F}_{S \wedge T} \subset \sigma(\mathcal{F}_S, \mathcal{F}_T)$. For the converse notice first that $\sigma(F_S, F_T)$ is generated by events in $\mathcal{F}_S$ and $\mathcal{F}_T$, hence by the monotone class theorem, it suffices to check that $\mathcal{F}_S$ and $\mathcal{F}_T$ are included in $\mathcal{F}_{S \wedge T}$. Let $A \in \mathcal{F}_S$, then it suffices to show that $A \cap \{S \wedge T > t\} \in \mathcal{F}_t$ for each $t \geqslant 0$. Notice that

$$A \cap \{S \wedge T > t\} = A \cap \{S > t\} \cap \{T > t\}.$$

Now $B := A \cap \{S > t\} \in \mathcal{F}_t$, by definition of $\mathcal{F}_S$, and as T is a stopping time $B \cap \{T > t\} \in \mathcal{F}_t$ and hence $A \in \mathcal{F}_{S \wedge T}$. Similarly for $A \in \mathcal{F}_T$.

5. Suppose that $X_n \to X$ in L^1. Then by Markov's inequality $X_n \to X$ in probability:

$$\mathbb{P}\big(|X_n - X| > \epsilon\big) \leqslant \epsilon^{-1}\mathbb{E}\big[|X_n - X|\big] \to 0 \quad \text{as } n \to \infty.$$

Fix $\epsilon > 0$, then there exists an $N \in \mathbb{N}$ such that $\mathbb{E}\big[|X_n - X|\big] < \epsilon$ for $n \geqslant N$. The sequence $X, X_1, ..., X_N$ is finite and hence uniformly integrable, so there exists a $K > 0$ such that $\mathbb{E}\big[|X_n|\mathbf{1}_{|X_n|>K}\big] < \epsilon$ for all $n \leqslant N$ and $\mathbb{E}\big[|X|\mathbf{1}_{[X|>K}\big] < \epsilon$. For $n > N$ we have

$$\mathbb{E}\big[|X_n|\mathbf{1}_{|X_n|>K}\big] \leqslant \mathbb{E}\big[|X_n - X|\mathbf{1}_{|X_n|>K}\big] + \mathbb{E}\big[|X|\mathbf{1}_{|X_n|>K}\big] < \epsilon + \mathbb{E}\big[|X|\mathbf{1}_{|X_n|>K}\big].$$

Then the second term is small if $\mathbb{P}\big(|X_n| > K\big)$ is small, uniformly in n. But then by Markov's inequality and the fact that $\mathbb{E}[|X_n|] \leqslant \mathbb{E}[|X|] + \epsilon$ we have

$$\mathbb{P}\big(|X_n| > K\big) \leqslant K^{-1}\mathbb{E}\big[|X_n|\big] \leqslant K^{-1}(\mathbb{E}\big[|X|\big] + \epsilon)$$

which can be made small by choosing K large.

[1] Here I will use the fact that $\mathbf{1}_{T>(k+1)N}\mathbf{1}_{T>kN} = \mathbf{1}_{T>(k+1)N}$.

For the converse suppose that X_n is UI and $X_n \to X$ in probability. Consider the following approximation

$$\mathbb{E}\big[|X_n - X|\big] \leqslant \mathbb{E}\big[|X_n - X|\mathbf{1}_{|X_n-X|\leqslant K}\big] + \mathbb{E}\big[|X_n - X|\mathbf{1}_{|X_n-X|>K}\big] \leqslant K + \mathbb{E}\big[|X_n - X|\mathbf{1}_{|X_n-X|>K}\big].$$

Now by the uniform integrability the term on the RHS can be made small given that $\mathbb{P}\big(|X_n - X| > K\big)$ is small. Pick $K = \epsilon$ small and let $n > N$ be sufficiently large such that $\mathbb{E}\big[|X_n - X|\mathbf{1}_{|X_n-X|>K}\big] < \epsilon$.

6. Suppose that $\mathbb{P} << \mathbb{Q}$ then by the Radon-Nikodym theorem we have that $\mathbb{P}(A) = \mathbb{E}_{\mathbb{Q}}[X\mathbf{1}_A]$ where $X \in L^1(\mathbb{Q})$ and $0 \leqslant X \leqslant 1$, so in particular $\mathbb{P}(A) \leqslant \mathbb{Q}(A)$.

Conversely let $\mathbb{Q}(A) = 0$, then for each epsilon $\mathbb{P}(A) < \epsilon$, i.e. $\mathbb{P}(A) = 0$.

7. Suppose first that $\mathbb{Q} << \mathbb{P}$, then by the Radon-Nikodym theorem $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[X\mathbf{1}_A]$ for all $A \in \mathcal{F}$ with $X \in L^1(\mathbb{P})$ and $0 \leqslant X \leqslant 1$. In particular we have that $M_n = \mathbb{E}[X|\mathcal{F}_n]$ and hence M_n is uniformly integrable.

Suppose on the other hand that M_n is uniformly integrable (with respect to $\mathbb{P}$), then M_n converges in $L^1(\mathbb{P})$ and a.s. to some M_∞, so that $M_n = \mathbb{E}[M_\infty|\mathcal{F}_n]$.

8. The idea is to prove that $\mathcal{T}$ is independent of itself. To that end define $\mathcal{F}_n := \sigma(X_k : k \leqslant n)$, then $\mathcal{F}_n$ is independent of $\sigma(X_k : k > n)$ (as the random variables are independent), and in particular independent from $\mathcal{T}$. This holds for all $n \in \mathbb{N}$ and hence $\mathcal{T}$ is independent of $\mathcal{F}_\infty := \bigvee_{n\geqslant 1} \mathcal{F}_n$. However $\mathcal{T} \subset \mathcal{F}_\infty$ and hence $\mathcal{T}$ is independent of itself. Now for any $A \in \mathcal{T}$, we have that $\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A)$ so $\mathbb{P}(A)$ is either 0 or 1.

The trivial counterexample to when X_i are not independent is by considering $X_i = X$ for some non-trivial random variable, then $\mathcal{T} = \sigma(X)$ which is non-trivial.

9. a) Let $X \in \mathcal{F}_\infty$ be bounded, then $X_n := \mathbb{E}[X|\mathcal{F}_n]$ makes sense and is bounded by the same bound. Then by the martingale convergence $X_n \to X$ in L^1 and hence the result.

b) By part a), the bounded elements of $L^1(\mathcal{F}_\infty)$ are limit points of $\mathbb{E}[\cdot|\mathcal{F}_n] \in \bigcup_{k\geqslant 0} \mathcal{F}_k$. Now if $X \in L^1(\mathcal{F}_\infty)$ is not bounded, then it can be approximated by bounded functions.

c) Kolmogorov's 0-1 Law: We have that for any $A \in \mathcal{T}$

$$\mathbb{E}[\mathbf{1}_A|\mathcal{F}_n] \to \mathbf{1}_A$$

so as before A is independent of $\mathcal{F}_n$, hence $\mathbb{E}[\mathbf{1}_A|\mathcal{F}_n] = \mathbb{P}(A)$.

d) In the case $\mathcal{F}_n$ is finite, then

$$\mathbb{E}[X|\mathcal{F}_n] = \sum \frac{\mathbb{E}[X\mathbf{1}_{A_n}]}{\mathbb{P}(A_n)}\mathbf{1}_{A_n}$$

which is computable. So then the limits also may be computed explicitly.

(i) Suppose that the measure space is separable. First note that $L^1(\mathcal{F}_n)$ has countably many simple functions with rational coefficients and they are dense. Now $\bigcup L^1(\mathcal{F}_n)$ has a countable dense subset. By using double approximation, this set is also dense in $L^1(\mathcal{F}_\infty)$.

10. a) Notice that S_n is a submartingale and $S_n^{T_{ab}}$ is bounded and hence by the optional stopping theorem

$$\mathbb{E}[S_0] = 0 \leqslant \mathbb{E}[S_{T_{ab}}] = a\,\mathbb{P}(T_a \leqslant T_b) + b\,\mathbb{P}(T_b < T_a).$$

The equation above gives a lower bound

$$\mathbb{P}(T_b < T_a) \geqslant \frac{-a}{b-a}.$$

Now as $a \to -\infty$, $T_a \to \infty$ and the right hand side converges to 1, which gives that $\mathbb{P}(T_b < \infty) \geqslant 1$. From this it follows that both T_b and T_{ab} are finite.

(i) Direct computation shows that

$$\mathbb{E}\left[\left(\frac{q}{p}\right)^{S_n} - \left(\frac{q}{p}\right)^{S_{n-1}} |\mathcal{F}_{n-1}\right] = \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n-1}}\left(\left(\frac{q}{p}\right)^{X_n} - 1\right)|\mathcal{F}_{n-1}\right] = \left(\frac{q}{p}\right)^{S_{n-1}}\left(\mathbb{E}\left(\frac{q}{p}\right)^{X_n} - 1\right).$$

It suffices to check that $\mathbb{E}[(q/p)^{X_n}] = 1$:

$$\mathbb{E}[(q/p)^{X_n}] = p\frac{q}{p} + q\frac{p}{q} = p + q = 1.$$

The martingale $X_n := (q/p)^{S_n}$ is bounded by 1, hence we may apply the optional stopping theorem to obtain

$$\mathbb{E}[X_0] = 1 = \mathbb{E}[X_{T_{ab}}] = (q/p)^a\mathbb{P}(T_a < T_b) + (q/p)^b\mathbb{P}(T_b < T_a).$$

Rearranging the above gives that

$$\mathbb{P}(S_{T_{ab}} = a) = \mathbb{P}(T_a < T_b) = \frac{1-(q/p)^b}{(q/p)^a - (q/p)^b}.$$

(ii) Let $X_n : S_n - n(p-q)$, then X_n is a martingale. Notice that $X^{T_{ab}}$ is bounded by $-a \vee b$, so by the optional stopping theorem

$$\mathbb{E}[X_0] = 0 = \mathbb{E}[X_{T_{ab}}] = \mathbb{E}[S_{T_{ab}}] - (p-q)\mathbb{E}[T_{ab}].$$

11. a) Let X_i^n be i.i.d Bernoulli $\{0,2\}$ with equal probability and $\mathcal{F}_n := \sigma(X_i^k : i \geqslant 1, k \leqslant n)$ then $Z_n := \sum_{i=1}^{Z_{n-1}} X_i^n$. Then $\mathbb{E}[Z_n|\mathcal{F}_{n-1}] = Z_{n-1}\mathbb{E}[X_1^n] = Z_{n-1}$ so Z_n is a martingale. The martingale Z_n is positive so by the martingale convergence theorem it converges to some Z_∞ a.s. Now we show that the limit must be 0. For any $k > 0$ we have that $\mathbb{P}(Z_{n+1} = k|Z_n = k) = 1/2$ and so

$$\mathbb{P}(Z_n = k; ...; Z_{n+N} = k) \leqslant 2^{-N}.$$

But now $\mathbb{P}(\lim Z_n = k) \leqslant 2^{-N}$ for each $N \in \mathbb{N}$.

(ii) The convergence again follows from non-negative martingale convergence. First consider the case $\mu < 1$. Then we have that $\mathbb{E}[Z_n] = \mu^n$ and so

$$\mathbb{P}(Z_n > 0) = \sum_{k \geqslant 1} \mathbb{P}(Z_n = k) \leqslant \sum_{k \geqslant 1} k\mathbb{P}(Z_n = k) = \mu^n.$$

By taking $n \to \infty$ we see that $Z_n = 0$ a.s.

Now for the case $\mu = 1$ we ignore the case $\mathbb{P}(Z_1 = 1) = 1$ otherwise the result does not hold, nor do we have any interesting activities. So then $p := \mathbb{P}(Z_1 = 0) > 0$ as the expectation is 1. Following the idea as above, let $k > 0$, then we have instead

$$\mathbb{P}(Z_n = k; ...; Z_{n+N} = k) \leqslant (1-p)^N$$

and hence $\mathbb{P}(\lim Z_n = k)$ which is the union of events of the form $\{\forall n \geqslant N, Z_n = k\}$ is zero.

(iii) Let $p = \mathbb{P}(M_\infty = 0)$. There are a possible number of cases to consider. If $p = 0$ or 1, then the result follows easily. If $0 < p < 1$ then on the set $\{M_\infty = 0\}$, $Z_n \to 0$ and hence $p^{Z_n} \to 1$. On the set $\{M_\infty > 0\}$ we have that as $\mu > 1$, $Z_n \to \infty$, now as $p < 1$ this implies that $p^{Z_n} \to 0$. Thus $p^{Z_n} \to \mathbf{1}_{M_\infty = 0}$. Z_n roughly behaves like $M_\infty \mu^n$ asymptotically.

c) First by the tower law

$$Var[Z_n] = \mathbb{E}\big[Var[Z_n^2|Z_{n-1}]\big] = \mathbb{E}[Z_{n-1}\sigma^2] = \mu^{n-1}\sigma^2$$

so then $Var(M_n) = \mu^{-n-1}\sigma^2$ which shows the bound in L^2. An application of Cauchy-Schwartz gives that $\mathbb{E}[Z_n \mathbf{1}_{Z_n>0}]^2 \leqslant \mathbb{P}(Z_n > 0)\mathbb{E}[Z_n^2]$ so that

$$\mathbb{P}(Z_n > 0) \geqslant \frac{\mathbb{E}[Z_n]^2}{\mathbb{E}[Z_n^2]} = \frac{\mu^{2n}}{\mu^{n-1}\sigma^2 + \mu^{2n}} \geqslant \frac{1}{1+\sigma^2} > 0.$$

Thus the probability of survival is strictly positive.

12. Take $\{X_i\}_{i\geqslant 1}$ to be i.i.d. Bernoulli $\{0, 2\}$, i.e. $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 2) = 1/2$. Consider $M_n := \prod_{i=1}^n X_i$, then we have that

$$\mathbb{E}[M_n - M_{n-1}|\mathcal{F}_{n-1}] = \mathbb{E}\big[(X_n - 1)M_{n-1}|\mathcal{F}_n\big] = \big(\mathbb{E}[X_n] - 1\big)M_{n-1} = 0$$

so M_n is a martingale. Next notice that $\mathbb{E}[M_n] = 1$ by the independence of the X_i, so that M_n cannot converge in L^1 to 0. On the other hand observe that

$$\mathbb{P}(M_n \neq 0) = \prod_{i=1}^n \mathbb{P}(X_i \neq 0) = \frac{1}{2^n}$$

and hence by Borel-Cantelli $M_n \to 0$ a.s.

13. First as M is bounded in L^1, it converges in L^1 to M_∞. On the event $\{T = \infty\}$, $M_T = M_\infty \in L^1$, and on the event $\{T < \infty\}$, by dominated convergence $\mathbb{E}\big[|M_T|\big] = \lim_{t\to\infty} \mathbb{E}\big[|M_{T\wedge t}|\big]$ and as $|M|$ is a submartingale $\mathbb{E}\big[|M_{T\wedge t}|\big] \leqslant \mathbb{E}\big[|M_t|\big] \leqslant \mathbb{E}\big[|M_\infty|\big]$.

For the counterexample take $M_t = B_t$ a standard Brownian motion and $T := \inf\{t \geqslant 0 : B_t = 1\}$, then $\mathbb{E}[B_T] = 1 \neq 0 = \mathbb{E}[B_0]$.

14. Let $\mathcal{F}_n := \sigma([a, b) : a, b \in \mathbb{D}_n)$, then F_n increases to $\mathcal{F}_\infty$ which is the Borel sigma-algebra. With this set up M_n is nothing but the projection of f' on to $\mathcal{F}_n$, i.e. $M_n = \mathbb{E}[f'|\mathcal{F}_n]$. Indeed for any $[a, b)$ being a basic set in $\mathcal{F}_n$, we have that $\int_a^b f_n'(x)dx = \int_a^b f'(x)dx$. So now by Lévy's upward theorem $M_n \to \mathbb{E}[f'|\mathcal{F}_\infty] = f'$ a.s and in L^1 as f' is continuous and hence Borel measurable.

15. Let e_n be an orthonormal basis of H, we wish to show that $\sum_{k=1}^n h_k G_k \to X_h$ in L^2 and a.s., where G_k are i.i.d. normal and $h = \sum h_n e_n$. We have seen before that $X_h \in L^2$. Let $\mathcal{F}_n := \sigma(G_1, ..., G_n)$, then consider the martingale $M_n := \mathbb{E}[X_h|\mathcal{F}_n] = \sum_{k=1}^n h_k G_k$. Now by theorem 72, the convergence is a.s. as well.

16. The Borel σ-algebra of $\mathcal{C}\big([0, 1], \mathbb{R}\big)$ is generated by the coordinate process, with elementary events $\{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}$, for $0 \leqslant t_1 < \cdots < t_n \leqslant 1$ and B_i measurable subsets of $\mathbb{R}$. It is also generated by the events of the form $A = \{X_{t_1} - X_0 \in C_1\, X_{t_2} - X_{t_1} \in C_2, \ldots, X_{t_n} - X_{t_{n-1}} \in C_n\}$, for C_i measurable subsets of $\mathbb{R}$. Can you prove it? To prove that $\mathbb{P}^1$ is absolutely continuous with respect to $\mathbb{P}$ and find its Radon-Nikodym derivative, it suffices then to compare $\mathbb{P}^1(A)$ and $\mathbb{P}(A)$. Using the independence of the increments and their Gaussian nature, you can easily see that $\mathbb{P}^1(A) =$

$\mathbb{E}\big[e^{-aX_{t_n}-\frac{a^2t_n^2}{2}}\mathbf{1}_A\big] = \mathbb{E}\big[e^{-aX_1-\frac{a^2}{2}}\mathbf{1}_A\big]$, since the process $\big(e^{-aX_t-\frac{a^2t^2}{2}}\big)_{0\leqslant t\leqslant 1}$ is a martingale. It follows that $\frac{d\mathbb{P}^1}{d\mathbb{P}} = e^{-aX_1-\frac{a^2}{2}}$.

17. This is pretty much the same argument as Corollary 79. Let $\mathbb{P}$ be the uniform measure on $\mathfrak{S}_n$, X be the coordinate map and $\mathcal{F}_k := \sigma(X_1, ..., X_k)$. Then we are done if we can estimate $\big|\mathbb{E}[f|\mathcal{F}_{k+1}] - \mathbb{E}[f|\mathcal{F}_k]\big|$. Notice that $\mathbb{E}[f|\mathcal{F}_k]$ is the average of f on $\{\sigma : \sigma_i = x_i, i \leqslant k\}$ so that moving between the two averages, the function f could then at most differ by one change of coordinate, and hence $\big|\mathbb{E}[f|\mathcal{F}_{k+1}] - \mathbb{E}[f|\mathcal{F}_k]\big| \leqslant 1$ as f is a contraction. Hence by the Theorem 78, the result follows.

18. The idea is to construct a set which can be determined by $f(t)$ for any $t > 0$. So take $\{f : \inf\{t \geqslant 0 : f(t) \neq f(0)\} = 0\}$. Notice that

$$\{f : \inf\big\{t \geqslant 0 : f(t) \neq f(0)\big\} = 0\} = \bigcap_{t>0}\big\{f : f(t) \neq f(0)\big\} \in \mathcal{F}_t$$

for any $t > 0$. However this set cannot be in $\mathcal{F}_0$ as this cannot be determined by sets of the form $\big\{f : f(0) \in A\big\}$.

19.a) Take $A \in \cap_{n\geqslant 1}\sigma(\mathcal{G}, \mathcal{G}_n, ...)$ and consider $X := \mathbf{1}_A - \mathbb{E}[\mathbf{1}_A|\mathcal{G}]$. Now as $\mathbb{E}[X|\mathcal{G}] = 0$, X is independent of $\mathcal{G}$. By definition $X \in \sigma(\mathcal{G}, \mathcal{G}_n, ...)$ for each $n \geqslant 1$ and hence $X \in \sigma(\mathcal{G}_n, ...)$, therefore $X \in \cap_{n\geqslant 1}\sigma(\mathcal{G}_n, ...)$.[2]

Then Kolmogorov's 0-1 law gives that X is constant, but $\mathbb{E}[X] = 0$, so $X = 0$ a.s. In other words $\mathbf{1}_A = \mathbb{E}[\mathbf{1}_A|\mathcal{G}]$, i.e. there exists a set $B \in \mathcal{G}$ such that $\mathbf{1}_A = \mathbf{1}_B$ a.s.

b) By the independence of the increments $\mathcal{T}_n := \sigma\big(B_{t+1/n} - B_{t+\frac{1}{n+1}}\big)$ are independent from each other and from $\mathcal{G}_t$. Then as $\mathcal{G}_{t+} = \cap_{n\geqslant 1}\sigma\big(\mathcal{G}_t, \mathcal{T}_n, ...\big)$ from the previous part we have that $\mathcal{G}_t$ and $\mathcal{G}_{t+}$ coincide up to null events. The result now follows as they both contain all the null events.

[2]This is true as the L^2 is the sum of orthogonal components.