PERCOLATION PROCESSES AND DIMENSIONALITY

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Abstract

We discuss bond percolation on the cubic lattice $\mathbb{Z}^d$ in dimensions $d = 1,2,3$, paying particular attention to the ways in which such processes "evolve" as the dimension increases from $d = 1$ through $d = 2$ to $d = 3$.. There are many conjectures.

Contents

1. Introduction to bond percolation
2. One-dimensional percolation models and long-range effects
3. Two-dimensional percolation and the transition from one to two dimensions
4. Three-dimensional percolation and the transition from two to three dimensions
References

1. Introduction to bond percolation

There are three dimensions to the space which we inhabit, and consequently most spatial models of statistical physics are three-dimensional. On the other hand, the two-dimensional case is often more popular with mathematicians and the following trite table may indicate something of why this is so.

dimension	degree of difficulty	observation
1	none	no critical phenomena
2	considerable	critical phenomena
3	very great	critical phenomena

Thus physical processes in one dimension are generally soluble, often having exact solutions which exhibit no critical behaviour as the underlying parameters vary. At the other extreme, three-dimensional processes are often very difficult to study and much of our information about such processes is derived by approximation methods such as Monte Carlo simulation, series expansions and renormalization rather than by rigorous analytical techniques. In the intermediate case of two dimensions, however, such processes may sometimes be studied usefully, though with some difficulty, by rigorous mathematical arguments. Also, two is the least number of dimensions for which there is a truly spatial aspect, generally exhibiting critical phenomena. Two-dimensional processes can thus both be challenges and yield rewards, since they are often rather difficult but may sometimes reveal their secrets under pressure. It is interesting to study particular physical processes in detail as their dimensions increase from one, through two, to three, and it is the purpose of this paper to do this for the bond percolation process on the cubic lattice $\mathbb{Z}^d$.

The bond percolation process was introduced by Broadbent and Hammersley (1957) as a model for a porous stone. More recently it has become a fashionable subject for contributors to physics journals, and one reason for this is that it may be considered to be a model for ferromagnetism in which the interaction has been reduced to a minimum. Recent reviews include Kesten (1982), Smythe and Wierman (1978), Essam (1980) and Stauffer (1979).

Let $\mathbb{Z}^d$ be the d-dimensional cubic lattice with vertex set $\{(i_1,i_2,\dots,i_d) : i_k = \dots,-1,0,1,\dots,\ k = 1,2,\dots,d\}$ and edges joining vertices $\underline{i}$ and $\underline{j}$ whenever

$$\sum_{k=1}^{d} |i_k - j_k| = 1,$$

and let p be a number satisfying $0 < p < 1$. We declare each edge of $\mathbb{Z}^d$ to be <u>open</u> with probability p and <u>closed</u> otherwise, independently of all other edges. The set of vertices of $\mathbb{Z}^d$ together with the open edges (only) forms a subgraph of the cubic lattice, and the components of this subgraph are called <u>open clusters</u>. Percolation theory is concerned with properties of these open clusters and particularly with the ways in which such properties change as the density p of open edges increases from 0 to 1. Of great interest and importance is the so-called critical phenomenon which takes place for a particular intermediate value of p. Writing I for the event that there is an infinite open cluster and P_p for the probability function corresponding to a given value of p, it is not too difficult to show that there exists a number $\pi(d)$, depending on the dimension d, such that

$$P_p(I) = \begin{cases} 0 & \text{if } 0 \le p < \pi(d), \\ 1 & \text{if } \pi(d) < p \le 1, \end{cases}$$

and $\pi(d)$ is called the <u>critical probability</u> of the process. It is easy to see that $\pi(1) = 1$, so that there is no interesting critical phenomenon in one dimension. On the other hand, it may be shown that $0 < \pi(d) < 1$ if $d \ge 2$, so that all dimensions exceeding 1 exhibit critical behaviour. It seems to be exceedingly difficult to calculate $\pi(d)$ exactly, although the now celebrated Harris-Russo-Seymour-Welsh-Kesten theorem states that $\pi(\frac{1}{2}) = \frac{1}{2}$. It is thought that the sequence $\pi(1),\pi(2),\dots$ is strictly decreasing and satisfies

$$d\pi(d) \to \tfrac{1}{2} \quad \text{as} \quad d \to \infty$$

(see Gaunt and Ruskin (1978)).

We place the emphasis of this paper upon the dimensionality of the bond percolation process, beginning with a discussion of the case $d = 1$. The usual percolation process is of little interest here, but we shall see that there is a more general type of process involving "long-range interactions" which enjoys a critical phenomenon; this new process is akin to a well-known one-dimensional model for ferromagnetism. We study the transition from $d = 1$ to $d = 2$ in some detail, paying particular attention to the problem of determining the "effective dimensions" of subsets of the square lattice $\mathbb{Z}^2$. Much less is known about the transition from $d = 2$ to $d = 3$ and there are several related entertaining conjectures.

2. One-dimensional percolation models and long-range effects

In this section we consider a more general one-dimensional percolation model than the usual bond percolation process. Let $\underline{p} = (p(1),p(2),\dots)$ be a sequence of numbers satisfying $0 \le p(n) < 1$ for all n, and let $\mathbb{Z}$ be the set of all integers. We examine each distinct unordered pair (i,j) in turn, and either we join this pair by an edge with probability $p(|i-j|)$ or we leave the pair disconnected otherwise; this is done independently of all other unordered pairs. We denote by G the ensuing (random) graph. The usual bond percolation process on $\mathbb{Z}$ is retrieved by setting $p(n) = 0$ if $n \ge 2$, and thus the general model may be thought of as a percolation-type process with (possibly) long-range interaction.

Whereas the bond percolation process on $\mathbb{Z}$ is trivial to study and shows no critical behaviour, the model described above has a rich structure which depends largely upon the tail behaviour of the sequence $p(1),p(2),\dots$. The following theorem describes the threshold between connectedness and disconnectedness of G.

Theorem 1. *The graph* G *is almost surely connected if*

$$\sum_{n=1}^{\infty} p(n) = \infty \tag{2.1}$$

and the greatest common divisor of $\{n : p(n) > 0\}$ *equals* 1; *if either of these two conditions fails to hold then* G *is almost surely disconnected*.

See Grimmett, Keane and Marstrand (1984) for the proof of this theorem and for more details of the threshold. Theorem 1 may be generalized in the obvious way to higher dimensions, replacing $\mathbb{Z}$ by $\mathbb{Z}^d$ where $d \ge 2$. It is believed that the corresponding result is valid for the subgraph of G on the smaller vertex set

$\{0,1,2,\ldots\}^d$, but no proof of this is known if $d \geq 2$. S. Kalikow has found an easier approach (unpublished) to the proof of Theorem 1; he extends the conclusion of the theorem to the subgraph of G on $\{0,1,2,\ldots\}$ but his argument is not easy to generalize to higher dimensions.

Even when $\Sigma_n p(n) < \infty$ it is possible that G contains an infinite component. Let $\theta(\underline{p})$ be the probability that the component of G containing the vertex 0 is infinite. It is not difficult to show that

$$P_{\underline{p}}(\text{G contains an infinite component}) = \begin{cases} 0 & \text{if } \theta(\underline{p}) = 0, \\ 1 & \text{if } \theta(\underline{p}) > 0, \end{cases}$$

where $P_{\underline{p}}$ is the probability function corresponding to the sequence $\underline{p}$ of edge-probabilities. This threshold between non-existence and existence of infinite components is akin to the usual bond percolation threshold described in the introduction. It turns out that there is a non-trivial threshold for the long-range process. First, there is a simple argument (see Schulman (1984)) showing that

$$\theta(\underline{p}) = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} np(n) < \infty \tag{2.2}$$

and implying that all components are almost surely finite if $\Sigma_n np(n) < \infty$. The remaining cases are those for which the p(n)'s satisfy the conditions of neither (2.1) nor (2.2), and the natural case to study is when

$$p(n) \simeq \frac{c}{n^s} \quad \text{for large } n, \tag{2.3}$$

for constants c and s satisfying $1 < s \leq 2$. With a little thought we may see that, unlike in Theorem 1, both short-range and long-range interactions are important here; consequently we think of the sequence $\underline{p}$ as containing two "independent" quantities, being the number p(1) and the sequence (p(2),p(3),...) satisfying (2.3). The usual branching process argument (see Schulman (1984)) gives that all components are almost surely finite if p(1) and c are small enough:

$$\theta(\underline{p}) = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} p(n) \leq \tfrac{1}{2}.$$

On the other hand, C. Newman and L. Schulman have shown (unpublished) that it is possible to have infinite components with positive probability.

Theorem 2. *Suppose that the sequence* $(p(2),p(3),\ldots)$ *satisfies* (2.3).

(i) *If* $1 < s < 2$ *and* $c > 0$, *then* $\theta(\underline{p}) > 0$ *for large values of* $p(1)$.

(ii) *If* $s = 2$ *and* $c > 2$, *then* $\theta(\underline{p}) > 0$ *for large values of* $p(1)$.

It seems that the case $s = 2$ provides the most interesting critical behaviour, especially in the light of the following result (unpublished) of M. Aizenman and C. Newman.

Theorem 3. *Under the hypothesis of Theorem 2,*

(i) *if* $s = 2$ *and* $c < 1$, *then* $\theta(\underline{p}) = 0$,

(ii) *if* $s = 2$, *then either* $\theta(\underline{p}) = 0$ *or* $\theta(\underline{p}) > c^{-\frac{1}{2}}$.

Thus, in the extreme case when $s = 2$, $\theta(\underline{p})$ is a discontinuous function of c at the critical point. It is currently an open problem to ascertain the critical value σ of c given by

$$\sigma = \inf\{c : \theta(\underline{p}) > 0 \text{ for large } p(1)\},$$

although we have from Theorems 2 and 3 that $1 \le \sigma \le 2$.

In summary, the usual bond percolation process in one dimension is of little or no interest, but a contrasting long-range model has a complicated and interesting theory. It is revealing to note the similarity between this model and a well-known one-dimensional model of statistical physics in which two vertices which are distance n apart enjoy a ferromagnetic interaction with strength $J(n) \approx Jn^{-s}$ for constants J and s. See Fröhlich and Spencer (1982) for recent results about the critical behaviour of this process.

3. Two-dimensional percolation and the transition from one to two dimensions

We return to the usual bond percolation process on $\mathbb{Z}^d$ and concentrate in this section on the planar case when $d = 2$. It is now well-known that the critical probability $\pi(2)$ in two dimensions is equal to $\frac{1}{2}$. If A is a subgraph of $\mathbb{Z}^2$, we may define the critical probability of the bond percolation process on A to be the number $\pi(A)$ given by

$$\pi(A) = \sup\{p : P_p(I(A)) = 0\}$$

where $I(A)$ is the event that A contains an infinite open cluster. The critical probability of A is an indication of the "effective dimension" of the bond percolation process on A, two extreme cases being $\pi(\mathbb{Z}) = 1$ and $\pi(\mathbb{Z}^2) = \frac{1}{2}$. It turns out that $\pi(A)$ takes all values in the interval $[\frac{1}{2},1]$ as A varies over subsets of $\mathbb{Z}^2$.

Theorem 4. Let $0 \le c < \infty$ and let $f(x) = c \log(x+1)$ for $x \ge 0$. The subgraph of $\mathbb{Z}^2$ on the set

$$A(c) = \{(i,j) \in \mathbb{Z}^2 : 0 \le j \le f(i),\ i \ge 0\}$$

has critical probability $\pi(A(c)) = \nu(c)$ where ν is a continuous and strictly decreasing function which maps $[0,\infty)$ onto $(\frac{1}{2},1]$.

See Grimmett (1981, 1983) for a proof of this result. Fig. 1 contains a sketch of the function ν. Theorem 4 may be read as saying that the transition from one dimension to two dimensions is smooth, in the sense that the critical probability of $A(c)$ varies continuously between $\pi(\mathbb{Z}) = 1$ and $\pi(\mathbb{Z}^2) = \frac{1}{2}$ as c varies from 0 to ∞.

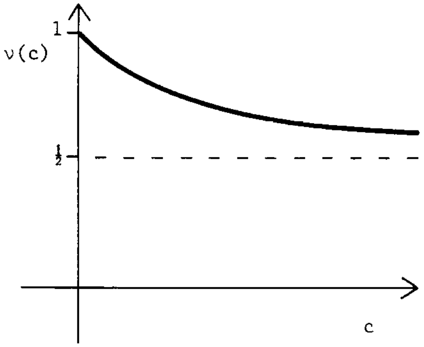

Fig. 1 A sketch of the function ν.

J. van den Berg has pointed out that the strict monotonicity of ν may be obtained by applying a general theorem of reliability theory (see Barlow and Proschan (1975)) as well as by the ad hoc argument of Grimmett (1983).

The conclusion of Theorem 4 may be contrasted with the results of Kesten (unpublished) and Hammersley and Whittington (1984) who have counted self-avoiding walks in wedges of $\mathbb{Z}^2$. Let w_n be the number of self-avoiding walks of length n in $\mathbb{Z}^2$ starting from the origin, and let $w_n(f)$ be the number of such walks which are confined to the subset

$$A(f) = \{(i,j) \in \mathbb{Z}^2 : 0 \le j \le f(i),\ i \ge 0\}$$

of $\mathbb{Z}^2$ where f is a non-negative function on $[0,\infty)$. It is well-known that the limit

$$\kappa = \lim_{n\to\infty} \frac{1}{n} \log w_n$$

exists; κ is called the connective constant of $\mathbb{Z}^2$. Hammersley and Whittington (1984) show the following theorem.

Theorem 5. If $f(x) \to \infty$ as $x \to \infty$, then

$$\frac{1}{n} \log w_n(f) \to \kappa \text{ as } n \to \infty.$$

That is to say, the connective constant of the wedge A(f) is the same as the connective constant of the whole square lattice whenever the height of the wedge tends to infinity as one looks progressively to the right. In other words, "$f(x) \to \infty$ as $x \to \infty$" is a sufficient condition for A(f) to be "effectively two-dimensional" with regard to the number of its self-avoiding walks.

4. Three-dimensional percolation and the transition from two to three dimensions

Bond percolation in three dimensions is a process which is perhaps as rich in conjectures and open problems as it is in known results. Pride of place amongst conjectures should go to the question of the uniqueness of the critical probability. Let $W(\underline{0})$ be the number of vertices of $\mathbb{Z}^3$ which are joined to the origin $\underline{0}$ by open paths (that is, by paths containing open edges only). The critical probability $\pi(3)$ may be defined by

$$\pi(3) = \sup\{p : P_p(W(\underline{0}) = \infty) = 0\}.$$

We may also define

$$\gamma(3) = \sup\{p : E_p(W(\underline{0})) < \infty\},$$

the supremum of all values of p for which $\underline{0}$ belongs to an open cluster with finite mean size. It is clear that $\gamma(3) \leq \pi(3)$.

Conjecture 6. Using the above notation, it is the case that $\gamma(3) = \pi(3)$.

The transition from two to three dimensions is not well understood at present. There are (at least) two ways of formulating the problem, and the more important way is as follows. Let $\mathbb{Z}^2(k)$ denote the slice $\mathbb{Z}^2 \times \{1,2,\dots,k\}$ with thickness k cut from $\mathbb{Z}^3$, and let $\rho(k)$ be the critical probability of the bond percolation process on $\mathbb{Z}^2(k)$. Thus $\rho(1) = \pi(2) = \frac{1}{2}$ and $\rho(k+1) \leq \rho(k)$ for $k = 1,2,\dots$. It is conjectured that the k-slice behaves more and more like the whole lattice $\mathbb{Z}^3$ as $k \to \infty$.

Conjecture 7. The limit $\rho = \lim_{k\to\infty} \rho(k)$ is given by $\rho = \pi(3)$.

It is clear that $\rho \geq \pi(3)$, but it does not seem to be at all easy to show equality here (see Kesten (1982) and Aizenman et al. (1983)).

An alternative approach to the transition from two to three dimensions resembles more the method of the previous section. Let f_2 and f_3 be non-negative, non-decreasing functions on $[0,\infty)$ and define the f-*wedge* of $\mathbb{Z}^3$ to be the subgraph of $\mathbb{Z}^3$ on the vertex set $\{(i,j,k) \in \mathbb{Z}^3 : 0 \le j \le f_2(i),\ 0 \le k \le f_3(i),\ i \ge 0\}$. We write $\pi(f)$ for the critical probability of the bond percolation process on this f-wedge. Very little is known about the behaviour of $\pi(f)$ in terms of the functions f_2 and f_3.

Theorem 8. *Let* $g(i) = (f_2(i) + 1)(f_3(i) + 1)$ *for* $i = 0,1,2,\ldots$.

(i) *If* $g(i)/\log i \to 0$ *then* $\pi(f) = 1$, *and*

(ii) *if* $g(i)/\log i \to \infty$ *then* $\pi(f) \le \frac{1}{2}$,

where the limits are taken as $i \to \infty$.

Hammersley and Whittington (1984) have proved part (i); part (ii) is a minor extension of another of their results. A slightly more sophisticated treatment provides some information about the behaviour of $\pi(f)$ in the critical case when $g(x) = a\log(x+1)$ and $a > 0$, giving that there exists $\nu(a) < 1$ such that

$$\max\{\pi(3),\ 1 - e^{-1/a}\} \le \pi(f) \le \nu(a)$$

in this case.

It is an open question to ascertain attractive conditions on f_2 and f_3 which imply that $\pi(f) = \pi(3)$. Perhaps it is enough to require that

$$f_2(i) \to \infty,\quad f_3(i) \to \infty,\quad f_2(i)f_3(i)/\log i \to \infty$$

as $i \to \infty$.

Another intriguing open question is whether or not there can be more than one infinite open cluster in $\mathbb{Z}^3$. Writing N for the number of such clusters, it is not difficult to show that almost surely exactly one of the following three possibilities must hold for any given value of p:

$$\text{(i)}\quad N = 0,\qquad \text{(ii)}\quad N = 1,\qquad \text{(iii)}\quad N = \infty.$$

Clearly $N = 0$ almost surely if $p < \pi(3)$, and it is generally believed that $N = 1$ almost surely if $p > \pi(3)$.

Conjecture 9. $P_p(N = 0 \text{ or } N = 1) = 1$ *for all* p.

This conjecture is bound up with Conjecture 7, since H. Kesten can show (unpublished) that $P_p(N = 1) = 1$ if $p > \rho = \lim_{k\to\infty}\rho(k)$. See Newman and Schulman (1981 a,b) for some results and speculations about the possibility that there are infinitely many infinite open clusters for some value of p.

To every percolation process there corresponds a dual process. An observation of great importance in two dimensions is that the dual of a two-dimensional bond percolation process is also a bond percolation process, and this fact is extremely useful for planar models. Unfortunately, the dual of bond percolation on $\mathbb{Z}^3$ is a type of "random surface" process, and this is rather different from the original process. There are some quite difficult topological complications which have to be taken into account in studying this random surface model, and some of the first steps in doing so have been taken by Aizenman et al. (1983). We finish this survey with a simple related conjecture.

Conjecture 10. The number $W(\underline{0})$ *of vertices of* $\mathbb{Z}^3$ *which are joined to* $\underline{0}$ *by open paths satisfies*

$$E_p(W(\underline{0})1(W(\underline{0}) < \infty)) < \infty \tag{4.1}$$

for $p \neq \pi(3)$, *where* $1(A)$ *is the indicator function of the event* A.

That is to say, we conjecture that the mean size of the cluster containing $\underline{0}$ is finite whenever this cluster is finite itself, so long as $p \neq \pi(3)$. This is evidently true if $p < \gamma(3)$ but we know of no proof if $p > \pi(3)$, although it should not be too difficult to prove this if $p > \rho$. If this conjecture is true, then so are certain conjectured central limit theorems of Cox and Grimmett (1984). For instance, if $p > \pi(3)$ and (4.1) holds then the number of vertices of $\mathbb{Z}^3$ which are contained within a cube with side-length n and are joined by open paths to the boundary of this cube is asymptotically normally distributed as $n \to \infty$.

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