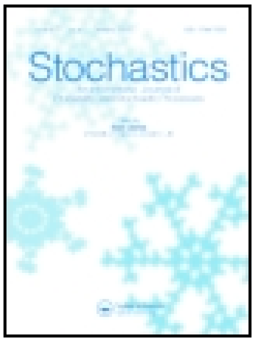

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Flow in Networks with Random Capacities

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We study the problem of finding the maximum flow through a capacitated network in which the set of capacities is a collection of independent random variables, drawn from some known distribution. We find limit theorems when the underlying graph is either a branching tree or a complete graph. Finally, we discuss the difficulty of the problem of finding the expected maximum flow, using the language of computational complexity, and propose an easy approximate solution to the closely related reliability problem.

1. INTRODUCTION

Finding the maximum flow in a capacitated network is one of the most fundamental and discussed problems of operations research. It has a well known, straightforward, fast algorithm as given in the book by Ford and Fulkerson [4]. However, when the edge capacities are not deterministic but are random variables drawn from a given known distribution, the problem of finding either the distribution or the moments of the maximum flow seems to be formidable.

Several authors have studied various forms of this problem: see for example Doulliez and Jamoulle [3], Gaul [6] and Spelde [21]. Many other authors have studied problems which may be seen as disguises of the random flow question. For example, classical percolation theory (see the introductory paper of Broadbent and Hammersley [2] and the reviews of Welsh [23], Hammersley and Welsh [10] and Kesten [13]) posed the question of finding conditions which ensure strictly positive probability of a flow from the origin to infinity in a regular lattice when the edge capacities have the Bernoulli distribution. First passage percolation too (see the exposition of Hammersley and Welsh [9] and the review of Smythe and Wierman [20]) may be seen as a flow problem. Suppose that each edge of a given network has an associated random time co-ordinate, representing the time required to traverse the edge, and ask for the time necessary to move from one given vertex to another. This is the problem of finding the shortest route between two given vertices; by the max-flow min-cut theorem, this is essentially equivalent to the flow problem on the dual network, whenever the original network is planar.

Alternatively, the maximum random flow problem may be regarded as an important special case of stochastic linear programming, though this approach seems to be of little help in the mathematical theory of the problem.

In this paper we consider first the problem of studying the maximum flow in two extreme cases: when the underlying network is a *regular tree* or *Bethe lattice*, and when it is a *complete graph*. Even in special cases the problem is difficult; we illustrate this by relating its difficulty to some of the existing complexity classes as described in Garey and Johnson [5]. We close by proposing an approximate solution to the reliability problem, which relates this problem to the max-flow min-cost problem.

The precise problem considered is the following. Let G be a directed or undirected graph with specified vertices s as *source* and t as *sink*. Each edge e of G is independently assigned a capacity $B(e)$, which is a non-negative random variable drawn from a known probability distribution function F_B. The capacity $B(e)$ associated with edge e represents the maximum amount of flow which that edge can accommodate (in the direction of its orientation if it is directed). We write Ω for the collection of all possible assignments of capacities to the endges of G, and note that Ω is the sample space of the natural probability space $(\Omega, \mathscr{F}, P)$ describing the ensuing capacitated network; the probability measure P is just product measure based on the distribution F_B. For $\omega \in \Omega$, the *capacity* $C(\omega)$ of G is the maximum flow from s to t which is attainable when edge capacities are specified according to ω, and Kirchhoff's laws are obeyed at each vertex other than the source and sink.

In the deterministic case, this is a well understood problem with many applications; see for example the book by Ford and Fulkerson [4]. In principle, the max-flow min-cut theorem [4] shows C to be given by

$$C(\omega) = \min_{A \in \mathscr{A}} C(A)$$

where $C(A)$ denotes the random capacity of A and $\mathscr{A}$ is the collection of cutsets between s and t. The principal difficulty is that the $C(A)$'s are usually *dependent* random variables, and it seems to be a hopeless task to determine the distribution of C by this method.

2. THE CAPACITY OF A BETHE LATTICE

The principal difficulty in percolation and other flow processes arises out of the stochastic dependence between flows along collections of routes which have edges in common.

Most percolation type problems seem to be easiest to solve when the underlying network has a tree-like structure; in particular, Hammersley has made extensive use of the idea of bounding various percolation variables on regular lattices by their analogous quantities on the underlying Bethe lattice (see for example [8]). Accordingly, we attempt first to settle the flow problem for this traditionally easiest case.

Let r be a given positive integer greater than one. We denote by T (or T^r, if the value of r is to be emphasized) the Bethe lattice with a root 0, called the *origin*, which is joined to exactly r new vertices, each of which is joined in turn to r further vertices; this process continues. The vertices of the *nth generation* are those which are distance n from the origin. With each edge e of the lattice we associate a non-negative random variable $B(e)$ called the *capacity* of e; the edge capacities are independent random variables with the common distribution function F_B.

Let T_n (or T_n^r) be the graph obtained by connecting each nth generation vertex to some "point at infinity" by an edge of infinite capacity, and let C_n (or C_n^r) be the random capacity between the origin and the point at infinity in the graph T_n. We are concerned with the sequence $\{C_n\}$ of random variables.

LEMMA 2.1 *The sequence* $\{C_n\}$ *is monotone non-increasing and converges for all realizations. That is,*

$$C_n(\omega) \geqq C_{n+1}(\omega);\ C(\omega) = \lim_{n\to\infty} C_n(\omega) \quad \textit{exists for all} \quad \omega. \tag{2.1}$$

Proof All the flow which reaches an $(n+1)$th generation vertex must have passed through the nth generation. C is called the *capacity* of T.

Next, we derive a recurrence relation involving the distributions of the members of the sequence $\{C_n\}$. In the tree T_{n+1}, C_{n+1} is the aggregate flow through those edges which are incident with the origin; it follows that

$$C_{n+1} = \sum_{i=1}^{r} \min(C_n(i), B_i) \tag{2.2}$$

where the $C_n(i)$ $(i=1,2,\dots,r)$ are independent variables distributed like C_n, and B_i is the capacity of the ith edge incident with the origin.

LEMMA 2.3 *The characteristic function ϕ of C satisfies*

$$\phi(t)=(\phi(t)+\psi(t)-\int \exp(itu)d(F_BF_C))^r, \tag{2.3}$$

where ψ is the characteristic function of the distribution F_B of edge capacities and F_C is the distribution function of C.

Proof Let ϕ_n and F_n be the characteristic and distribution functions of C_n. From (2.2), $\phi_{n+1}(t)=(\gamma(t))^r$ where

$$\gamma(t)=-\int \exp(itu)d(P(C_n>u)P(B>u))$$

$$=\phi_n(t)+\psi(t)-\int \exp(itu)d(F_nF_B)$$

is the characteristic function of $\min(C_n(i),B_i)$ and B is a typical edge capacity. Now let $n\to\infty$ and use (2.1) to obtain the result.

We have not been able to solve the functional equation (2.3) even when $r=2$, and will return to this problem later. It is possible however to deduce certain properties of the law of the limit C.

LEMMA 2.4 *If C is concentrated at a point, then this point must be zero, provided that the edge capacities are not almost surely equal to a strictly positive constant.*

Proof (2.2) implies that C is the sum of r independent random variables $M_1,M_2,\dots,M_r$, each distributed like $\min(C,B)$ where B is a typical edge capacity which is independent of C. If C is concentrated at k then each M_i is concentrated at k/r. Either $P(B=k/r)=1$, or $k=k/r$ and so $k=0$.

LEMMA 2.5 *Provided that the edge capacities have no atom at zero, $P(C=0)$ is either zero or one.*

Proof Using the notation of the previous proof we see that $\alpha=P(C=0)$ satisfies

$$\alpha=(P(M_1=0))^r$$

$$=(\alpha+F_B(0)-\alpha F_B(0))^r.$$

If $F_B(0)=0$, then α equals zero or one.

We shall see later (see the discussion after (2.15)) that $P(C=0)=0$ whenever $F_B(0)=0$, and shall need this fact to prove the next lemma.

LEMMA 2.6 *The capacity C has a continuous distribution whenever F_B is continuous.*

Proof Let $p(a)=P(C=a)$ and suppose that F_B is continuous. From (2.2) we know that

$$p(a)=\sum q(a_1)q(a_2),\ldots,q(a_r) \tag{2.7}$$

where the summation is over all sequences $\{a_i\}$ such that $\sum_1^r a_i=a$, and

$$\begin{aligned} q(a) &= P(\min(C,B)=a) \\ &= P(C=a)P(B\geqq a) \\ &= p(a)g(a), \end{aligned}$$

where $g(a)=1-F_B(a)$. Hence

$$\begin{aligned} \sum_a p(a)g(a) &= \sum_a g(a)\sum_{\{a_i\}} p(a_1)g(a_1)p(a_2)g(a_2),\ldots,p(a_r)g(a_r) \\ &\leqq \left(\sum_a p(a)g(a)\right)^r. \end{aligned}$$

But $0\leqq\sum_a p(a)g(a)\leqq 1$, and hence

$$\sum_a p(a)g(a)=0 \quad \text{or} \quad 1.$$

If $\sum p(a)g(a)=1$, then $g(a)=1$ whenever $p(a)>0$. Hence $P(B\geqq A)=1$ where $A=\sup\{a:p(a)>0\}$. But a minimum edge capacity A implies that $C\geqq rA$ and hence $A\geqq rA$, which is impossible unless $A=0$. Hence C has no atoms except possibly at zero; this possibility is excluded by the remark immediately preceding (2.6). If $\sum p(a)g(a)=0$ then $P(B\leqq a)=1$ whenever $p(a)>0$. Suppose that the set $\mathscr{A}=\{a:p(a)>0\}$ is nonempty. Let $A'=\inf\{a:a\in\mathscr{A}\}$; certainly $A'>0$, for if $A'=0$ then $P(B\leqq 0)=1$ contradicting the assumption that F_B is continuous. Pick $a\in\mathscr{A}$ such that $a<2A'$; such a choice may certainly be made if $\mathscr{A}$ is nonempty. It is not hard to see from (2.7) that C cannot have an atom at a, because all sequences $\{a_i\}$ satisfying $\sum a_i=a$ are such that $q(a_i)=0$ for all i. This contradiction completes the proof.

As specific examples we consider the two special cases when the edge capacity distribution is either Bernoulli or exponential. The first of these cases turns out to be useful in establishing bounds for the mean flow for arbitrary edge distributions.

Bernoulli edge capacities

Suppose that each edge e has a capacity $B(e)$ which may take the values 0 and a (>0) with probabilities q and p ($=1-q$) respectively. Then C is a discrete random variable with the binomial distribution,

$$P(C=ka)=\binom{r}{k}(p(1-\alpha))^k(q+p\alpha)^{r-k}\ (0\leqq k\leqq r), \tag{2.8}$$

where α is the smallest positive root of the equation

$$x=(q+px)^r.$$

This follows from a fundamental result in the theory of branching processes (see Harris [11], p. 7); α is the probability that the origin is joined to only finitely many vertices of T when edges of zero capacity are removed. Clearly, $P(C=0)=1$ if and only if $\alpha=1$; this occurs if and only if $pr\leqq 1$. Moreover

$$E(C)=apr(1-\alpha). \tag{2.9}$$

Note that $E(C)$ varies continuously with p, since $\alpha=\alpha(p)$ is continuous.

Exponential edge capacities

Suppose that the edge capacity distribution F_B is the exponential distribution with parameter λ. Equation (2.2) implies that the Laplace transform $m_n(t)=E(\exp(-tC_n))$ of C_n satisfies

$$m_{n+1}(t)=\left(\frac{\lambda+tm_n(\lambda+t)}{\lambda+t}\right)^r. \tag{2.10}$$

Hence the Laplace transform m of C satisfies

$$m(t)=\left(\frac{\lambda+tm(\lambda+t)}{\lambda+t}\right)^r. \tag{2.11}$$

Even in this special case we are unable to solve (2.11) for m. It is not difficult however to see that (2.11) has a unique solution which is the Laplace transform of a distribution on the positive real numbers; information about this solution is available by numerical methods. We have been able to use (2.11) to establish the mean value of C.

LEMMA 2.12 *When F_B is exponential with parameter λ, the capacity C has expected value $\mu=E(C)$ given by $\mu=\lambda^{-1}r(1-x)$ where x is the limit of the sequence*

$$x_n=\left(\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}+\frac{2}{3}\left(\frac{1}{4}+\frac{3}{4}\left(\cdots\left(\frac{1}{n-1}+\frac{n-2}{n-1}\left(\frac{1}{n}+\frac{n-1}{n}\left(\frac{1}{(n+1)}\right)^r\right)^r\cdots\right)^r.\qquad(2.12)$$

We are indebted to J. M. Hammersley for showing us how to evaluate the limit of this sequence. His calculation, reproduced in the appendix, shows that in the case $r=2$

$$E(C)=\lambda^{-1}1.3371\ldots$$

It is clear that his argument may be adapted to deal with other values of r.

Proof We will show that the result holds for $\lambda=1$; the general case follows by a scaling argument. Differentiate (2.10) at $t=0$ to obtain

$$\begin{aligned}E(C_{n+1})&=-m'_{n+1}(0)\\&=r(1-m_n(1)).\end{aligned}$$

But, by (2.10),

$$m_n(1)=(\tfrac{1}{2}+\tfrac{1}{2}m_{n-1}(2))^r;\qquad(2.13)$$

repeated application of (2.10) yields

$$m_n(1)=x_n$$

where x_n is given in (2.12), since the final term in the iterative procedure is

$$\begin{aligned}m_1(n)&=(E(\exp(-nB)))^r\\&=(n+1)^{-r}.\end{aligned}\qquad(2.14)$$

The result follows by monotone convergence.

General edge capacities

Let F_C and F_B be the distribution functions of C and B as before; we suppose that F_B is not concentrated at zero. We may obtain bounds for $E(C)$ as follows.

Lower bounds for $E(C)$ may be found by considering flows with discrete edge distributions which are embeddable in the original random flow in a certain way. This is done as follows. Pick $a>0$, and suppose $F_B(a)<1$. If an edge has capacity not exceeding a, then delete it; otherwise replace it by an edge of capacity exactly a. The ensuing network flow has edge capacities B' taking values 0 and a with probabilities $F_B(a)$ and $1-F_B(a)$ respectively. Let C' be the capacity function in the modified process. Then $C \geqq C' = C'(a)$, and so by (2.9),

$$E(C) \geqq \sup_{a \geqq 0} E(C'(a)) = \sup_{a \geqq 0} ar(1-\alpha)(1-F_B(a)) \tag{2.15}$$

where $\alpha = \alpha(a)$ is the smallest positive root of the equation

$$x = (F_B(a) + x(1-F_B(a))^r.$$

This lower bound equals zero if and only if $\alpha(a)=1$ for all $a>0$. The theory of branching processes tells us that this occurs if and only if $r(1-F_B(0)) \leqq 1$. If this inequality is not satisfied then there exists $a>0$ such that $\alpha(a)<1$ and $F_B(a)<1$, and we deduce that $E(C)>0$ in this case; this fact, combined with (2.5), implies that if $F_B(0)=0$ then $P(C=0)$ is zero, as stated before Lemma 2.6. Such lower bounds may be improved by constructing other discrete flows, more complex than the Bernoulli flow, which are also embeddable in the actual flow.

An alternative lower bound is a consequence of the observation that

$$E(C) \geqq \int_0^\infty P(C' \geqq a)da$$

$$= \int_0^\infty (1-\alpha(a))da$$

where α is given as before. Again, note that this bound equals zero if and only if $\alpha(a)=0$ for all a.

Upper bounds for $E(C)$ may in principle be calculated explicitly from the inequalities

$$E(C) \leqq E(C_n) \quad \text{for all } n.$$

We consider next the limiting behaviour of the capacity C^r of the Bethe tree T^r with degree r, as $r \to \infty$. First note the intuitively obvious result.

LEMMA 2.16 *If $F_B(0) < 1$, then $P(C^r = 0) \to 0$ as $r \to \infty$.*

Proof Pick $\varepsilon > 0$ such that $F_B(\varepsilon) < 1$. Then

$$P(C^r \geqq \varepsilon) \geqq 1 - \alpha(\varepsilon)$$

where $\alpha = \alpha(\varepsilon)$ is the smallest positive root of

$$x = (F_B(\varepsilon) + x(1 - F_B(\varepsilon)))^r.$$

But $\alpha \downarrow 0$ as $r \to \infty$.

Next we have a law of large numbers.

THEOREM 2.17 *If the capacity distribution F_B has a first moment μ then $\lim_{r \to \infty} C^r/r = \mu$ a.e. and in L^1.*

Proof We may suppose that $F_B(0) < 1$, since the result holds trivially otherwise.

For each vertex of the tree T^r we may label the descending edges in some arbitrary order picking distinct labels from the set $\{1, 2, \ldots, r\}$. Let $T_{n,m}$ $(1 \leqq n < m \leqq r)$ denote the tree branching from the origin which is obtained by deleting all edges of T^r except those labelled between n and m inclusive. Let $C_{n,m}$ be the capacity of $T_{n,m}$; clearly $C_{n,m}$ is distributed like C^{m-n+1}. As r increases we may label the new edges of T^r consistently with existing labels, so that $C_{n,m}$ is a random variable which is well defined independently of the value of r, so long as $r \geqq m$. It is also clear that the process $\{C_{n,m}: 1 \leqq n < m < \infty\}$ is a superadditive stochastic process since

$$C_{n,n+r+s} \geqq C_{n,n+r} + C_{n+r+1,n+r+s}$$

for any non-negative integers n, r, s, and

$$n^{-1} E(C_{1,n}) \leqq E(B) < \infty.$$

(For a formal definition of subadditive stochastic processes see [14], [15] or (3.4)–(3.6) below.) Of course, $C^n = C_{1,n}$. It follows from the ergodic theorem of Kingman [15] that the limit

$$\lim_{r \to \infty} r^{-1} C^r = \lambda = \sup_k (k^{-1} E(C^k)) \tag{2.18}$$

exists a.e. and in L^1. Note that λ is constant a.e. since it is $\mathscr{F}$-measurable where $\mathscr{F}$ is the trivial tailfield of a sequence of independent random variables.

We will complete the proof by showing that $\lambda = E(B)$. From (2.2), if $\phi^r(t)$ is the characteristic function of C^r, then the characteristic function $v_r(t)$ of $r^{-1}C^r$ satisfies

$$v_r(t) = \phi^r(t/r) = \left(E\left(\exp\left(\frac{it}{r} \min(C^r, B) \right) \right) \right)^r$$

where C^r and B are independent. Now,

$$v_r(t) = \left(1 + \frac{it}{r} E\left(\min(C^r, B) \right) + 0\left(\frac{1}{r} \right) \right)^r$$

$$\rightarrow \exp(it\gamma) \quad \text{as} \quad r \rightarrow \infty$$

where

$$\gamma = \lim_{r \rightarrow \infty} E(\min(C^r, B)).$$

This limit γ exists since C^r is monotone in r and has finite first moment. Thus

$$\lambda = \lim_{r \rightarrow \infty} E(\min(C^r, B)) \text{ a.e.}$$

But this limit is just $E(B)$, since (2.18) implies that $C^r \rightarrow \infty$ a.e. so long as $E(C^r) > 0$ for some r; hence

$$\lim_{r \rightarrow \infty} E(\min(C^r, B)) = E(B)$$

by monotone convergence. This completes the proof that $\lambda = E(B)$.

Finally, we show that the sequence $\{C^r\}$ satisfies a central limit theorem.

THEOREM 2.19 *If the edge capacities are not almost surely constant and have a finite* $(2+\delta)$*th moment for some* $\delta > 0$, *then*

$$\frac{C^r - E(C^r)}{\sqrt{\operatorname{var}(C^r)}} \tag{2.19}$$

is asymptotically normally distributed as $r \rightarrow \infty$.

Proof Using (2.2) we see that

$$C^r = \sum_{k=1}^{r} X_{kr}$$

where $X_{kr} = \min(C^r(k), B_k)$, the B_k are independent edge capacities and the $C^r(k)$ are independent of each other and the B_k's, and are distributed like C^r. The result of (2.19) will then follow by a standard theorem (Loève [16], p. 295) if we can establish that $S_r(\varepsilon) \to 0$ as $r \to \infty$ for every $\varepsilon > 0$, where

$$S_r(\varepsilon) = \sum_{k=1}^{r} \int_{|x| \geqq \varepsilon} x^2 \, dF_r \tag{2.20}$$

where F_r is the distribution function of the random variable

$$N_{kr} = (X_{kr} - E(X_{kr}))/(\operatorname{var} C^r)^{1/2}.$$

Writing $y = x/a_r$ in (2.20), where $a_r = (\operatorname{var} C^r)^{1/2}$, yields

$$\begin{aligned} S_r(\varepsilon) &= (r/a_r^2) \int_{|y| \geqq \varepsilon a_r} y^2 \, dG_r \\ &= (\operatorname{var} X_{kr})^{-1} \int_{|y| \geqq \varepsilon a_r} y^2 \, dG_r \end{aligned} \tag{2.21}$$

because $a_r^2 = \operatorname{var} C^r = r \operatorname{var} X_{kr}$, where G_r is the distribution function of $X_{kr} - E(X_{kr})$. As we remarked in the previous proof, $C^r \to \infty$ a.e., and so as $r \to \infty$ the sequence X_{kr} converges monotonely to B_k, the capacity of the kth edge incident to the origin. Hence

$$E(X_{kr}) \uparrow E(B), \ \operatorname{var}(X_{kr}) \to \operatorname{var}(B), \tag{2.22}$$

where B is a typical edge capacity. Consider the domain of the integral in (2.21). G_r is concentrated on $[-E(B), \infty)$ and $a_r \to \infty$; thus, for large enough r, the domain may be replaced by the condition $y \geqq \varepsilon a_r$, omitting the modulus sign. Hence, for large r,

$$\begin{aligned} S_r(\varepsilon) &= (\operatorname{var} X_{kr})^{-1} \int_{y \geqq \varepsilon a_r} y^2 \, dG_r \\ &\leqq (\operatorname{var} X_{kr})^{-1} \int_{z \geqq \varepsilon a_r + \mu_r} z^2 \, dH_r \end{aligned}$$

where H_r and μ_r are the distribution function and mean of X_{kr}. By a standard inequality

$$S_r(\varepsilon) \leqq (\operatorname{var} X_{kr})^{-1}(\varepsilon a_r + \mu_r)^{-\delta} E(X_{kr}^{2+\delta})$$

for any $\delta > 0$, showing that $S_r(\varepsilon) \to 0$ as $r \to \infty$; this last statement follows because $\operatorname{var} X_{kr}$ remains bounded away from zero as $r \to \infty$ by (2.22), $0 \leqq \mu_r \leqq E(B)$, $E(X_{kr}^{2+\delta}) \leqq E(B^{2+\delta}) < \infty$ for some $\delta > 0$, and $a_r \to \infty$ as $r \to \infty$. This completes the proof.

3. THE CAPACITY OF THE COMPLETE GRAPH

Next, we consider the maximum flow through a complete graph on $n+1$ vertices, each edge of which has a random capacity drawn independently from the distribution function F_B. We consider two cases. In the first, the complete graph is undirected, whilst in the second each edge is directed in a specified manner.

Let K denote the complete undirected labelled graph on the vertex set $\{0, 1, 2, \ldots\} \cup \{\infty\}$; ∞ denotes a distinguished vertex. To each edge of K we assign a random capacity drawn, independently of all other capacities, from the specified distribution function F_B. Sometimes we shall consider the case when the edges of K are directed according to the following rule: the edge which joins vertex i to vertex j, with $i < j$, is directed from i to j. The ensuing directed network is denoted by DK. We denote a typical capacity by the letter B.

Let K_n (resp. DK_n) be the undirected (resp. directed) capacitated network on the complete subgraph of K (resp. DK) induced by the vertex set $\{0, 1, \ldots, n-1\} \cup \{\infty\}$; K_n has $n+1$ vertices. We are interested in flows between the source 0 and the sink ∞, and we write X_n and Y_n for the values of the maximum flows through K_n and DK_n, respectively. Clearly, X_n provides an upper bound for the flow through any other undirected graph on $n+1$ vertices with capacity distribution F_B. First, we note that $Y_n \leqq X_n$, since flows through DK_n are subject to more restrictions than flows through K_n. The problem of ascertaining the distributions of Y_n and X_n, for fixed n, seems hopeless. However, certain inequalities for their mean values are available immediately. Let μ_B denote the mean value of an edge capacity, and let $\mu_M = E(M)$, where $M = \min(B_1, B_2)$ is the minimum of two independent edge capacities; thus

$$\mu_B = \int x \, dF_B(x), \quad \mu_M = \int x \, dF_M(x)$$

where

$$F_M(x) = 1 - (1 - F_B(x))^2. \tag{3.1}$$

Certainly $\mu_M \leqq \mu_B$; the inequality is strict in all interesting cases.

LEMMA 3.2 $\mu_B+(n-1)\mu_M \leqq E(Y_n) \leqq E(X_n) \leqq n\mu_B.$ (3.2)

Proof The first inequality arises by considering the maximum possible flow in DK_n along the edge $(0,\infty)$ and along the paths $\{(0,i),(i,\infty):1\leqq i\leqq n-1\}$ of length two. The final inequality follows from an application of the max-flow min-cut theorem (see Ford and Fulkerson [4]) to the cutset of K_n comprising all the edges incident with the vertex 0.

Lemma 3.2 suggests the possibility that X_n and Y_n increase linearly with n; this is borne out by the next result.

THEOREM 3.3 *If* $\mu_B<\infty$ *then there exists a constant* γ *satisfying* $\mu_M\leqq\gamma\leqq\mu_B$ *such that, as* $n\to\infty$,

$$\frac{1}{n}Y_n\to\gamma \quad \textit{and} \quad \frac{1}{n}X_n\to\mu_B, \textit{ a.e. and in } L^1. \tag{3.3}$$

We have not been able to find the exact growth rate γ of the sequence $\{Y_n\}$. Notice that the theorem shows that $E(X_n)\sim n\mu_B$; compare this with the result of (3.2).

Proof First we show the existence of $\lim(n^{-1}Y_n)$ and $\lim(n^{-1}X_n)$, and then we calculate the second limit exactly. To prove the existence of these limits, we shall appeal to the theory of subadditive stochastic processes. A collection $Z=\{Z_{mn}:1\leqq m<n<\infty\}$ of random variables is called *subadditive* if the following conditions hold.

$$\text{The distribution of } Z_{mn} \text{ depends only on } n-m. \tag{3.4}$$

$$-A_n\leqq E(Z_{1,n})<\infty \text{ for all } n \text{ and some fixed } A>0. \tag{3.5}$$

$$Z_{mn}\leqq Z_{mr}+Z_{rn} \text{ whenever } m<r<n. \tag{3.6}$$

See Kingman [14], [15] for discussions of the properties of subadditive processes, and for a proof of the following result: if Z is a subadditive process then there exists a random variable ξ such that

$$\frac{1}{n}Z_{1,n}\to\xi \text{ a.e. and in } L^1. \tag{3.7}$$

For $m<n$, let X_{mn} (resp. Y_{mn}) be the maximum flow from 0 to ∞ in the subgraph of K (resp. DK) induced by the vertex set $\{0,m,m+1,\ldots,n-2,n$

$-1, \infty\}$. Writing $B_{0\infty}$ for the capacity of the edge $(0, \infty)$ joining the source to the sink directly, it is not difficult to see that the sequences $\{-(X_{mn} - B_{0\infty}): m<n\}$ and $\{-(Y_{mn}-B_{0\infty}): m<n\}$ are subadditive, and it follows from (3.7) that there exist random variables ξ_1 and ξ_2 such that, as $n \to \infty$,

$$\frac{1}{n} X_{1,n} \to \xi_1, \frac{1}{n} Y_{1,n} \to \xi_2 \text{ a.e. and in } L^1. \tag{3.8}$$

Actually, ξ_1 and ξ_2 are almost surely constant since they are measurable with respect to the tail σ-field of an independent collection of random variables. Also, $X_{1,n}=X_n$ and $Y_{1,n}=Y_n$, and so we have proved the existence of the limits in (3.3).

It is unusual to be able to ascertain a useful expression for the exact value of the limit of a subadditive process. For example, it remains an open question to calculate the limits of the original subadditive processes arising in first passage percolation; see Smythe and Wierman [20] for details of this. However, we have been able to do this for the sequence $\{X_n\}$, and show next that the limit is exactly μ_B. The proof is in three stages, dealing with the following three cases.

a) B is Bernoulli.
b) B is a discrete random variable, taking only finitely many values.
c) B has a general distribution F_B.

Case (*a*) Suppose that

$$P(B=0)=q, P(B=1)=p=1-q,$$

where $0<p<1$. Then X_n equals the number of edge-disjoint paths joining 0 to ∞ in the random graph on the vertex set $\{0, n, \ldots, n-1, \infty\}$. By the max-flow min-cut theorem,

$$P(X_n \leqq r) = P(E(n,r)) \tag{3.9}$$

where $E(n,r)$ is the event that there exists a set V of vertices of K_n such that $0 \in V$, $\infty \notin V$ and there are no more than r edges joining V to its complement V^c in K_n; we call such a set V an *r-cutset* of K_n. Then

$$P(E(n,r)) \leqq \sum_V P \text{ (} V \text{ is an } r\text{-cutset)}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} P(S_{(k+1)(n-k)} \leqq r), \tag{3.10}$$

where the first sum is over all subsets V of the vertex set of K_n, and S_m is a binomial variable with parameters m and p. Suppose $0<\varepsilon<p$ and write $r=r(n)=n(p-\varepsilon)$. Then $(k+1)(n-k)\geqq n$ whenever $0\leqq k\leqq n-1$, and so

$$P(S_{(k+1)(n-k)}\leqq r)\leqq P(S_{(k+1)(n-k)}\leqq (k+1)(n-k)(p-\varepsilon))$$
$$=P\left(\frac{T_1+\cdots+T_{(k+1)(n-k)}}{(k+1)(n-k)}\geqq\varepsilon\right),$$

where $T_i=E(U_i)-U_i$, and the U_i are independent Bernoulli variables with parameter p. Now apply Theorem 1 of Bahadur and Ranga Rao [1] to find that

$$P(S_{(k+1)(n-k)}\leqq n(p-\varepsilon))\leqq A\rho^{(k+1)(n-k)}$$

where A and ρ are constants and $0<\rho<1$. Hence, from (3.9) and (3.10),

$$P(X_n\leqq n(p-\varepsilon))\leqq A\sum_{k=0}^{n-1}\binom{n-1}{k}\rho^{(k+1)(n-k)}$$
$$\leqq 2A\sum_{k=0}^{[(1/2)n]}\binom{n-1}{k}\rho^{(1/2)n(k+1)}$$
$$\leqq 2A\rho^{(1/2)n}(1+\rho^{(1/2)n})^{n-1},$$

where $[x]$ is the integer part of x. But $\rho<1$ and so

$$\sum_n P(n^{-1}X_n\leqq p-\varepsilon)<\infty,$$

showing that

$$P\left(\liminf_{n\to\infty}(n^{-1}X_n)\geqq p\right)=1.$$

Use (3.2) and (3.8), with the fact that $p=\mu_B$ in this case, to deduce the required result for Bernoulli capacities.

Case (*b*) Let B be a discrete random variable with

$$P(B=b_k)=p_k,\quad 1\leqq k\leqq m,$$

where $0\leqq b_1<\cdots<b_m$, $\sum_1^m p_k=1$ and $p_k>0$ for each k.

Let $H_n(1)$, $H_n(2), \ldots, H_n(m)$ each be complete undirected graphs on $n+1$ vertices, each of whose vertex sets is labelled with the collection $\{0, 1, 2, \ldots, n-1\} \cup \{\infty\}$. The capacities of the edges of each $H_n(j)$ are specified by the following rule. If e is an edge of K_n with capacity $B(e)$ then the capacity of the similarly labelled edge e_j of $H_n(j)$ is

$$B_j(e) = \begin{cases} b_j - b_{j-1} & \text{if } B(e) \geqq b_j \\ 0 & \text{otherwise,} \end{cases}$$

where $b_0 = 0$. For each j, $H_n(j)$ is a capacitated network with the Bernoulli-type capacity distribution

$$P(B_j(e)=0) = \sum_1^{j-1} p_k, \quad P(B_j(e) = b_j - b_{j-1}) = \sum_j^m p_k.$$

Writing $X_n(j)$ for the maximum flow from 0 to ∞ in $H_n(j)$ and using case (a) above, we find that

$$\begin{aligned} \frac{1}{n} E(X_n) \geqq \frac{1}{n} \sum_1^m E(X_n(j)) \to \sum_1^m E(B_j(e)) \quad \text{as} \quad n \to \infty \\ = \sum_1^m (b_j - b_{j-1}) \sum_j^m p_k \\ = \sum_1^m b_j p_j = E(B). \end{aligned}$$

Now use (3.2) and (3.8) to deduce the result for discrete capacities.

Case (*c*) If B is a random variable with mean $E(B)$ and $\varepsilon > 0$, then there exists a discrete variable B', taking only finitely many values, such that

i) $B' \geqq 0$ and B' is stochastically smaller than B,
ii) $E(B) - \varepsilon \leqq E(B') \leqq E(B)$.

Apply the result of case (b) above to the capacitated network whose capacities are independent copies of B' to find that

$$\liminf_{n \to \infty} \frac{1}{n} E(X_n) \geqq E(B) - \varepsilon.$$

But ε was arbitrary; use (3.2) and (3.8) to deduce the general result.

4. COMPLEXITY AND RELIABILITY

We have had almost no success in finding exact results for the random capacity problem, even in the very special cases treated above. Accordingly, we consider briefly the degree of difficulty of the problem of calculating the expected capacity of some specified graph with respect to a given distribution; we use the terminology of computational complexity; and refer the reader to the books of Garey and Johnson [5] and Hopcroft and Ullman [12].

First we should try to place the problem in the complexity hierarchy of P, NP, P-space, and so on, and shall consider a special case of the general problem; the general problem is at least as hard as the special case. We define the problem EXPECTED CAPACITY as follows:

Input: Directed graph G with specified vertices s and t and rational numbers $p \in (0, 1)$ and $k \in [0, n-1]$, where n is the number of vertices of G.

Question: If each edge of G has a capacity drawn independently from the Bernoulli distribution parameter p, is the expected maximum flow from s to t at least k?

We denote this problem by Π. Then $\Pi \in P$ (resp. $\Pi \in NP$) if there exists a polynomial-time algorithm for deciding Π on a deterministic (resp. non-deterministic) Turing machine. An algorithm is "polynomial-time" if it runs in a time which is bounded uniformly by a polynomial function of the input length. We should like to think of this input length as being governed largely by the size n of the graph G, but must take account of the fact that *arbitrary* rationals p and k may have a substantial effect on the input length. Accordingly, we consider an even more restricted problem, for which p is constrained to take the value $1/2$ and k is constrained to be an *integer* in the range $[0, n-1]$; we denote this problem by $\Pi_{1/2}$. Still we can say very little about $\Pi_{1/2}$.

LEMMA 4.1 $\Pi_{1/2} \in P$-space. (4.1)

This is easy to prove. It seems likely that $\Pi_{1/2}$ does not belong to NP or co-NP, and therefore not to P as well; also, a proof that $\Pi_{1/2} \notin NP$, for example, would imply that $NP \neq P$-space.

Furthermore, we are unable to show that $\Pi_{1/2}$ belongs to any member of the polynomial-time hierarchy (see Meyer and Stockmeyer [17]); indeed, we predict that $\Pi_{1/2}$ lies in none of the categories of this hierarchy.

The problem $\Pi_{1/2}$ is closely related to the hard enumeration problems of Gill[7], Simon [19] and Valiant [22]. Consider Valiant's problem [22] of $S-T$ CONNECTEDNESS:

Input: A directed (resp. undirected) graph G and a pair of specified vertices s and t.

Question: How many subgraphs of G contain a directed (resp. undirected) path from s to t?

This is equivalent to the problem of finding the probability that the flow through G from s to t is strictly positive when the edge capacities are Bernoulli with parameter 1/2. Valiant shows that $S-T$ CONNECTEDNESS is $\#P$-complete. Since $\#P$ contains problems such as evaluating the permanent of a matrix and finding the number of Hamiltonian circuits in a graph, it seems unlikely that polynomial-time algorithms for these problems will be found soon.

Thus, we regard the problem of finding an efficient algorithm to calculate the expected flow through a random capacitated network to be intractable. We move on to discuss a method of handling a closely related problem. This problem may be closer to reality than Π, and is called the *reliability problem*; it concerns the probability that a flow of given value may be supported in a random network.

Let G be a directed graph with specified vertices s as source and t as sink. We assume that the capacity $B(e)$ of each edge e is a random variable with some *unknown* distribution which may depend upon the choice of edge e. However, with each edge e we are given a triple $(m(e), \mu(e), M(e))$, where $m(e)$ and $M(e)$ are lower and upper bounds for possible values of $B(e)$ and $\mu(e)$ is the expected value $E(B(e))$.

As usual, an $s-t$ *flow* in G is a mapping $f: E(G) \to \mathbb{R}$ ($E(G)$ denotes the edge set of G here, rather than an expectation) which satisfies Kirchhoff's laws at all vertices except s and t. The *reliability* $R(f)$ of f is defined to be the probability that the random edge capacities of G can support f:

$$R(f) = \prod_{e \in E(G)} P(B(e) \geqq f(e)).$$

Suppose that V is a prescribed real number. We seek an $s-t$ flow f of value V such that $R(f)$ is a maximum; that is, f is the flow of value V which is most likely to be supported by the random capacities. Such a flow is a solution to the nonlinear programme

$$\text{maximize } R(f) \tag{4.2}$$

subject to the constraint that f be an $s-t$ flow of value V.

It is easy to see that, in this form, the problem is a nonlinear programme with a very special constraint matrix. We have the choice of using either the standard methods of nonlinear programming or the decomposition method of Doulliez and Jamoulle [3], but neither of these approaches seems suitable for large networks. We describe briefly a method which is very fast and which seems (to us at least) to have the merit of being a reasonable approximation to real life situations—it is exactly the flow analogue of the technique used in critical path analysis, an area in which the problems concern longest paths instead of maximum flows.

We have supposed that the capacity $B(e)$ of the edge e satisfies

$$P(m(e) \leqq B(e) \leqq M(e)) = 1, \quad E(B(e)) = \mu(e) \tag{4.3}$$

and we propose to describe a method solution to (4.2) subject to the following assumptions about the parameters and capacities:

$$m(e) \text{ and } M(e) \text{ are integers for each edge } e, \tag{4.4}$$

for each edge e, the capacity $B(e)$ is integer valued and takes the value $m(e)+k$ with probability

$$\binom{n(e)}{k} p(e)^k (1-p(e))^{n(e)-k}, \quad (0 \leqq k \leqq n(e)) \tag{4,5}$$

where $n(e) = M(e) - m(e)$, and $p(e)$ is chosen to satisfy $E(B(e)) = \mu(e) = m(e) + n(e)p(e)$.

These assumptions are not unreasonable; the integrity condition (4.4) can be achieved to any prescribed degree of accuracy, except possibly perfection, by a linear scaling, and the assumption that $B(e) - m(e)$ is binomially distributed corresponds to the assumption of a beta type distribution in the PERT network model.

Under assumptions (4.3), (4.4) and (4.5) we may solve the reliability problem (4.2) as follows. Replace G by a new network G' in which each edge e of G is replaced by $m(e)$ green edges and $M(e) - m(e)$ amber edges. We stipulate that each green edge has capacity one with probability one, whilst each amber edge has capacity one or zero with probabilities $p(e) = (\mu(e) - m(e))/n(e)$ and $(1 - p(e))$ respectively. It is clear that the problems of finding most reliable flows of a prescribed value in G and G' are equivalent in the sense that each flow through G' corresponds to a unique flow through G, and each flow through G corresponds to at least one flow through G'. Thus, we shall restrict our attention to G'. Each edge e of G' has an associated probability $q(e)$, such that e has capacity one or zero with probabilities $q(e)$ and $(1 - q(e))$ respectively ($q(e) = 1$ if e is green, and is given by the appropriate formula if e is amber). If f is a flow through

G' which assigns flows of zero or one to each edge, then

$$R(f)=\prod_{e\in E(G')} q(e)^{f(e)}.$$

Maximizing $R(f)$ is equivalent to maximizing $\log R(f)$, and hence the problem recuces to the following:

$$\text{minimize} \sum_e a(e)f(e)$$

subject to f being a flow of value V, taking the values zero or one on each edge, where

$$a(e)=-\log q(e).$$

This is the max-flow min-cost problem, for which there exist well known fast algorithms; see for example Ford and Fulkerson [4].

See Shogan [18] for another approach to the reliability problem.

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Appendix

We are grateful to J. M. Hammersley for showing the following.

THEOREM *The limit x of the sequence*

$$x_n=\left(\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}+\frac{2}{3}\left(\frac{1}{4}+\frac{3}{4}\left(\cdots\left(\frac{1}{n-1}+\frac{n-2}{n-1}\left(\frac{1}{n}+\frac{n-1}{n}\frac{1}{(n+1)^2}\right)^2\right)^2\cdots\right)^2\right.\right.\right.$$

is given by $x=0.3314\ldots$ to an accuracy of four significant figures.

We reproduce his proof here.

Proof For $0\leqq z\leqq 1$ define

$$f_n(z)=\left(\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}+\frac{2}{3}\left(\cdots\left(\frac{1}{n}+\frac{n-1}{n}z^2\right)^2\cdots\right)^2\right.\right.. \tag{A1}$$

Thus

$$x_n = f_n\left(\frac{1}{n+1}\right) = f_{n+1}(0). \tag{A2}$$

For fixed $n \geqq 2$, $f_n(z)$ is a polynomial in z with non-negative coefficients; so $f_n(z)$ and its derivative $f'_n(z)$ are both strictly increasing functions of z on $0 \leqq z \leqq 1$.

We have that

$$f_n(1) = 1 \quad \text{for} \quad n = 2, 3, \ldots \tag{A3}$$

Also, from (A1),

$$f_n(z) = f_{n-1}\left(\frac{1}{n} + \frac{n-1}{n} z^2\right), \quad (n \geqq 3). \tag{A4}$$

Hence

$$f'_n(z) = \frac{2(n-1)}{n} z f'_{n-1}\left(\frac{1}{n} + \frac{n-1}{n} z^2\right). \tag{A5}$$

Put $z = 0$ in (A4). Then

$$f_n(0) = f_{n-1}\left(\frac{1}{n}\right) > f_{n-1}(0). \tag{A6}$$

So $\{x_n\}$ is an increasing sequence, bounded above by 1 in view of (A3), and hence x exists. Also

$$f_n(0) < x, \quad (n \geqq 2). \tag{A7}$$

Put $z = 1/n-1$ in (A4). This gives

$$f_n\left(\frac{1}{n}\right) < f_n\left(\frac{1}{n-1}\right) = f_{n-1}\left(\frac{1}{n} + \frac{1}{n(n-1)}\right) = f_{n-1}\left(\frac{1}{n-1}\right). \tag{A8}$$

Hence $\{f_n(1/n)\}$ is a decreasing sequence bounded below by 0 and there exists

$$y = \lim_{n \to \infty} f_n\left(\frac{1}{n}\right). \tag{A9}$$

From (A8) we have

$$f_n\left(\frac{1}{n}\right) > f_{n+m}\left(\frac{1}{n+m}\right) > f_{n+m}(0). \tag{A10}$$

Letting $m \to \infty$ in this equation, we obtain

$$f_n\left(\frac{1}{n}\right) > x, \tag{A11}$$

whence

$$y \geqq x. \tag{A12}$$

Also, put $z = 1/n - 1$ in (A5). We get

$$f'_n\left(\frac{1}{n}\right) < f'_n\left(\frac{1}{n-1}\right) = \frac{2}{n} f'_{n-1}\left(\frac{1}{n-1}\right). \tag{A13}$$

Hence

$$\begin{aligned} f'_n\left(\frac{1}{n}\right) &< \frac{2}{n} \cdot \frac{2}{n-1} \cdots \frac{2}{3} f'_2\left(\frac{1}{2}\right) \\ &= \frac{2^{n-1}}{n!} f'_2\left(\frac{1}{2}\right) = \frac{5}{32} \cdot \frac{2^n}{n!}. \end{aligned} \tag{A14}$$

Consequently

$$\begin{aligned} 0 < f_n\left(\frac{1}{n}\right) - f_n(0) &= \frac{1}{n} f'_n\left(\frac{\theta_n}{n}\right) \quad \text{for some} \quad 0 < \theta_n < 1 \\ &< \frac{1}{n} f'_n\left(\frac{1}{n}\right) < \frac{5}{32n} \cdot \frac{2^n}{n!}. \end{aligned} \tag{A15}$$

Let $n \to \infty$ in (A14), and deduce that

$$y = x. \tag{A16}$$

We have so far proved that

$$f_n\left(\frac{1}{n+1}\right) < x < f_n\left(\frac{1}{n}\right). \tag{A17}$$

We can however sharpen the lower bound as follows. Suppose $n \geqq 2$. Then

$$0 < n^6 + 12n^4(n-2) + 20n(n-2) + 4$$

$$< n^6 + 12n^5 - 19n^4 + 20n^2 - 16n + 4.$$

Hence

$$n^6(n^2-4n+1) < n^8 - 4n^7 + 2n^6 + 12n^5 - 19n^4 + 20n^2 - 16n + 4$$

$$= (n-1)^4(n^2-2)^2.$$

So

$$\frac{n^2-4n+1}{(n-1)^4} < \left(\frac{n^2-2}{n^3}\right)^2,$$

giving that

$$f_n\left(\frac{n^2-2}{n^3}\right) > f_n\left[\left(\frac{n^2-4n+1}{(n-1)^4}\right)^{1/2}\right]$$

$$= f_{n-1}\left[\frac{1}{n} + \frac{n-1}{n} \cdot \frac{n^2-4n+1}{(n-1)^4}\right]$$

$$= f_{n-1}\left[\frac{(n-1)^2-2}{(n-1)^3}\right].$$

Consequently

$$\left\{f_n\left(\frac{n^2-2}{n^3}\right)\right\}$$

is an increasing sequence. Also

$$f_n(0) < f_n\left(\frac{n^2-2}{n^3}\right) < f_n\left(\frac{1}{n}\right).$$

Letting $n \to \infty$ here, we deduce

$$f_n\left(\frac{n^2-2}{n^3}\right) \to x.$$

Therefore

$$f_n\left(\frac{n^2-2}{n^3}\right)<x<f_n\left(\frac{1}{n}\right), \quad (n\geqq 2), \tag{A18}$$

which is the sharpened form of (A17).

Also, as in (A15),

$$0<f_n\left(\frac{1}{n}\right)-f_n\left(\frac{n^2-2}{n^3}\right)<\frac{2}{n^3}f'_n\left(\frac{1}{n}\right)<\frac{5}{16n^3}\cdot\frac{2^n}{n!},$$

whence

$$f_n\left(\frac{1}{n}\right)-\frac{5}{16n^3}\cdot\frac{2^n}{n!}<x<f_n\left(\frac{1}{n}\right). \tag{A19}$$

For $n=10$, this gives

$$f_{10}\left(\frac{1}{10}\right)-\frac{1}{22680000}<x<f_{10}\left(\frac{1}{10}\right). \tag{A20}$$

So $x=f_{10}(1/10)$ correct to 7 places of decimals. For numerical purposes (A18) will be better than (A20). Thus

$$f_{10}(0.098)<x<f_{10}(0.100), \tag{A21}$$

or

$$0.3314015256<x<0.3314015419. \tag{A22}$$

It is not difficult to prove that

$$\lim_{n\to\infty} f_n(z)=\begin{cases} z(0\leqq z<1) \\ 1(z=1).\end{cases}$$