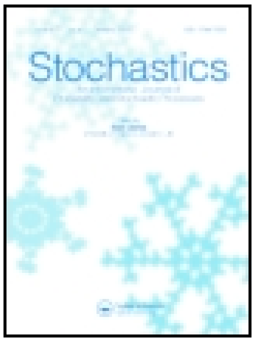

Stochastics: An International Journal of Probability and Stochastic Processes

ISSN: 0090-9491 (Print) Journal homepage: www.tandfonline.com/journals/gssr18

The maximal flow through a directed graph with random capacities

G. R. Grimmett & W-C. S. Suen

Stochastics, 1982, Vol. 8, pp. 153–159
0090-9491/82/0802-0153 $06.50/0

Printed in Great Britain

The Maximal Flow Through a Directed Graph with Random Capacities

G. R. GRIMMETT and W-C. S. SUEN
School of Mathematics, University of Bristol, University Walk, Bristol BS8 ITW, UK

(*Accepted for publication June 21, 1982*)

We consider the maximal flow through a randomly capacitated network. We show that the maximal flow F_n between vertices 0 and ∞ of the complete graph on $\{0, 1, 2, \ldots, n-1, \infty\}$, whose edges (i, j) are directed from i to j if and only if $i<j$, satisfies $n^{-1}F_n \to E(B)$ almost surely and in mean, as $n \to \infty$; here $E(B)$ is the mean value of a typical edge capacity. This answers a question posed by Grimmett and Welsh [1].

1. THE RESULTS

Let G be a directed or undirected graph with specified vertices 0 as *source* and ∞ as *sink*. Each edge e of G is independently assigned a capacity $B(e)$, which is a non-negative random variable drawn from a known probability distribution function F_B. The capacity $B(e)$ associated with edge e is an upper bound for the amount of fluid which may pass along e (in the appropriate direction, if e is directed). The *capacity* $C(G)$ of the network G is defined to be the maximum flow from 0 to ∞ which is attainable, subject to these capacities. Grimmett and Welsh [1] studied the capacities of certain types of networks, including branching trees and undirected complete graphs, and we refer the reader to [1] for a general discussion of the problem. In this note, we answer a question posed in [1], concerning the capacity of a directed complete graph. We shall follow the notation of [1] wherever possible.

Let DK be the directed complete graph on the vertex set $\{0, 1, 2, \ldots\} \cup \{\infty\}$, whose edges are directed according to the following rule: if $0 \leqq i < j \leqq \infty$, then the edge joining i and j is directed from i to j. To each edge e of DK, we assign a random capacity $B(e)$, where $\{B(e)\}$ is a family of independent variables with common distribution function F_B and mean value μ_B. Let DK_n denote the capacitated subgraph of DK induced by the vertex set $\{0, 1, 2, \ldots, n-1\} \cup \{\infty\}$, and we denote by Y_n the capacity of

DK_n between source 0 and sink ∞. The random variables $\{Y_n: n \geqq 1\}$ are defined on the common probability space $(\Omega, \mathscr{F}, P)$, where Ω is the collection of all possible assignments of edge capacities to DK, $\mathscr{F}$ is the obvious σ-field, and P is product measure based on the distribution function F_B. Our main result is the following theorem.

THEOREM 1 *If* $\mu_B < \infty$ *then, as* $n \to \infty$, $(1/n)Y_n \to \mu_B$ *almost surely and in* L^1.

Grimmett and Welsh [1] showed that $n^{-1}Y_n \to \gamma$ as $n \to \infty$, for some constant γ satisfying $\mu_M \leqq \gamma \leqq \mu_B$, where $\mu_M = E(\min(B_1, B_2))$ is the mean value of the minimum of two independent edge capacities. They used the limit theorem for subadditive processes, but were unable to calculate the exact value of the limit γ. Here, we show directly that $n^{-1}Y_n$ converges, without appealing to subadditive theory, and that γ is given by $\gamma = \mu_B$. Note that Theorem 1 generalises Theorem 3.3 of [1], since we have imposed extra conditions upon the edge capacities without changing the asymptotic behaviour of the maximal flow.

There is an alternative natural way in which the edge-directions of the complete graph may be specified. Let K be the complete graph on $\{0, 1, 2, \dots\} \cup \{\infty\}$; we direct the edges of K according to the following rule: for $0 \leqq i < j \leqq \infty$, the edge joining i and j is directed in one of the two possible directions, each being picked with probability $\frac{1}{2}$ and independently of all other edge-directions. Let $\mathbf{K}$ denote the ensuing randomly-directed graph. As before, we assign a random capacity $B(e)$, with mean value μ_B, to each edge e and write $\mathbf{K}_n$ for the subgraph of $\mathbf{K}$ induced by the vertex set $\{0, 1, 2, \dots, n-1\} \cup \{\infty\}$. The asymptotic behaviour of the maximum flow $\mathbf{Y}_n$ through $\mathbf{K}_n$ between 0 and ∞ is given by the next theorem. This is an easy consequence of Theorem 1.

THEOREM 2 *If* $\mu_B < \infty$ *then, as* $n \to \infty$, $(1/n)\mathbf{Y}_n \to \frac{1}{2}\mu_B$ *almost surely and in* L^1.

If the directions of the edges of $\mathbf{K}$ are chosen with the probabilities p and $1-p$, in the obvious way, then the proof of this theorem is easily adapted to show that

$$\frac{1}{n}\mathbf{Y}_n \to p\mu_B \text{ a.s. and in } L^1, \text{ as } n \to \infty.$$

The conclusions of Theorems 1 and 2 do not surprise us, since the cutsets of DK_n and $\mathbf{K}_n$ which involve fewest edges are the two families of edges which are incident to 0 and to ∞ respectively; each involves n edges and has mean capacity $n\mu_B$ in DK_n and $\frac{1}{2}n\mu_B$ in $\mathbf{K}_n$.

2. THE PROOFS

Proof of Theorem 1. A special case of Theorem 1 deals with the situation when the edge capacities have the Bernoulli distribution:

$$P(B(e)=1)=p,\ P(B(e)=0)=1-p,$$

where $0 \leqq p \leqq 1$. We show that the theorem holds in this instance. The result for general distributions follows exactly by the approximation argument in [1], and we recall this method briefly here. It is easy to show that if the result holds for Bernoulli capacities, then it holds for capacities taking values in any finite set $\{b_1, b_2, \ldots, b_m\}$; if F_B is a distribution function with finite mean, we may approximate to F_B, to any prescribed degree of accuracy, by such a discrete distribution. We shall concentrate on the case $0<p<1$ only, since the theorem holds trivially for $p=0$ and $p=1$.

First we show almost sure convergence. By considering the cutset of edges incident to the vertex 0, it is obvious that $Y_n \leqq B_1 + \ldots + B_n$ where the B's are independent Bernoulli variables with parameter p. The law of large numbers gives

$$P\left(\limsup_{n\to\infty} \frac{1}{n} Y_n \leqq p\right)=1. \tag{1}$$

We claim that

$$P\left(\liminf_{n\to\infty} \frac{1}{n} Y_n \geqq p\right)=1. \tag{2}$$

To show this, we make use of the maximum-flow minimum-cut theorem, which implies that

$$P(Y_n \leqq r)=P(E(n,r))$$

where $E(n, r)$ is the event that there exists a set V of vertices of DK_n such that $0 \in V$, $\infty \notin V$, and such that the total capacity $C(V)$ of those edges directed from V to its complement V^c in DK_n is no more than r. Therefore

$$\begin{aligned} P(Y_n \leqq r) &= P(E(n,r)) \\ &\leqq \sum_V P(C(V) \leqq r) \\ &= \sum_{k=0}^{n-1} \sum_i P(S(d(i,k)) \leqq r) \end{aligned} \tag{3}$$

where the second summation sign sums over all sequences $i_1, i_2, \ldots, i_k$ satisfying $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n-1$, $S(m)$ is a binomial variable with parameters m and p, and $d(\boldsymbol{i}, k)$ is the number of edges directed out of $V = \{0, i_1, i_2, \ldots, i_k\}$. Now

$$d(\boldsymbol{i}, k) = (n-k) + \sum_{l=1}^{k} (n - i_l - (k-l))$$

$$= n(k+1) - \tfrac{1}{2}k(k+1) - \Sigma(\boldsymbol{i}) \tag{4}$$

where

$$\Sigma(\boldsymbol{i}) = \sum_{l=1}^{k} i_l.$$

Note that

$$d(\boldsymbol{i}, k) \geq n \quad \text{for all } \boldsymbol{i} \text{ and } \quad 0 \leq k \leq n-1.$$

Suppose $0 < \varepsilon < p$. From (3)

$$P(Y_n \leqq n(p-\varepsilon)) \leqq \sum_{k=0}^{n-1} \sum_{\boldsymbol{i}} P(S(d(\boldsymbol{i}, k)) \leqq d(\boldsymbol{i}, k)(p-\varepsilon)).$$

We use a standard inequality to bound the last term: if $g : \mathbb{R} \to (0, \infty)$ is a non-decreasing function, and $\eta > 0$, then

$$P(Y \geqq \eta) \leqq \frac{E(g(Y))}{g(\eta)}$$

for any random variable Y. Set $Y = mp - S(m)$, $\eta = m\varepsilon$, and $g(x) = e^{tx}$ where $t > 0$, to find that

$$P(S(m) \leqq m(p-\varepsilon)) \leqq \rho(t, \varepsilon)^m$$

where

$$\rho(t, \varepsilon) = e^{-t\varepsilon} E(e^{t(p - S(1))})$$

$$= 1 - t\varepsilon + o(t) \quad \text{as} \quad t \downarrow 0.$$

Choose $\tau > 0$ such that $\rho = \rho(\tau, \varepsilon) < 1$ to deduce that

$$P(S(m) \leqq m(p-\varepsilon)) \leqq \rho^m \quad \text{for all} \quad m \geqq 1,$$

where $0<\rho<1$. Thus, from (4),

$$P(Y_n \leqq n(p-\varepsilon)) \leqq \sum_{k=0}^{n-1} \sum_{i} \rho^{n(k+1)-\frac{1}{2}k(k+1)-\Sigma(i)}$$

$$= \sum_{k=0}^{n-1} \rho^{n-\frac{1}{2}k(k+1)} C_n(k)$$

$$\leqq \sum_{k=0}^{n-1} \rho^{n-\frac{1}{2}k(k+1)} C_{\infty}(k) \tag{5}$$

where $C_n(k)$ is the coefficient of x^k in the product $\prod_{l=1}^{n-1}(1+x\rho^l)$. In order to find $C_{\infty}(k)$, note that

$$\prod_{l=1}^{\infty}(1+x\rho^l)=(1+x\rho)\prod_{l=1}^{\infty}(1+(x\rho)\rho^l),$$

and so

$$C_{\infty}(k)=\rho^k C_{\infty}(k)+\rho^k C_{\infty}(k-1),$$

giving that

$$C_{\infty}(k)=\frac{\rho^k}{1-\rho^k}C_{\infty}(k-1)$$

$$=\frac{\rho^{\frac{1}{2}k(k+1)}}{(1-\rho)\dots(1-\rho^k)} \leqq A(\rho)\rho^{\frac{1}{2}k(k+1)}$$

where

$$A(\rho)=\left(\prod_{l=1}^{\infty}(1-\rho^l)\right)^{-1}.$$

By (5) we have that

$$P(Y_n \leqq n(p-\varepsilon)) \leqq A(\rho)n\rho^n \quad \text{for all } n,$$

giving that $P(Y_n \leqq n(p-\varepsilon))$ decays geometrically in n, and so

$$\sum_n P(Y_n \leqq n(p-\varepsilon)) < \infty;$$

Eq. (2) follows by the Borel-Cantelli lemma, and the proof of almost sure convergence is complete. Convergence in L^1 follows easily, since $0 \leq n^{-1} Y_n \leq 1$ for all n.

Proof of Theorem 2. It is clear that

$$P\left(\limsup_{n \to \infty} \frac{1}{n} \mathbf{Y}_n \leq \tfrac{1}{2}\mu_B \right) = 1;$$

to see this, use the corresponding argument in the previous proof. To show that

$$P\left(\liminf_{n \to \infty} \frac{1}{n} \mathbf{Y}_n \geq \tfrac{1}{2}\mu_B \right) = 1,$$

we proceed as follows. **K** is a directed graph, each edge e of which has some random capacity $B(e)$. We alter these capacities in the following way: if e is the edge joining vertices i and j, where $i<j$, we replace the capacity $B(e)$ by $B'(e)$ where

$$B'(e) = \begin{cases} B(e) & \text{if } e \text{ is directed from } i \text{ to } j \\ 0 & \text{if } e \text{ is directed from } j \text{ to } i. \end{cases}$$

The ensuing capacitated network is equivalent to the graph DK, each edge e of which is assigned some random capacity $D(e)B(e)$, where $D(e)$ is a Bernoulli random variable taking values 0 and 1 with equal probabilities. Let Z_n be the maximal flow through the latter network. By Theorem 1, we have that

$$\frac{1}{n} \mathbf{Y}_n \geq \frac{1}{n} Z_n \to E(DB) = \tfrac{1}{2}\mu_B \quad \text{almost surely.}$$

L^1 convergence follows as before, and the proof is complete.

Note added in proof With reference to Theorem 2, suppose that each edge $e=(i,j)$, where $i<j$, is directed from i to j with probability p and from j to i with probability $1-p$. We are grateful to Colin McDiarmid for pointing out that, in the case $p=\frac{1}{2}$, $\mathbf{Y}_n$ has the same distribution as the flow through an undirected graph with edge capacities distributed as $D(e)B(e)$, above (see Theorem 3.3 of [2]). This provides an alternative proof of Theorem 2, but fails to hold for more general values of p.

References

[1] G. R. Grimmett and D. J. A. Welsh, Flow in networks with random capacities. *Stochastics* **7** (1982), 205–229.

[2] C. J. H. McDiarmid, General first passage percolation, *Advances in Applied Probability*, to appear.