B denotes a real-valued or $\mathbb{R}^d$-valued Brownian motion constructed on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$; the distribution of $x + X$ is denoted by $\mathbb{P}_x$.

1. Kolmogorov's $0-1$ law. *This exercise is the companion to exercise 8 of example sheet 2.* Let us work in $\mathbb{R}^d$. Define the *tail σ-algebra* : $\mathcal{T} = \bigcap_{t \geqslant 0} \sigma(B_{s+t}\,;\, s \geqslant 0)$. Using the inversion property of Brownian motion and Blumenthal's $0-1$ law, prove that all the events of $\mathcal{T}$ are trivial under $\mathbb{P}$.

2. Let A be an open subset of the $(d-1)$-dimensional sphere and U the cone $\{ta\,;\, a \in A, 0 \leqslant t \leqslant \varepsilon\}$ of vertex 0 (for some $\varepsilon > 0$). Prove that the hitting time $\tau_U = \inf\{t > 0\,;\, B_t \in U\}$ of U for a Brownian motion starting from 0 is almost-surely equal to 0. *This result is useful to solve Dirichlet problem by the probabilistic method in concrete cases as it ensures that all points of the boundary of an open set O are regular if any point of ∂O is the vertex of a cone contained in O^c.*

3. Using the martingale property of Brownian motion, prove that we have for any positive a, b

$$\mathbb{P}(H_{-a} < H_b) = \frac{b}{b+a} \qquad \text{and} \qquad \mathbb{E}[H_{-a} \wedge H_b] = ab.$$

4. Let B be a real-valued Brownian motion and $\sigma \in \mathbb{R}$.

a) Show that the process $\left(e^{\sigma B_t - \frac{\sigma^2}{2} t}\right)_{t \geqslant 0}$ is a martingale with respect to the filtation of B.

b) Deduce, by differentiating with respect to σ, that the following processes are also martingales: $\left(B_t^2 - t\right)_{t \geqslant 0}$, $\left(B_t^3 - 3tB_t\right)_{t \geqslant 0}$, $\left(B_t^4 - 6tB_t^2 + 3t^2\right)_{t \geqslant 0}$.

5. Given $c \in \mathbb{R}$, the process $B_t^c = B_t + ct$, is called the *Brownian motion with drift c.* For fixed $x > 0$, set $H_x^c = \inf\{t \geqslant 0\,;\, B_t^c = x\}$.

a) Fix $\lambda > 0$. Under which conditions on $\theta \in \mathbb{R}_+$ is the process $\exp\left(\theta B_t^c - \lambda t\right)$ a martingale?

b) Supposing θ chosen appropriately, deduce from **a)** that

$$\mathbb{E}\left[e^{-\lambda H_x^c}\right] = \exp\left(-x(\sqrt{c^2 + 2\lambda} - c)\right),$$

and so, that the distribution of H_x^c has density $\frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{(x-ct)^2}{2t}\right)$. Is it surprising?

c) Conclude that

$$\mathbb{P}(H_x^c < \infty) = 1 \text{ if } c \geqslant 0, \text{ and } e^{-2|c|x} \text{ if } c < 0.$$

6. **a)** Given $a > 0$, set $H_a = \inf\{s \geqslant 0\,;\, B_s = a\}$. Prove that the distribution of H_a has a density with respect to Lebesgue measure on $\mathbb{R}_+$, equal to $\frac{a}{(2\pi t^3)^{1/2}} \exp\left(-\frac{a^2}{2t}\right)$.

b) Prove that the process of hitting times $(H_a)_{a \geqslant 0}$ has stationnary independent increments. Is it a Lévy process?

7. Given any $a \geqslant 0$, set $S_a = \inf\{t \geqslant 0\,;\, B_t > a\}$ and $T_a = \inf\{t \geqslant 0\,;\, B_t \geqslant a\}$.

a) Prove that S_a and T_a are almost-surely equal.

b) Let L be a non-negative random time independent of the filtration generated by B. Prove that the event $\{T_L \neq S_L\}$ is measurable and $\mathbb{P}(T_L \neq S_L) = 0$.

c) Find a random time L for which $\mathbb{P}(T_L = S_L) = 0$.

8. Occupation time. Write D for $B(0,r) \subset \mathbb{R}^d$.

a) Prove that $\mathbb{P}\big(\int_0^\infty \mathbf{1}_D(B_t)\,dt = \infty\big) = 1$, if $d = 1$ or 2.

b) Let $x \in \mathbb{R}^d$ be any starting point. Prove that $\mathbb{E}_x\big[\int_0^\infty \mathbf{1}_D(B_t)\,dt\big] < \infty$, for $d \geqslant 3$.

9. Let $B = (B^1, B^2)$ be a 2-dimensional Brownian motion starting from the point with coordinates $(0,1)$. Setting $T = \inf\{t \geqslant 0\,;\, B^2_t = 0\}$, what is the law of B^1_T?

10. Let B be here an $\mathbb{R}^d$-valued Brownian motion, $r > 0$ and $x \in \mathbb{R}^d$ with $\|x\| < r$. Set $H = \inf\{s \geqslant 0\,;\, \|B_s\| = r\}$. Prove that $\mathbb{E}_x[T] = \frac{r^2 - \|x\|^2}{d}$.

11. Uniqueness in Dirichlet problem. Let O be a connected and bounded open set and g be a solution to Dirichlet problem, with continuous boundary condition f. Prove that

$$\max_{x \in O} g(x) = \max_{y \in \partial O} g(y) \ \big(= \max_{y \in \partial O} f(y)\big).$$

Conclude that the Dirichlet problem has at most one solution.

12. Let N be a Poisson process of intensity λ. Prove that the number of jumps of N by time $t > 0$ is a Poisson random variable with parameter λt.

13. Prove that a Poisson process is a Lévy process.

14. A Poisson process of rate λ is observed by someone who believes that the first holding time if longer than all the other holding times. How long on average will it take before the observer is proved wrong?

15. Let N be a Poisson process of intensity λ. Given any time $t > 0$, denote by $T_t = \inf\{s \geqslant t\,;\, N_s \neq N_t\}$ the next jump time after time t.

a) Prove that we have almost-surely $T_t > t$.

b) Prove that $T_t - t$ is exponentially distributed, with parameter λ. *This is surprising as the interval $[t, T_t - t]$ is contained in one of the intervals between jumps, all of which are exponentially distributed, with parameter λ(!).* Can you explain that paradox?

16. Is the sum of two Lévy processes always a Lévy process?

17. Can a process with stationnary and independent increments not be a Lévy process?

18. Given a Lévy process X, set $\Delta X_t := X_t - X_{t^-}$. Prove that we have almost-surely $\Delta X_t = 0$ for any fixed $t > 0$, so Lévy processes do not have jumps at fixed times. *This result generalizes the corresonding result for Poisson processes proved in question **a)** in exercise 15.*

19. Let X be a Lévy process with jump measure Λ_X of finite mass.

a) Prove that X has almost-surely finitely many jumps in any bounded interval of time.

b) Denote by $(\Delta X)_n$ the n^{th} jump of X, and let $(\epsilon_n)_{n \geqslant 1}$ a collection of independent Bernoulli random variables , with parameter $p \in (0,1)$, independent of X. Let Y be the process obtained from X by removing from X all the jumps of X for which $\epsilon_n = 0$, at the time when they occur: If X has made n_t jumps by time t we have $Y_t = X_t - \sum_{j=1}^{n_t}(1 - \epsilon_j)(\Delta X)_j$. The process Y is càdlàg . Prove that Y is a Lévy process and find its jump measure Λ_Y.

20. Using the same method as was used for Brownian motion in the course, state and prove the strong Markov property for a Lévy process.

21. Using the same method as in exercise 19 in example sheet 2, prove that the filtration generated by a Lévy process, completed with null sets, is continuous on the right.

22. **a)** Prove that one can always associate to a given Lévy process an independent copy which can be decomposed as the sum of two independent Lévy processes.

b) Deduce from **a)** and exercise 17 that a Lévy process is almost-surely continuous iff it is a Brownian motion with drift.