THE STOCHASTIC RANDOM-CLUSTER PROCESS, AND THE UNIQUENESS OF RANDOM-CLUSTER MEASURES

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Abstract. The random-cluster model is a generalisation of percolation and ferromagnetic Potts models, due to Fortuin and Kasteleyn (see [29]). Not only is the random-cluster model a worthwhile topic for study in its own right, but also it provides much information about phase transitions in the associated physical models. This paper serves two functions. First, we introduce and survey random-cluster measures from the probabilist's point of view, giving clear statements of some of the many open problems. Secondly, we present new results for such measures, as follows. We discuss the relationship between weak limits of random-cluster measures and measures satisfying a suitable DLR condition. Using an argument based on the convexity of pressure, we prove the uniqueness of random-cluster measures for all but (at most) countably many values of the parameter p. Related results concerning phase transition in two or more dimensions are included, together with various stimulating conjectures. The uniqueness of the infinite cluster is employed in an intrinsic way, in part of these arguments. In the second part of this paper is constructed a Markov process whose level-sets are reversible Markov processes with random-cluster measures as unique equilibrium measures. This construction enables a coupling of random-cluster measures for all values of p. Furthermore it leads to a proof of the semicontinuity of the percolation probability, and provides a heuristic probabilistic justification for the widely held belief that there is a first-order phase transition if and only if the cluster-weighting factor q is sufficiently large.

1. Introduction

The Ising model [39] is well known to probabilists as a model for ferromagnetism; it exhibits a phase transition and provides a host of beautiful problems for the mathematician and the physicist. Whereas the Ising model allows only two possible spins at each site, the Ashkin–Teller and Potts models permit a general number of spin values ([4, 57]). In the late 1960s, Kasteleyn observed that electrical networks, percolation processes, and Ising/Potts models have certain features in common, namely versions of the series and parallel laws. In joint work with Fortuin, he formulated a class of measures which includes the percolation, Ising, and Potts measures. This class is simple to describe and has rich structure; it is the class

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of *random-cluster measures*, sometimes known as Fortuin–Kasteleyn measures; see [19, 20, 21, 22, 29, 40] for the early work on this topic.

The random-cluster model is a process on the *edges* of a graph rather than on its *vertices*. Through studying its properties, we obtain information about phase transitions in physical systems. The model incorporates a unifying description of certain physical processes, and provides a natural setting for various techniques of value. Indeed it is now recognised as a standard tool in studying Ising/Potts systems ([1, 7, 10, 18, 23, 28, 44, 56, 59, 60]).

Whereas a Potts model has a strength J of interaction and a number q of states, the corresponding random-cluster model has an edge parameter $p = 1 - e^{-J}$ and a 'cluster weighting factor' q; we shall assume that $0 \le p \le 1$ (so that $J \ge 0$) and q is a real number satisfying $0 < q < \infty$. The relationship between random-cluster models and their physical counterparts is well documented elsewhere, and we shall not repeat this material here; see [18, 30]. It has proved valuable to study the random-cluster model in its own right (see, for example, [1, 7, 10, 18, 23, 28, 30, 52, 56, 59]). Quite apart from its relevance to statistical physics, the model is of considerable intrinsic interest and has many beautiful mathematical questions of stochastic geometry associated with it.

This paper begins with an introduction to the random-cluster model, and a brief description of the main techniques of value. The purpose of this is to prepare the reader with a background in modern probability, and to tempt that reader to try to solve some of the beautiful open problems associated with the model. In addition this paper contains new results, as summarised later in this introduction.

We define a random-cluster measure on a finite graph $G = (V, E)$ as follows. Let $0 \le p \le 1$ and $q > 0$. The relevant sample space is the finite set $\Omega_E = \{0, 1\}^E$, containing configurations that allocate 0's and 1's to the edges of G. For $\omega \in \Omega_E$, we call an edge e *open* if $\omega(e) = 1$, and *closed* otherwise. The random-cluster measure on G, having parameters p and q, is the probability measure $\phi_{G,p,q}$ on Ω_E given by

$$\phi_{G,p,q}(\omega) = \frac{1}{Z_{G,p,q}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \quad \omega \in \Omega_E, \tag{1.1}$$

where $k(\omega)$ is the number of open components of ω (i.e., the number of components of the graph $(V, \eta(\omega))$, where $\eta(\omega)$ is the set of open edges under ω), and

$$Z_{G,p,q} = \sum_{\omega \in \Omega_E} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)} \tag{1.2}$$

is the normalising factor (or 'partition function'). Note that $\phi_{G,p,q}$ differs from product measure (i.e., percolation [26] or 'random graphs' [12]) only in the presence of the term $q^{k(\omega)}$.

The reader is referred to [29, 30] for some historical remarks and basic references pertaining to such measures. We note that percolation corresponds to the case $q = 1$, the Ising model to the case $q = 2$, and Potts models to the cases $q = 2, 3, \dots$.

In defining a random-cluster measure on an infinite lattice $\mathcal{L}$, we may follow either of two routes. The first is to take G to be a finite box in $\mathcal{L}$, and to pass to the infinite-volume limit (with suitable boundary conditions). The second is to follow the

Dobrushin–Lanford–Ruelle formalism, and to study measures which, conditional on the states of edges outside a finite subgraph G of $\mathcal{L}$, have the form (1.1) with appropriate boundary conditions. There are some difficulties in comparing these two approaches, which are explored in some detail in Section 3. When $q \geq 1$, we conjecture that there is a unique random-cluster measure $\phi_{p,q}$ (following either route) except at the critical point of a first-order phase transition (see below).

Assume for the moment that $q \geq 1$. An infinite-volume random-cluster measure $\phi_{p,q}$ has a phase transition. More specifically, the probability $\theta(p,q) = \phi_{p,q}(0 \leftrightarrow \infty)$, that the origin lies in an infinite open path, satisfies

$$\theta(p,q) \begin{cases} = 0 & \text{if } p < p_c(q), \\ > 0 & \text{if } p > p_c(q), \end{cases} \tag{1.3}$$

for some critical value $p_c(q)$ $(\in (0,1))$ that depends on the lattice. It is hopeless to expect an exact calculation of $p_c(q)$ for a general lattice, but there are certain tempting conjectures for some two-dimensional lattices. For example, for the square lattice it is believed that

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}, \quad \text{if } q \geq 1; \tag{1.4}$$

this conjecture is based on the self-duality of the square lattice (see Section 5). This exact calculation is known to be valid for the cases $q = 1$, $q = 2$, and for large values of q ([41, 54, 43, 45]).

One of the principal features of random-cluster measures is the discontinuity of the phase transition for large q. It is believed that, for any lattice $\mathcal{L}$ in at least two dimensions, there exists $Q = Q(\mathcal{L})$ such that the 'order parameter' $\theta(p,q)$ (defined with an appropriate boundary condition) is continuous at $p = p_c(q)$ if $q < Q$, and is discontinuous if $q > Q$. This amounts to the conjecture that

$$\theta^1(p_c(q),q) \begin{cases} = 0 & \text{if } q < Q, \\ > 0 & \text{if } q > Q, \end{cases} \tag{1.5}$$

where $\theta^1(p,q) = \phi^1_{p,q}(0 \leftrightarrow \infty)$ and $\phi^1_{p,q}$ is the maximal random-cluster measure (with the usual stochastic ordering of measures). Furthermore one expects that

$$Q(\mathbb{L}^d) = \begin{cases} 4 & \text{if } d = 2, \\ 2 & \text{if } d \geq 6, \end{cases} \tag{1.6}$$

where $\mathbb{L}^d$ denotes the d-dimensional hypercubic lattice. For any lattice in two or more dimensions, it is known that $\theta(\cdot,q)$ is discontinuous at the critical point so long as q is sufficiently large; see [44]. This is in contrast to the state of knowledge for small q. In particular it is widely believed but currently unproven that, in the case $q = 1$,

$$\theta(p_c(1),1) = 0 \quad \text{for all lattices}, \tag{1.7}$$

and this is one of the main open problems of percolation theory (see [5, 6, 26, 32, 34]). We call a phase transition *first-order* if $\theta(\cdot,q)$ is discontinuous at the critical point, and *second-order* otherwise.

There are numerous other open questions for random-cluster measures, such as the exponential decay of the pair connectivity function throughout the subcritical phase (i.e., when $p < p_c(q)$), and so on. Many partial results are known, but few complete theorems.

Having given a taste of the open problems for these measures, we move on to summarise the material presented in detail in this paper. Throughout the article we shall encounter references and discussion related to the above issues.

There are two main mathematical targets, and a number of lesser results. The first three principal sections (3–5) are devoted to a study of 'random-cluster measures' in their generality. Here we study the relationship between weak limits of such measures on finite boxes, and the associated measures on the infinite lattice which satisfy a type of Dobrushin–Lanford–Ruelle (DLR) condition. We prove a partial uniqueness theorem for random-cluster measures, and make certain conjectures about uniqueness and translation-invariance.

The second main target of this paper is to construct Markov processes on the infinite lattice having invariant measures which are random-cluster measures. Such constructions have been obtained for a host of interacting particle systems (see [48] for example). In the present instance, the usual general theory from interacting particle systems cannot be applied, since the natural 'speed functions' are not continuous in the product topology; we adopt here an alternative strategy based on FKG orderings of measures. We pursue this strategy at a level of generality sufficient to produce also a level-set representation of random-cluster measures for different values of p (the second parameter q is fixed and assumed to satisfy $q \geq 1$). Such couplings of processes for different values of p have applications for percolation and the Ising model also (see [9, 26, 35]).

We terminate this introduction with an outline of the contents of the remainder of the paper. In Section 2 we introduce some necessary notation, and sketch the main techniques, namely the FKG inequality and the comparison inequalities. Section 3 contains two definitions. The first of these is a definition of a random-cluster measure as a probability measure satisfying a certain DLR condition via an appropriate 'specification' (see [24]). The second definition is of weak limits of such measures defined on finite boxes. It is proved that all translation-invariant weak limits are indeed random-cluster measures, and that a certain pair of weak limits, $\phi^0_{p,q}$ and $\phi^1_{p,q}$, are extremal when $q \geq 1$. The theorem of Burton and Keane [14] concerning uniqueness of infinite clusters is employed here, and this uniqueness takes the role played by 'quasilocality' for Gibbs states (see [24]).

In Section 4, we adapt an argument first used by Lebowitz and Martin-Löf [47] in order to prove that there is a unique random-cluster measure for almost every value of p, so long as $q \geq 1$. Further results are available for the special case of two dimensions, and some progress is achieved in the 'non-FKG' regime when $0 < q < 1$.

Phase transition is the theme of Section 5. In particular, the semicontinuity of certain percolation probabilities is noted, as are further partial results concerning the uniqueness of random-cluster measures. It is noted that the critical point $p_c = p_c(q)$ is a Lipschitz-continuous and strictly increasing function of q on $[1, \infty)$.

Time-evolutions and couplings are the subjects of Sections 6 and 7. The appropriate graphical representation is established in Section 6, together with an account of the Markov processes on finite boxes whose level sets form stochastic random-cluster processes with different values of p. Certain monotonicities are established which enable the thermodynamic limit to be taken (in Section 7) at the level of *processes*, thereby yielding Markov processes on the infinite lattice with appropriate level-set properties. It is interesting that *two different* Markov semigroups turn out to be relevant for the evolution in time of random-cluster processes. As a consequence of this work, we obtain a heuristic explanation suitable for probabilists of the widely held belief that 'first-order phase transition occurs if and only if q is sufficiently large'. Certainly it is known that the percolation probability is discontinuous at the critical point if q is large ([43, 44, 45]), but it is an open problem to prove the existence of a critical value of q marking the onset of this discontinuity. The Markov processes of Section 7 have a structure which hints strongly at this belief, in that atoms in the marginals of the unique equilibrium measure of a certain process appear to increase as q increases. Of course, this phenomenon of discontinuity is fully understood for the *mean-field* random-cluster measures ([13]).

Other general accounts of the area have been published. Much of the basic methodology appeared first in the papers of Fortuin and Kasteleyn listed above. In addition, Aizenman et al. [1] have provided a useful modern account of some of this material; see also [29, 30].

2. Fundamental techniques, and notation

One of the most valuable properties of random-cluster measures $\phi_{G,p,q}$, defined in (1.1), is the FKG inequality, which is valid if and only if $q \geq 1$. There appears to have been no serious study of the case $0 < q < 1$, presumably because the FKG inequality does not hold in this regime; we include certain results about this case in Section 4, particularly in Theorem 4.5. Before stating the FKG inequality, we require some notation in addition to that given around (1.1).

There is a partial order on Ω_E given by: $\omega \leq \omega'$ if and only if $\omega(e) \leq \omega'(e)$ for all $e \in E$. A function $f : \Omega_E \to \mathbb{R}$ is called *increasing* if $f(\omega) \leq f(\omega')$ whenever $\omega \leq \omega'$, and is called *decreasing* if $-f$ is increasing. An event A ($\subseteq \Omega_E$) is called *increasing* (resp. *decreasing*) if its indicator function 1_A is increasing (resp. decreasing).

If ν is a probability measure and g is a random variable, we denote by $\nu(g)$ the expectation of g under ν. Further notation will be introduced as necessary.

Theorem 2.1 (FKG inequality). *Suppose that $q \geq 1$. If f and g are increasing functions on Ω_E, then*

$$\phi_{G,p,q}(fg) \geq \phi_{G,p,q}(f)\phi_{G,p,q}(g). \tag{2.1}$$

Replacing f and g by $-f$ and $-g$, we deduce that (2.1) holds for decreasing f and g. Specialising to indicator functions, we obtain that

$$\phi_{G,p,q}(A \cap B) \geq \phi_{G,p,q}(A)\phi_{G,p,q}(B) \quad \text{for increasing events } A, B, \tag{2.2}$$

whenever $q \geq 1$. It is easy to see, by example, that the FKG inequality is not generally valid when $0 < q < 1$.

A second valuable property of random-cluster measures is the pair of 'comparison inequalities', as follows. Given two probability mass functions μ_1 and μ_2 on Ω_E, we say that μ_2 *dominates* μ_1, and write $\mu_1 \leq \mu_2$, if

$$\mu_1(f) \leq \mu_2(f) \quad \text{for all increasing functions } f : \Omega_E \to \mathbb{R}. \tag{2.3}$$

Certain domination inequalities may be established, involving the measures $\phi_{G,p,q}$ for different values of the parameters p and q.

Theorem 2.2 (Comparison inequalities). *We have that*

$$\phi_{G,p',q'} \le \phi_{G,p,q} \qquad \text{if} \quad q' \ge q,\ q' \ge 1,\ p' \le p, \tag{2.4}$$

$$\phi_{G,p',q'} \ge \phi_{G,p,q} \qquad \text{if} \quad q' \ge q,\ q' \ge 1,\ \frac{p'}{q'(1-p')} \ge \frac{p}{q(1-p)}. \tag{2.5}$$

For proofs of the above inequalities, see [1, 30]. Comparison inequality (2.4) may be improved somewhat, using a technique developed in [2, 10, 51] to prove the strict inequality of critical points. More precisely, there exists a function γ such that

$$\phi_{G,p',q'} \le \phi_{G,p,q} \quad \text{if } q' \ge q \ge 1 \text{ and } p' \le p + \gamma(p,q,q'); \tag{2.6}$$

moreover $\gamma(p,q,q') > 0$ if $q' > q \ge 1$ and $0 < p < 1$. The function γ depends on G only through the maximum degree of its vertices. Inequality (2.6) is proved in [31], and applied there to obtain the forthcoming Theorem 5.1(c).

There is one further general property of random-cluster measures, namely the effect of conditioning on the absence or presence of some given edge. For $e \in E$, we denote by $G\backslash e$ (resp. $G.e$) the graph obtained from G by deleting (resp. contracting) e. We write $\Omega'_E = \{0,1\}^{E\backslash\{e\}}$; for $\omega \in \Omega_E$ we define $\omega' \in \Omega'_E$ by $\omega'(f) = \omega(f)$ for $f \ne e$. Recall that the event $\{e \text{ is open}\}$ is the set of configurations ω with $\omega(e) = 1$, and similarly for the event $\{e \text{ is closed}\}$; we write $J_e = \{e \text{ is open}\}$ and J_e^{c} for the complement of J_e.

Theorem 2.3. *We have that*

$$\phi_{G,p,q}\big(\omega \bigm| J_e^{\mathrm{c}}\big) = \phi_{G\backslash e,p,q}(\omega'), \quad \text{for } \omega \notin J_e, \tag{2.7}$$

$$\phi_{G,p,q}\big(\omega \bigm| J_e\big) = \phi_{G.e,p,q}(\omega'), \quad \text{for } \omega \in J_e. \tag{2.8}$$

That is to say, the effect of conditioning on the absence or presence of an edge e is to replace the measure $\phi_{G,p,q}$ by the random-cluster measure on the respective graph $G \setminus e$ or $G.e$. The proof is elementary and is omitted.

We turn now to the notation of this paper. The results which follow are valid for general lattices, but for the sake of definiteness we shall consider only the d-dimensional hypercubic lattice $\mathbb{L}$ having vertex set $\mathbb{Z}^d$ and edge set $\mathbb{E}$ containing all pairs of vertices which are euclidean distance 1 apart; we assume throughout that $d \ge 2$. We shall write $x = (x_1, x_2, \dots, x_d)$ for $x \in \mathbb{Z}^d$, and denote by $\langle x, y\rangle$ an edge joining vertices x and y. A path of $\mathbb{L}$ is an alternating sequence $x_0, e_0, x_1, e_1, \dots$ of distinct vertices x_i and edges e_j such that $e_j = \langle x_j, x_{j+1}\rangle$ for each j. If this path terminates at some x_n then it is said to join x_0 to x_n and to have length n; if a path has infinitely many vertices then it is said to connect x_0 to ∞.

The basic configuration space is $\Omega = \{0,1\}^{\mathbb{E}}$ endowed with the σ-field $\mathcal{F}$ generated by the finite-dimensional cylinders of Ω. In Sections 6 and 7 we shall study Markov processes on the larger state space $X = [0,1]^{\mathbb{E}}$, and particularly the level sets of such processes under the projection mappings $\pi^p, \pi_p : X \to \Omega$ given by

$$\pi^p\alpha(e) = \begin{cases} 1 & \text{if } 1-p \le \alpha(e), \\ 0 & \text{if } 1-p > \alpha(e), \end{cases} \qquad \pi_p\eta(e) = \begin{cases} 1 & \text{if } 1-p < \eta(e), \\ 0 & \text{if } 1-p \ge \eta(e), \end{cases} \qquad e \in \mathbb{E},$$

where $\alpha \in X$. The complement of an event A will be denoted by A^{c}.

A configuration ω ($\in \Omega$) is an assignment of 0 or 1 to each edge e ($\in \mathbb{E}$), and may be put into one–one correspondence with the set

$$\eta(\omega) = \{e \in \mathbb{E} : \omega(e) = 1\}$$

of 'open' edges in ω. The 'open paths' of a configuration ω are those paths of $\mathbb{L}$ all of whose edges are open. If A and B are sets of vertices, we write $\{A \leftrightarrow B\}$ for the event that there exists an open path joining some vertex of A to some vertex of B. Similarly we write $\{A \leftrightarrow \infty\}$ for the event that some vertex of A is the endpoint of an infinite open path. For any set S of edges (or vertices), we write $\{A \overset{S}{\leftrightarrow} B\}$ for the event that there exists an open path joining some vertex of A to some vertex of B and using only edges (or vertices) lying in S. The complements of such events are denoted using the symbol $\nleftrightarrow$.

For any subset E of $\mathbb{E}$, we write $\mathcal{F}_E$ for the σ-field of subsets of Ω generated by the finite-dimensional cylinders of E, so that $\mathcal{F} = \mathcal{F}_{\mathbb{E}}$. A *box* Λ is a subset of $\mathbb{Z}^d$ of the form

$$\Lambda = \prod_{i=1}^{d} [x_i, y_i]$$

for some $x, y \in \mathbb{Z}^d$, and where $[x_i, y_i]$ is interpreted as $[x_i, y_i] \cap \mathbb{Z}$. The box Λ generates a subgraph of $\mathbb{L}$ with vertex set Λ and edge set $\mathbb{E}_\Lambda$ containing all edges $\langle u, v\rangle$ with $u, v \in \Lambda$. We write $\mathcal{T}_\Lambda = \mathcal{F}_{\mathbb{E} \setminus \mathbb{E}_\Lambda}$, the 'external' σ-field of Λ, and

$$\mathcal{T} = \bigcap_{\Lambda} \mathcal{T}_\Lambda$$

for the tail σ-field. The *boundary* ∂V of a set V of vertices is the set of all vertices x ($\in V$) which are adjacent to some vertex of $\mathbb{L}$ not in V. The complement of V is denoted by V^{c}.

3. Random-cluster measures

As in the case of Gibbs states, there are two candidates for the definition of a random-cluster measure on the infinite lattice $\mathbb{L}$; the first is in terms of a 'specification', and the second is as a weak limit of measures defined on finite regions.

For $\xi \in \Omega$ ($= \{0, 1\}^{\mathbb{E}}$) and a box Λ, we write Ω_Λ^ξ for the (finite) subset of Ω containing all configurations ω satisfying $\omega(e) = \xi(e)$ for $e \notin \mathbb{E}_\Lambda$. For $\xi \in \Omega$ and values of p, q satisfying $0 \le p \le 1$, $q > 0$, we define $\phi_{\Lambda,p,q}^\xi$ to be the random-cluster measure on the finite graph $(\Lambda, \mathbb{E}_\Lambda)$ 'with boundary condition ξ'; this is the equivalent of a 'specification' for Gibbs states. More precisely, let $\phi_{\Lambda,p,q}^\xi$ be the probability measure on $(\Omega, \mathcal{F})$ satisfying

$$\phi_{\Lambda,p,q}^\xi(\omega) = \frac{1}{Z_{\Lambda,p,q}^\xi} \Big\{ \prod_{e \in \mathbb{E}_\Lambda} p^{\omega(e)} (1-p)^{1-\omega(e)} \Big\} q^{k(\omega,\Lambda)} \quad \text{for } \omega \in \Omega_\Lambda^\xi, \tag{3.1}$$

where $k(\omega, \Lambda)$ is the number of components of the graph $(\mathbb{Z}^d, \eta(\omega))$ which intersect Λ, and where $Z^{\xi}_{\Lambda,p,q}$ is the appropriate normalising constant

$$Z^{\xi}_{\Lambda,p,q} = \sum_{\omega \in \Omega^{\xi}_{\Lambda}} \left\{ \prod_{e \in \mathbb{E}_{\Lambda}} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)}. \tag{3.2}$$

Note that $\phi^{\xi}_{\Lambda,p,q}(\Omega^{\xi}_{\Lambda}) = 1$. There follows the definition of a random-cluster measure, based upon the usual Dobrushin–Lanford–Ruelle (DLR) definition of a Gibbs state ([16, 46]). After this is the definition of a weak limit.

Definition 3.1. A probability measure ϕ on $(\Omega, \mathcal{F})$ is called a *random-cluster measure* with parameters p and q if

$$\phi(A \mid \mathcal{T}_{\Lambda}) = \phi^{\cdot}_{\Lambda,p,q}(A) \quad \phi\text{-a.s., for all } A \in \mathcal{F} \text{ and boxes } \Lambda. \tag{3.3}$$

The set of such measures is denoted by $\mathcal{R}_{p,q}$.

Definition 3.2. A probability measure ϕ on $(\Omega, \mathcal{F})$ is called a *limit random-cluster measure* with parameters p and q if there exists $\xi \in \Omega$ and an increasing sequence $(\Lambda_n : n \geq 1)$ of boxes, satisfying $\Lambda_n \to \mathbb{Z}^d$ as $n \to \infty$, such that

$$\phi^{\xi}_{\Lambda_n,p,q} \Rightarrow \phi \quad \text{as } n \to \infty \tag{3.4}$$

where '$\Rightarrow$' denotes weak convergence. The set of all such measures is denoted by $\mathcal{W}_{p,q}$, and the closed convex hull of $\mathcal{W}_{p,q}$ by $\overline{\mathrm{co}\,\mathcal{W}_{p,q}}$.

No extra generality is obtained by allowing a sequence (ξ_n) of configurations in such a way that

$$\phi^{\xi_n}_{\Lambda_n,p,q} \Rightarrow \phi$$

in place of (3.4) in the latter definition. This is so since, for any ξ $(\in \Omega)$ and any box Λ, there exists a configuration ψ $(\in \Omega)$ and a box Δ containing Λ such that $\phi^{\xi}_{\Lambda,p,q}$ and $\phi^{\psi'}_{\Lambda,p,q}$ induce the same measure on Λ, for all configurations ψ' which agree with ψ on $\mathbb{E}_{\Delta}$. It follows that, if $\phi^{\xi_n}_{\Lambda_n,p,q} \Rightarrow \phi$, then there exists ξ $(\in \Omega)$ and a subsequence $(\Lambda_{n_k} : k \geq 1)$ of $(\Lambda_n : n \geq 1)$ such that $\phi^{\xi}_{\Lambda_{n_k},p,q} \Rightarrow \phi$ as $k \to \infty$.

We note that $\mathcal{W}_{p,q} \neq \varnothing$ for all $0 \leq p \leq 1$, $q > 0$, by the usual compactness argument.

It is well known that limit random-cluster measures for integral q (≥ 2) may be constructed from Gibbs measures with Potts interactions (having q spin-values available at each vertex), but it is important to note that Definition 3.2 does not cover every such possibility. For example, consider the Ising measure on the box Λ, with plus boundary conditions on the upper half U and minus boundary conditions on the lower half L. The corresponding random-cluster measure on Λ is the measure $\phi^{1}_{\Lambda,p,q}$ (where $p = 1 - e^{-\beta J}$), having boundary condition $\xi \equiv 1$, *conditioned on the event* that there is no open path from U to L. This last event may be thought of as 'negative information', and such events play no part in Definition 3.2. Thus Definition 3.2 excludes certain possibilities which are relevant to, for example, the construction of non-translation-invariant Gibbs states (see [1, 8, 17, 23, 52] for related work).

We write 0 (resp. 1) for the configuration in Ω which takes the value 0 (resp. 1) on every edge.

Theorem 3.1. *Suppose* $0 \le p \le 1$ *and* $q \ge 1$.
(a) *The weak limits*

$$\phi_{p,q}^b = \lim_{\Lambda \to \mathbb{Z}^d} \phi_{\Lambda,p,q}^b, \quad \textit{for } b = 0, 1, \tag{3.5}$$

exist and are translation-invariant.
(b) *We have that* $\phi_{p,q}^0, \phi_{p,q}^1 \in \mathcal{R}_{p,q}$, *and furthermore*

$$\phi_{p,q}^0 \le \phi \le \phi_{p,q}^1 \quad \textit{for all } \phi \in \mathcal{R}_{p,q} \cup \mathcal{W}_{p,q}. \tag{3.6}$$

(c) *The probability measures* $\phi_{p,q}^0$ *and* $\phi_{p,q}^1$ *are ergodic.*

We interpret the limit in (3.5) as being along any increasing sequence of boxes Λ with limit $\mathbb{Z}^d$. The stochastic inequalities of (3.6) are to be interpreted in the usual way; see (2.3). Part (a) of this theorem is well known (see [1, 30]).

Theorem 3.1 implies that $\mathcal{R}_{p,q}$ is non-empty when $q \ge 1$, and also the important and useful fact that

$$|\mathcal{R}_{p,q}| = |\mathcal{W}_{p,q}| = 1 \quad \text{if and only if} \quad \phi_{p,q}^0 = \phi_{p,q}^1. \tag{3.7}$$

Later we shall state conditions under which $\phi_{p,q}^0 = \phi_{p,q}^1$, thereby obtaining sufficient conditions for the uniqueness of random-cluster measures. Further properties of $\mathcal{R}_{p,q}$ and $\mathcal{W}_{p,q}$ are as follows.

Theorem 3.2. *Suppose that* $0 \le p \le 1$ *and* $q > 0$.
(a) $\mathcal{R}_{p,q}$ *is non-empty and convex, and contains at least one translation-invariant probability measure.*
(b) *All extremal members of* $\mathcal{R}_{p,q}$ *are trivial on the tail* σ*-field* $\mathcal{T}$ *and lie in* $\mathcal{W}_{p,q}$.
(c) *All translation-invariant members of* $\mathcal{W}_{p,q}$ *lie in* $\mathcal{R}_{p,q}$.
(d) *If* $q \ge 1$, *then* $\phi_{p,q}^0$ *and* $\phi_{p,q}^1$ *are extremal elements of* $\mathcal{R}_{p,q}$.

In proving Theorems 3.1(b, c) and 3.2 we shall make use of the following result concerning the uniqueness of the infinite cluster. For $\omega \in \Omega$, let $I = I(\omega)$ be the number of infinite components of the graph $(\mathbb{Z}^d, \eta(\omega))$, and let J_e be the event $\{\omega(e) = 1\}$.

Theorem 3.3. *Let* $\phi \in \overline{\mathrm{co}\,\mathcal{W}_{p,q}}$, *where* $0 \le p \le 1$ *and* $q > 0$.
(a) *If* $0 < p < 1$, *then* ϕ *has the 'finite-energy property', which is to say that*

$$0 < \phi\big(J_e \,\big|\, \mathcal{F}_{\mathbb{E}\setminus\{e\}}\big) < 1 \quad \phi\text{-a.s., } \textit{for all } e \in \mathbb{E}. \tag{3.8}$$

(b) *If* ϕ *is translation-invariant, then* $\phi(I \in \{0, 1\}) = 1$.
(c) *If* ϕ *is ergodic, then*

$$\textit{either } \phi(I = 0) = 1 \textit{ or } \phi(I = 1) = 1. \tag{3.9}$$

Theorem 3.3(a, b) will be used directly in the proof that translation-invariant weak limits are indeed random-cluster measures (part (c) of Theorem 3.2). In the present context, the uniqueness of the infinite cluster takes the role played by 'quasilocality' for Gibbs states (see [24]); however, we note that this uniqueness is a property of measures, whereas quasilocality is a property of specifications. Our proof of Theorem 3.2 constitutes an essential application of the Burton–Keane uniqueness theorem ([14]), and leads to hitherto unknown conclusions (cf. [50]).

We begin the proofs with that of Theorem 3.3.

Proof of Theorem 3.3. Parts (b) and (c) are obvious if $p = 0, 1$, and so we assume that $0 < p < 1$. It is a consequence of the Burton–Keane theorem [14] that (a) implies (b) and (c), and so we need only prove part (a). For related literature on the finite-energy property, see [14, 23, 53].

The basic fact we shall use is the following. Let ϕ_G be the random-cluster measure with parameters p and q on a finite graph $G = (V, E)$; see (1.1). Then, for any edge e and configuration ζ,

$$\phi_G\big(J_e \,\big|\, \omega(f) = \zeta(f) \text{ for } f \neq e\big) = \begin{cases} p & \text{if } \zeta \notin D \\ \dfrac{p}{p + (1-p)q} & \text{if } \zeta \in D, \end{cases} \tag{3.10}$$

where D is the event that there exists no open path of $E \setminus \{e\}$ joining the endpoints of e. This fact is easily checked by reference to the definition (1.1) of random-cluster measures (see also [1, 30]). Define the constants α, β by

$$\alpha = \min\left\{p, \frac{p}{p + (1-p)q}\right\}, \quad \beta = \max\left\{p, \frac{p}{p + (1-p)q}\right\}$$

so that $0 < \alpha \le \beta < 1$.

Suppose first that $\phi \in \mathcal{W}_{p,q}$. As in (3.4), let ξ ($\in \Omega$) and $(\Lambda_n : n \ge 1)$ be such that

$$\phi = \lim_{n\to\infty} \phi^{\xi}_{\Lambda_n,p,q}. \tag{3.11}$$

For any finite set F of edges of $\mathbb{L}$, and any $\zeta \in \Omega$, we write $[\zeta]_F$ for the cylinder event $\{\omega \in \Omega : \omega(f) = \zeta(f) \text{ for } f \in F\}$. By the martingale convergence theorem (or otherwise),

$$\phi\big(J_e \,\big|\, [\zeta]_{\mathbb{E}\setminus\{e\}}\big) = \lim_{\Lambda\to\mathbb{Z}^d} \phi\big(J_e \,\big|\, [\zeta]_{\mathbb{E}_\Lambda\setminus\{e\}}\big) \quad \text{for } \phi\text{-a.e. } \zeta. \tag{3.12}$$

Also, by (3.11), if $e \in \mathbb{E}_\Lambda$,

$$\phi\big(J_e \,\big|\, [\zeta]_{\mathbb{E}_\Lambda\setminus\{e\}}\big) = \lim_{n\to\infty} \phi^{\xi}_{\Lambda_n,p,q}\big(J_e \,\big|\, [\zeta]_{\mathbb{E}_\Lambda\setminus\{e\}}\big). \tag{3.13}$$

We have from (3.10) and Theorem 2.3 that

$$\alpha \le \phi^{\xi}_{\Lambda_n,p,q}\big(J_e \,\big|\, [\zeta]_{\mathbb{E}_\Lambda\setminus\{e\}}\big) \le \beta \quad \text{for all large } n \tag{3.14}$$

and therefore

$$\alpha \le \phi\big(J_e \mid [\zeta]_{\mathbb{E}\setminus\{e\}}\big) \le \beta \quad \text{for } \phi\text{-a.e. } \zeta$$

by (3.12). Therefore ϕ satisfies (3.8).

Assume next that

$$\phi = \sum_{i=1}^{m} \gamma_i \phi_i,$$

for positive reals γ_i having sum 1, and measures $\phi_i \in \mathcal{W}_{p,q}$. The measures ϕ_i satisfy (3.13) and (3.14) (for suitable $\xi = \xi_i$ and $\Lambda_n = \Lambda_{n,i}$), whence

$$\phi\big(J_e \,\big|\, [\zeta]_{\mathbb{E}_\Lambda \setminus \{e\}}\big) = \frac{\sum_i \gamma_i \phi_i\big(J_e \cap [\zeta]_{\mathbb{E}_\Lambda \setminus \{e\}}\big)}{\sum_i \gamma_i \phi_i\big([\zeta]_{\mathbb{E}_\Lambda \setminus \{e\}}\big)} \in [\alpha, \beta]. \tag{3.15}$$

Take the limit as $\Lambda \to \mathbb{Z}^d$ to obtain (3.8).

Finally, suppose that $\phi = \lim_{n\to\infty} \phi_n$ for measures ϕ_n lying in the convex hull of $\mathcal{W}_{p,q}$. Then

$$\phi\big(J_e \,\big|\, [\zeta]_{\mathbb{E}_\Lambda \setminus \{e\}}\big) = \lim_{n\to\infty} \phi_n\big(J_e \,\big|\, [\zeta]_{\mathbb{E}_\Lambda \setminus \{e\}}\big),$$

which lies in the interval $[\alpha, \beta]$, by (3.15). Pass to the limit as $\Lambda \to \mathbb{Z}^d$ to obtain (3.8) as before. □

Proof of Theorem 3.1. We may assume that $0 < p < 1$ since the result is elementary otherwise.

(a) This is well known, but we include a sketch proof for the sake of completeness. Let Λ and Δ be two boxes satisfying $\Lambda \subseteq \Delta$, and let A be the event that all edges in $\mathbb{E}_\Delta \setminus \mathbb{E}_\Lambda$ have state 0. Now $\phi^0_{\Lambda,p,q}$ may be thought of as the measure $\phi^0_{\Delta,p,q}$ *conditioned* on the event A (by repeated application of Theorem 2.3). Since A is a decreasing event, we have by the FKG inequality (see Theorem 2.1) that

$$\phi^0_{\Lambda,p,q}(B) = \phi^0_{\Delta,p,q}(B \mid A) \le \phi^0_{\Delta,p,q}(B) \tag{3.16}$$

for any increasing event B defined in terms of the edges in $\mathbb{E}_\Lambda$. It follows that the limit

$$\phi^0_{p,q}(B) = \lim_{\Lambda\to\mathbb{Z}^d} \phi^0_{\Lambda,p,q}(B)$$

exists for all increasing finite-dimensional cylinder events B. The collection of all such events B generates $\mathcal{F}$, whence $\phi^0_{p,q}$ exists.

To see that $\phi^0_{p,q}$ is translation-invariant, one argues as follows. Let B be an increasing event lying in $\mathcal{F}_F$ for some finite subset F of $\mathbb{E}$. Let τ be a translation of the lattice $\mathbb{L}$, and extend τ to be a shift $\tau : \Omega \to \Omega$ by $\tau\omega(e) = \omega(\tau e)$ for $e \in \mathbb{E}$. For any box Λ containing all endpoints of all edges in F, we have by the FKG inequality as in (3.16) that

$$\phi^0_{p,q}(B) \ge \phi^0_{\Lambda,p,q}(B) = \phi^0_{\tau\Lambda,p,q}(\tau B) \to \phi^0_{p,q}(\tau B) \quad \text{as } \Lambda \to \mathbb{Z}^d.$$

Applying the same argument with τ replaced by τ^{-1}, we find that $\phi^0_{p,q}(B) = \phi^0_{p,q}(\tau B)$.

Similar arguments are valid for $\phi^1_{p,q}$.

(b) Let Λ be a finite box, and let A be a cylinder event defined in terms of the states of edges in $\mathbb{E}_\Lambda$. We use a subsidiary lemma which will be of value later also.

Lemma 3.4. *Let $0 < p < 1$, $q > 0$, and let ϕ be a translation-invariant member of $\overline{\mathrm{co}\,\mathcal{W}_{p,q}}$. The random variable $g(\omega) = \phi^{\omega}_{\Lambda,p,q}(A)$ is ϕ-a.s. continuous, using the product topology on its domain Ω.*

Before proving this, we use it to establish that $\phi^b_{p,q} \in \mathcal{R}_{p,q}$ for $b = 0, 1$, as asserted in the theorem. Let $b \in \{0,1\}$, and let Δ be a box containing Λ. By the conditional-expectation property of random-cluster measures (Theorem 2.3),

$$\phi^{\cdot}_{\Lambda,p,q}(A) = \phi^b_{\Delta,p,q}(A \mid \mathcal{T}_\Lambda) \quad \phi^b_{\Delta,p,q}\text{-a.s.} \tag{3.17}$$

Let B be a cylinder event in $\mathcal{T}_\Lambda$. By part (a), and Lemma 3.4 applied to $\phi^b_{p,q}$, the function $1_B(\omega)\phi^{\omega}_{\Lambda,p,q}(A)$ is $\phi^b_{p,q}$-a.s. continuous (1_B is the indicator function of B), whence

$$\begin{aligned}
\phi^b_{p,q}\big(1_B(\cdot)\phi^{\cdot}_{\Lambda,p,q}(A)\big) &= \lim_{\Delta\to\mathbb{Z}^d} \phi^b_{\Delta,p,q}\big(1_B(\cdot)\phi^{\cdot}_{\Lambda,p,q}(A)\big) \\
&= \lim_{\Delta\to\mathbb{Z}^d} \phi^b_{\Delta,p,q}\big(1_B(\cdot)\phi^b_{\Delta,p,q}(A \mid \mathcal{T}_\Lambda)\big) \qquad \text{by (3.17)} \\
&= \lim_{\Delta\to\mathbb{Z}^d} \phi^b_{\Delta,p,q}(A \cap B) = \phi^b_{p,q}(A\cap B).
\end{aligned}$$

Since $\mathcal{T}_\Lambda$ is generated by the collection of all such B, we deduce that

$$\phi^{\cdot}_{\Lambda,p,q}(A) = \phi^b_{p,q}(A \mid \mathcal{T}_\Lambda) \qquad \phi^b_{p,q}\text{-a.s.,} \tag{3.18}$$

whence $\phi^b_{p,q} \in \mathcal{R}_{p,q}$ as required.

Turning to inequality (3.6), we note that, by thoughtful application of the FKG inequality,

$$\phi^0_{\Lambda,p,q}(A) \le \phi^{\omega}_{\Lambda,p,q}(A) \le \phi^1_{\Lambda,p,q}(A) \quad \text{for all } \omega \in \Omega,$$

and for all increasing A defined in terms of the states of $\mathbb{E}_\Lambda$. Using (3.4), this implies (3.6) for $\phi \in \mathcal{W}_{p,q}$. For $\phi \in \mathcal{R}_{p,q}$, use (3.3), take expectations, and let $\Lambda \to \mathbb{Z}^d$.

We complete the proof of part (b) by proving Lemma 3.4. Let ϕ be a translation-invariant member of $\overline{\mathrm{co}\,\mathcal{W}_{p,q}}$, and note from Theorem 3.3 that the number I of infinite clusters satisfies

$$\phi\big(I \in \{0,1\}\big) = 1. \tag{3.19}$$

Define the 'discontinuity set' D of the random variable $g(\omega) = \phi^{\omega}_{\Lambda,p,q}(A)$ by

$$D = \bigcap_{\Delta} \left\{ \omega : \sup_{\zeta:\zeta=\omega \text{ on } \Delta} |g(\zeta) - g(\omega)| > 0 \right\}$$

where the intersection is over all boxes Δ containing Λ, and we write '$\zeta = \omega$ on Δ' if $\zeta(e) = \omega(e)$ for all $e \in \mathbb{E}_\Delta$. For any such ζ, the difference $|g(\zeta) - g(\omega)|$ can be non-zero only if there exist two points $u, v \in \partial\Lambda$ such that both u and v are joined to $\partial\Delta$ by paths using open edges of ω lying in $\mathbb{E}_\Delta \backslash \mathbb{E}_\Lambda$, but that u is not joined to v by such a path (note that, if this event occurs for no such u, v, then $k(\omega', \Lambda) = k(\omega, \Lambda)$

for all ω' which agree with ω on $\mathbb{E}_\Delta$, so that $g(\zeta) = g(\omega)$). Denoting the last event by $D_{\Lambda,\Delta}$, we have that

$$D \subseteq \bigcap_{\Delta} D_{\Lambda,\Delta}.$$

Therefore

$$\phi(D) \le \phi\Big(\bigcap_{\Delta} D_{\Lambda,\Delta}\Big).$$

However,

$$\bigcap_{\Delta} D_{\Lambda,\Delta} \subseteq \Big\{\Lambda^{\mathrm{c}} \text{ contains two or more infinite open clusters}\Big\},$$

an event with zero probability by (3.19). This completes the proof of the lemma, since D contains all configurations ω at which g is discontinuous.

(c) Inequality (3.6) implies that $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are extremal random-cluster measures in the sense that, for $b = 0, 1$, there exists no $\alpha \in (0, 1)$ such that

$$\phi^b_{p,q} = \alpha\phi' + (1 - \alpha)\phi''$$

for some distinct $\phi', \phi'' \in \mathcal{R}_{p,q}$. It follows by [24, Thm. 7.7 and Remark 7.13] that $\phi^b_{p,q}$ is trivial on the tail σ-field $\mathcal{T}$ and hence ergodic, for $b = 0, 1$. □

Proof of Theorem 3.2. (a) The convexity of $\mathcal{R}_{p,q}$ follows from Definition 3.1 as for Gibbs states. That $\mathcal{R}_{p,q} \neq \varnothing$ follows from Theorem 3.1(b) when $q \ge 1$, but a different argument is needed when $q < 1$. Assume $q < 1$, and note that $\mathcal{W}_{p,q} \neq \varnothing$, by compactness. Let $\phi \in \mathcal{W}_{p,q}$, and let

$$\psi_m = \frac{1}{|\Delta_m|} \sum_{x \in \Delta_m} \tau_x \circ \phi$$

where $\Delta_m = [-m, m]^d$, and $\tau_x \circ \phi$ is the probability measure on $(\Omega, \mathcal{F})$ given by $\tau_x \circ \phi(A) = \phi(\tau_x A)$ for the shift $\tau_x(y) = x + y$ of the lattice. Clearly $\tau_x \circ \phi \in \mathcal{W}_{p,q}$ for all x, whence ψ_m belongs to the convex hull of $\mathcal{W}_{p,q}$. Let ψ be a limit point of the family $\{\psi_m : m \ge 1\}$ of measures. Certainly ψ is translation-invariant and lies in $\overline{\operatorname{co} \mathcal{W}_{p,q}}$, whence we may apply Lemma 3.4 to ψ.

We claim that $\psi \in \mathcal{R}_{p,q}$, and shall prove this in the same general way as we proved (3.18). Pick $\xi \in \Omega$ and a sequence Λ_n of boxes such that (3.11) holds. Let Λ be a box, let B be a cylinder event in $\mathcal{T}_\Lambda$, and let A be an event defined in terms of the edges in $\mathbb{E}_\Lambda$. Then, using Lemma 3.4 for the first step,

$$\begin{aligned}
\psi\big(1_B(\cdot)\phi^{\cdot}_{\Lambda,p,q}(A)\big) &= \lim_{m\to\infty} \psi_m\big(1_B(\cdot)\phi^{\cdot}_{\Lambda,p,q}(A)\big)\\
&= \lim_{m\to\infty} \frac{1}{|\Delta_m|}\sum_{x\in\Delta_m} \tau_x\circ\phi\big(1_B(\cdot)\phi^{\cdot}_{\Lambda,p,q}(A)\big)\\
&= \lim_{m\to\infty}\lim_{n\to\infty} \frac{1}{|\Delta_m|}\sum_{x\in\Delta_m} \tau_x\circ\phi^{\xi}_{\Lambda_n,p,q}\big(1_B(\cdot)\phi^{\cdot}_{\Lambda,p,q}(A)\big)\\
&= \lim_{m\to\infty}\lim_{n\to\infty} \frac{1}{|\Delta_m|}\sum_{x\in\Delta_m} \tau_x\circ\phi^{\xi}_{\Lambda_n,p,q}(A\cap B)\\
&= \lim_{m\to\infty} \frac{1}{|\Delta_m|}\sum_{x\in\Delta_m} \tau_x\circ\phi(A\cap B)\\
&= \psi(A\cap B),
\end{aligned}$$

whence (3.18) holds as before with $\phi^b_{p,q}$ replaced by ψ.
(b) The $\mathcal{T}$-triviality of extremal elements of $\mathcal{R}_{p,q}$ is a consequence of a general result [24, Thm. 7.7 and Remark 7.13]. That extremal elements of $\mathcal{R}_{p,q}$ lie in $\mathcal{W}_{p,q}$ is contained in part (b) of [24, Thm. 7.12].
(c) Let ϕ be a translation-invariant measure in $\mathcal{W}_{p,q}$. By Theorem 3.3, the number I of infinite open clusters satisfies $\phi(I\in\{0,1\})=1$. The proof of Theorem 3.1(b) may now be followed to obtain the claim.
(d) This was proved for Theorem 3.1. □

4. Uniqueness of random-cluster measures

In this section we address the question of the uniqueness (or not) of random-cluster measures for given values of p and q. To this end we introduce the notion of 'pressure'. Let $0<p<1$, $q>0$, $\xi\in\Omega$, and define the (finite box) partition functions $Z^{\xi}_{\Lambda,p,q}$ by (3.2). Rather than working with $Z^{\xi}_{\Lambda,p,q}$ itself, we work instead with

$$Y^{\xi}_{\Lambda,p,q} = (1-p)^{-|\mathbb{E}_\Lambda|} Z^{\xi}_{\Lambda,p,q} = \sum_{\omega\in\Omega^{\xi}_{\Lambda}} q^{k(\omega,\Lambda)}\exp\big\{\pi|\eta(\omega)\cap\mathbb{E}_\Lambda|\big\} \tag{4.1}$$

where $\pi=\log\{p/(1-p)\}$, and $\eta(\omega)$ is the set of open edges of ω as usual. The *pressure* $f(p,q)$ is defined in the following theorem.

Theorem 4.1. *Let $q>0$. The limits*

$$\lim_{\Lambda\to\mathbb{Z}^d}\left\{\frac{1}{|\mathbb{E}_\Lambda|}\log Y^{\xi}_{\Lambda,p,q}\right\} = f(p,q), \quad 0<p<1, \tag{4.2}$$

exist and are independent of ξ. Furthermore $f(p,q)$ is a convex function of $\pi=\log\{p/(1-p)\}$ for $\pi\in\mathbb{R}$, and therefore f is differentiable with respect to p except on some countable set $\mathcal{D}_q$ $(\subseteq(0,1))$.

As a consequence of this, one obtains a partial conclusion concerning the uniqueness of random-cluster measures when $q \geq 1$. We denote by $h^b(p,q)$ the edge-density under the measure $\phi^b_{p,q}$, that is

$$h^b(p,q) = \phi^b_{p,q}(\omega(e)=1), \quad b = 0,1, \tag{4.3}$$

for $q \geq 1$, and we note that $h^b(p,q)$ does not depend on the choice of e, by the translation-invariance of $\phi^b_{p,q}$.

Theorem 4.2. *Suppose that $0 < p < 1$ and $q \geq 1$. The following four statements are equivalent.*

(a) *The pressure $f(x,q)$ is differentiable with respect to x at the point $x = p$.*
(b) *The edge-density $h^b(x,q)$ is continuous at the point $x = p$, for $b = 0,1$.*
(c) *It is the case that $h^0(p,q) = h^1(p,q)$.*
(d) *There is a unique random-cluster measure with parameters p and q, i.e., $|\mathcal{R}_{p,q}| = 1$.*

Invoking Theorem 4.1, we deduce that (a)–(d) hold if and only if $x \notin \mathcal{D}_q$. Note that $h^b(x,q)$ is monotonic non-decreasing in x when $q \geq 1$ (see Proposition 4.4); the difference $h^1(p,q) - h^0(p,q)$ appears in Proposition 7.4 as the atom at the point $1-p$ of a certain probability measure on the interval $[0,1]$. The argument using convexity which leads to Theorem 4.2 has been pursued by others for Ising and other physical models; see [55] for recent results.

There is incomplete information about the countable set $\mathcal{D}_q$ of points of non-differentiability of the pressure $f(\cdot,q)$. It is thought to be the case that $\mathcal{D}_q$ is empty for small values of q (satisfying $q \geq 1$), and is a singleton point (i.e., the critical value of p, see Section 5) when q is large. Proofs of parts of this statement have been given in special cases ([36, 43, 44, 45, 49]), particularly for $d = 2$ and $q \geq 4$, and for $d \geq 2$ and sufficiently large q. We conjecture that there exists $Q = Q(\mathbb{L})$ such that

$$\mathcal{D}_q = \begin{cases} \varnothing & \text{if } q < Q, \\ \{p_c(q)\} & \text{if } q > Q. \end{cases}$$

This would imply in particular that $|\mathcal{R}_{p,q}| = 1$ unless $q \geq Q$ and $p = p_c(q)$. In those situations when $|\mathcal{R}_{p,q}| \neq 1$, we ask whether or not $\mathcal{R}_{p,q}$ is the set of convex combinations of $\phi^0_{p,q}$ and $\phi^1_{p,q}$. A weaker form of this conjecture is that, except possibly at a point of first-order transition, all random-cluster measures are translation-invariant; such a conjecture of translation-invariance may be made also about limit random-cluster measures.

Using a general conclusion of [1, p. 37], we may obtain a fairly complete picture when $d = 2$, which we summarise as follows (the proof is deferred to the end of Section 5).

Theorem 4.3. *Suppose that $d = 2$, and that $0 \leq p \leq 1$ and $q \geq 1$. Then*

$$|\mathcal{R}_{p,q}| = 1 \quad \text{if} \quad p \neq \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{4.4}$$

In the next section we discuss the phase transition for random-cluster models, and we shall recall the conjecture that $\kappa_q = \sqrt{q}/(1+\sqrt{q})$ is the critical value of

p in two dimensions (this value is the fixed point of a certain mapping involving graphical duality). The results of [43, 45] imply that

$$|\mathcal{R}_{\kappa_q,q}| > 1 \quad \text{if} \quad q > Q \tag{4.5}$$

in two dimensions, for some large Q. It is believed that

$$|\mathcal{R}_{\kappa_q,q}| \begin{cases} = 1 & \text{if } 1 \le q < 4 \\ > 1 & \text{if } q > 4; \end{cases} \tag{4.6}$$

see [36, 44].

Before proving the above results, we make two further remarks. The first concerns properties of $\phi^b_{p,q}$ for $b = 0, 1$ and $q \ge 1$.

Proposition 4.4. *Let $0 \le p \le 1$, $q \ge 1$, and $d \ge 2$.*

(a) *$\phi^b_{p,q}(A)$ is a non-decreasing function of p, for $b = 0, 1$, and for all increasing events A.*
(b) *$\phi^1_{p,q}(A)$ is a right-continuous function of p, for all increasing events A which are closed (in the product topology).*
(c) *$\phi^0_{p,q}(A)$ is a left-continuous function of p, for all increasing finite-dimensional events A.*

Part (b) refers to increasing *closed* events A, of which an important example is the event $A = \{0 \leftrightarrow \infty\}$. In order to see that A is closed, we argue as follows. If $\omega \in A^c$, then $\omega \in \{0 \not\leftrightarrow \partial\Lambda\}$ for some Λ, implying that $\omega' \in \{0 \not\leftrightarrow \partial\Lambda\}$ for all ω' which agree with ω on Λ. Therefore A^c is open.

The next remark of this section is interesting in that it is valid *for all* values of q, rather than for $q \ge 1$ only. It is proved by using the convexity of the pressure for all $q > 0$. Let $\mathcal{TR}_{p,q}$ denote the set of all translation-invariant members of $\mathcal{R}_{p,q}$, and recall from Theorem 3.2(a) that $\mathcal{TR}_{p,q} \neq \varnothing$.

Theorem 4.5. *Let $0 < p < 1$ and $q > 0$, and let $\mathcal{D}_q$ be given as in Theorem 4.1.*

(a) *The edge-density $\phi(\omega(e) = 1)$ is constant for all $e \in \mathbb{E}$ and all $\phi \in \mathcal{TR}_{p,q}$, if $p \notin \mathcal{D}_q$.*
(b) *If $0 < p < p' < 1$ and $\phi \in \mathcal{TR}_{p,q}$, $\phi' \in \mathcal{TR}_{p',q}$, then the respective edge-densities satisfy*

$$\phi(\omega(e) = 1) \le \phi'(\omega(e) = 1).$$

To place this in context, we recall that random-cluster measures satisfy the FKG inequality if $q \ge 1$, and not if $q < 1$ (see [1, 30] and Theorem 2.1). Even when the FKG inequality is invalid (i.e., $q < 1$), part (b) implies that the edge-density $\phi_p(\omega(e) = 1)$ is non-decreasing in p, where ϕ_p is an arbitrary member of $\mathcal{TR}_{p,q}$ for each p. It is not generally the case that $\phi_p(A)$ is non-decreasing in p for increasing events A having more complicated structures.

Proof of Proposition 4.4. (a) If A is finite-dimensional, this follows from the comparison inequalities; see Theorem 2.2. For general A, use Theorem 7.3 (or otherwise).
(b) For $\omega \in \Omega$ and the box $\Lambda_m = [-m, m]^d$, we write $(\omega, 1)_m$ for the configuration which agrees with ω on $\mathbb{E}_{\Lambda_m}$ and equals 1 elsewhere. Let A be an increasing closed

event, and let $A_m = \{\omega \in \Omega : (\omega, 1)_m \in A\}$. Clearly $A_m \supseteq A_n$ if $m \le n$, whence the limit

$$B = \lim_{n\to\infty} A_n = \bigcap_n A_n$$

exists. Furthermore $A \subseteq A_m$ for all m, so that $A \subseteq B$. If $\omega \in A_m$ for all m, then ω may be expressed as the (product topology) limit $\omega = \lim_{m\to\infty}(\omega, 1)_m$ of configurations in A; since A is closed, it follows that $\omega \in A$. We have proved that $A = B$.

Let $m \le n$. Using stochastic orderings of measures, we find that

$$\begin{aligned} \phi^1_{p,q}(A) \le \phi^1_{n,p,q}(A) \le \phi^1_{n,p,q}(A_m) &\quad \text{since } A \subseteq A_m \\ \to \phi^1_{p,q}(A_m) \qquad\qquad &\quad \text{as } n \to \infty \\ \to \phi^1_{p,q}(A) \qquad\qquad\ &\quad \text{as } m \to \infty, \end{aligned}$$

where $\phi^1_{n,p,q} = \phi^1_{\Lambda_n,p,q}$. Also,

$$\begin{aligned} \phi^1_{n,p,q}(A_n) &\ge \phi^1_{n+1,p,q}(A_n) &&\text{since } \Lambda_n \subseteq \Lambda_{n+1} \\ &\ge \phi^1_{n+1,p,q}(A_{n+1}) &&\text{since } A_n \supseteq A_{n+1}. \end{aligned}$$

The two sets of inequalities above imply that the sequence $(\phi^1_{n,p,q}(A_n) : n \ge 1)$ is decreasing with limit $\phi^1_{p,q}(A)$. However each $\phi^1_{n,p,q}(A_n)$ is a continuous function of p, whence $\phi^1_{p,q}(A)$ is upper semicontinuous, and hence right-continuous.
(c) If A is an increasing *cylinder* event, then $\phi^0_{\Lambda,p,q}(A)$ is (ultimately) non-decreasing as $\Lambda \to \mathbb{Z}^d$, whence the limit $\phi^0_{p,q}(A)$ is lower semicontinuous, and therefore left-continuous. □

Proof of Theorem 4.1. In the proofs of this and Theorem 4.2, we use a standard argument of statistical mechanics in a form related to that used in [47]. Fix the box Λ. For $\omega, \xi \in \Omega$ we define ω^ξ by

$$\omega^\xi(e) = \begin{cases} \omega(e) & \text{if } e \in \mathbb{E}_\Lambda \\ \xi(e) & \text{otherwise,} \end{cases}$$

and note that $\omega^\xi \in \Omega^\xi_\Lambda$. Clearly

$$k(\omega^1, \Lambda) \le k(\omega^\xi, \Lambda) \le k(\omega^0, \Lambda) \le k(\omega^1, \Lambda) + |\partial\Lambda|,$$

whence

$$Y^1_\Lambda \le Y^\xi_\Lambda \le Y^0_\Lambda \le Y^1_\Lambda\, q^{|\partial\Lambda|} \quad \text{if} \quad q \ge 1, \tag{4.7}$$

and with the inequalities reversed when $q < 1$. Take logarithms of (4.7) and divide by $|\mathbb{E}_\Lambda|$. The limits exist as $\Lambda \to \mathbb{Z}^d$, as in [25], and they are independent of the choice of ξ by (4.7) and the fact that $|\partial\Lambda|/|\mathbb{E}_\Lambda| \to 0$. Therefore $f(p, q)$ is well defined by (4.2).

The function

$$f_\Lambda^\xi(p,q) = \frac{1}{|\mathbb{E}_\Lambda|} \log Y_\Lambda^\xi$$

is a convex function of $\pi = \log\{p/(1-p)\}$, for any $\xi \in \Omega$; this is immediate from the form of Y_Λ^ξ, just differentiate twice and use Hölder's inequality. We note for later use that

$$\frac{df_\Lambda^\xi}{d\pi} = \frac{1}{|\mathbb{E}_\Lambda|} \phi_{\Lambda,p,q}^\xi\big(|\eta(\omega) \cap \mathbb{E}_\Lambda|\big). \tag{4.8}$$

Since, for any $\xi \in \Omega$, $(f_\Lambda^\xi(p,q))_\Lambda$ is a family of convex functions of $\pi = \pi(p)$ which converge to the finite limit function $f(p,q)$ as $\Lambda \to \mathbb{Z}^d$, it follows that $f(p,q)$ is a convex function of π. Therefore $f(p,q)$ is differentiable with respect to p except on some countable set $\mathcal{D}_q$ of values of p. □

Proof of Theorem 4.2. Fix $q \ge 1$, and let $\mathcal{D} = \mathcal{D}_q$ be the set of values of x $(\in (0,1))$ at which the pressure $f(x,q)$ is non-differentiable.

First we prove that (a) implies (c). Assume $0 < p < 1$. We have by the convexity of $f(\cdot,q)$ that

$$\frac{df_\Lambda^\xi}{d\pi} \to \frac{df}{d\pi} \quad \text{as } \Lambda \to \mathbb{Z}^d, \text{ for } \xi \in \Omega \text{ and } p \notin \mathcal{D}. \tag{4.9}$$

For any box Λ and any edge $e \in \mathbb{E}_\Lambda$,

$$\begin{aligned} \frac{1}{|\mathbb{E}_\Lambda|} \phi_{\Lambda,p,q}^0\big(|\eta(\omega) \cap \mathbb{E}_\Lambda|\big) &\le \phi_{p,q}^0(J_e) \\ &\le \phi_{p,q}^1(J_e) \le \frac{1}{|\mathbb{E}_\Lambda|} \phi_{\Lambda,p,q}^1\big(|\eta(\omega) \cap \mathbb{E}_\Lambda|\big), \end{aligned} \tag{4.10}$$

where $J_e = \{\omega(e) = 1\}$, and we have used the translation-invariance of $\phi_{p,q}^0$ and $\phi_{p,q}^1$, together with the stochastic orderings of certain measures. Using (4.8) and (4.9), we deduce by passing to the limit as $\Lambda \to \mathbb{Z}^d$ that

$$\frac{df}{d\pi} = \phi_{p,q}^0(J_e) = \phi_{p,q}^1(J_e) \quad \text{for } e \in \mathbb{E} \text{ and } p \notin \mathcal{D}. \tag{4.11}$$

This implies (c).

Suppose now that (c) holds. We claim that

$$\phi_{p,q}^0(A) = \phi_{p,q}^1(A) \quad \text{for all increasing cylinders } A, \tag{4.12}$$

which will imply (d), by (3.7). One way to see that (c) implies (4.12) is as follows. Since $q \ge 1$, the two measures $\phi_{p,q}^0$ and $\phi_{p,q}^1$ may be coupled in the way described by Holley [37]: there exists a probability measure μ on $\Omega_\Lambda^0 \times \Omega_\Lambda^1$ whose marginals are $\phi_{\Lambda,p,q}^0$ and $\phi_{\Lambda,p,q}^1$, and such that the μ-probability of the set of pairs (ω_0, ω_1)

($\in \Omega_\Lambda^0 \times \Omega_\Lambda^1$) with $\omega_0 \le \omega_1$ is one. For an increasing event A defined on the finite edge-set E ($\subseteq \mathbb{E}$), we have that

$$\begin{aligned}
\phi^1_{\Lambda,p,q}(A) - \phi^0_{\Lambda,p,q}(A) &= \mu(\omega_1 \in A,\ \omega_0 \notin A) \\
&\le \sum_{e\in E} \mu(\omega_1(e) = 1,\ \omega_0(e) = 0) \\
&= \sum_{e\in E} \left\{ \phi^1_{\Lambda,p,q}(J_e) - \phi^0_{\Lambda,p,q}(J_e) \right\} \\
&\to \sum_{e\in E} \left\{ \phi^1_{p,q}(J_e) - \phi^0_{p,q}(J_e) \right\} = 0
\end{aligned}$$

by (c).

Since $f(x,q)$ is a convex function of $\pi(x) = \log\{x/(1-x)\}$, it has right and left derivatives with respect to x, denoted by $df/dx^\pm$. Furthermore df/dx^+ (resp. df/dx^-) is right-continuous (resp. left-continuous) and non-decreasing. We shall prove that

$$\frac{df}{dp^+} - \frac{df}{dp^-} = \frac{1}{p(1-p)} \left\{ \phi^1_{p,q}(J_e) - \phi^0_{p,q}(J_e) \right\} \tag{4.13}$$

and that

$$\phi^1_{p,q}(J_e) = \lim_{p'\downarrow p} \phi^0_{p',q}(J_e), \quad \phi^0_{p,q}(J_e) = \lim_{p'\uparrow p} \phi^1_{p',q}(J_e). \tag{4.14}$$

In advance of proving (4.13) and (4.14), we note the following. Relation (4.13) yields that (d) implies (a), and we have proved that (a), (c), and (d) are equivalent. In conjunction with (4.14), it yields by the semicontinuity in p of $h^b(p,q) = \phi^b_{p,q}(J_e)$ (see Proposition 4.4) that (a) and (b) are equivalent.

Finally we prove (4.13) and (4.14). Equations (4.14) are a consequence of the semicontinuity and monotonicity of $\phi^b_{p,q}(J_e)$ (see Proposition 4.4), and the fact that $|\mathcal{R}_{p',q}| = 1$ for $p' \notin \mathcal{D}$, a countable set.

By (4.11), with $\pi = \pi(x)$,

$$\frac{df}{dx} = \frac{1}{x(1-x)} \frac{df}{d\pi} = \frac{1}{x(1-x)} \phi^b_{x,q}(J_e) \quad \text{for } b = 0,1 \text{ and } x \notin \mathcal{D}.$$

Writing f' for the derivative of $f(x,q)$ with respect to x,

$$\frac{df}{dp^+} = \lim_{\substack{x\downarrow p \\ x\notin\mathcal{D}}} f'(x) = \frac{1}{p(1-p)} \phi^1_{p,q}(J_e),$$

and

$$\frac{df}{dp^-} = \lim_{\substack{x\uparrow p \\ x\notin\mathcal{D}}} f'(x) = \frac{1}{p(1-p)} \phi^0_{p,q}(J_e),$$

whence (4.13) follows. $\square$

Proof of Theorem 4.5. Assume $\phi \in \mathcal{TR}_{p,q}$, and define the random variable

$$g_\Lambda(\omega) = \frac{1}{|\mathbb{E}_\Lambda|}\, |\eta(\omega) \cap \mathbb{E}_\Lambda|.$$

Then

$$\begin{aligned}
(4.15) \qquad \phi(J_e) &= \phi(g_\Lambda) && \text{by translation-invariance} \\
&= \phi(\phi^{\cdot}_{\Lambda,p,q}(g_\Lambda)) && \text{since } \phi \in \mathcal{R}_{p,q} \\
&= \phi\left(\frac{df^{\cdot}_\Lambda}{d\pi}\right) && \text{by (4.8).}
\end{aligned}$$

Now $(df^{\cdot}_\Lambda/d\pi)_\Lambda$ is a sequence of bounded random variables (since $|g_\Lambda| \le 1$) which converges as $\Lambda \to \mathbb{Z}^d$ to $df/d\pi$ so long as $p \notin \mathcal{D}_q$; this holds by (4.9), which is valid for all positive q. Letting $\Lambda \to \mathbb{Z}^d$, we find by the bounded convergence theorem that

$$\phi(J_e) = \phi\left(\frac{df}{d\pi}\right) = \frac{df}{d\pi} \quad \text{if } p \notin \mathcal{D}_q,$$

which implies (a).

As for part (b), pick $p'' \in (p, p')$ such that $p'' \notin \mathcal{D}_q$. By (4.15), (4.8), and the bounded convergence theorem,

$$\phi(J_e) \le \phi\left(\frac{df^{\cdot}_\Lambda}{d\pi}\bigg|_{p''}\right) \to \phi\left(\frac{df}{d\pi}\bigg|_{p''}\right) = \frac{df}{d\pi}\bigg|_{p''}$$

and

$$\phi'(J_e) \ge \phi'\left(\frac{df^{\cdot}_\Lambda}{d\pi}\bigg|_{p''}\right) \to \phi'\left(\frac{df}{d\pi}\bigg|_{p''}\right) = \frac{df}{d\pi}\bigg|_{p''}$$

as $\Lambda \to \mathbb{Z}^d$, where the derivatives are evaluated at $\pi = \pi(p'')$. $\square$

5. Phase transition

The phase transition in these models is marked by the onset of an infinite cluster. *We assume henceforth that* $q \ge 1$, and we concentrate here on the extremal random-cluster measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$. Let

$$(5.1) \qquad \theta^b(p,q) = \phi^b_{p,q}(0 \leftrightarrow \infty), \qquad b = 0, 1,$$

be the $\phi^b_{p,q}$ *percolation probability.*

The functions $\theta^0(p,q)$ and $\theta^1(p,q)$ play (respectively) the role of the magnetisation for Potts measures with free and constant-spin boundary conditions. More precisely, let σ_u be the spin at vertex u of a Potts model with q states (where q is now assumed to be integral). Then

$$\begin{aligned}
(1-q^{-1})\{\theta^0(p,q)\}^2 &= \lim_{|u|\to\infty} \Big\{\pi^{\mathrm{f}}(\sigma_0 = \sigma_u) - q^{-1}\Big\}, \\
(1-q^{-1})\theta^1(p,q) &= \pi^1(\sigma_0 = 1) - q^{-1},
\end{aligned}$$

where π^{f} and π^1 are the q-state Potts measures arising from free and spin-1 boundary conditions (respectively) with interaction J (>0), inverse-temperature β, and where $p = 1 - e^{-\beta J}$. It is standard that θ^1 satisfies the above equation (see [1, 18, 30]). The given statement for θ^0 may be proved similarly, making use of Theorem 3.2 and [24, Prop. 7.9]; the corresponding statement is valid for θ^1 also, with π^{f} replaced by π^1.

It is immediate from Proposition 4.4 that $\theta^b(\cdot, q)$ is non-decreasing, and therefore one may define the critical points

$$p_c^b(q) = \sup\{p : \theta^b(p, q) = 0\}, \qquad b = 0, 1. \tag{5.2}$$

We have by Theorems 4.1 and 4.2 that $\phi^0_{p,q} = \phi^1_{p,q}$ for almost every p, whence $\theta^0(p,q) = \theta^1(p,q)$ for almost every p, and therefore $p_c^0(q) = p_c^1(q)$. Henceforth we use the abbreviated notation

$$p_c(q) = p_c^0(q) = p_c^1(q), \tag{5.3}$$

and we record next some properties of $p_c(q)$. Parts (a) and (b) of the following theorem are well known (see [1]); part (c) is proved in [31] using the improved comparison inequality (2.6).

Theorem 5.1. *Let $d \geq 2$.*
(a) $0 < p_c(q) < 1$ *for all* $q \geq 1$.
(b) *If* $1 \leq q \leq q'$ *then*

$$\frac{1}{p_c(q')} \leq \frac{1}{p_c(q)} \leq \frac{q'/q}{p_c(q')} - \frac{q'}{q} + 1. \tag{5.4}$$

(c) $p_c(q)$ *is a Lipschitz-continuous and strictly increasing function of* q *on* $[1, \infty)$.

We turn our attention now to continuity properties of the percolation probabilities $\theta^b(p,q)$ for $b = 0, 1$. Of course, $\theta^0(p,q) = \theta^1(p,q) = 0$ if $p < p_c(q)$.

Theorem 5.2. *Let $d \geq 2$ and $q \geq 1$.*
(a) *The function* $\theta^0(\cdot, q)$ *is left-continuous on* $[0,1] \setminus \{p_c(q)\}$.
(b) *The function* $\theta^1(\cdot, q)$ *is right-continuous on* $[0,1]$.
(c) $\theta^0(p,q) = \theta^1(p,q)$ *if and only if* $p \notin \mathcal{D}_q$*, where* $\mathcal{D}_q$ *is given in Theorem 4.1.*
(d) *The functions* $\theta^0(\cdot, q)$ *and* $\theta^1(\cdot, q)$ *are continuous at the point* p ($\neq p_c(q)$) *if and only if* $p \notin \mathcal{D}_q$.

It is presumably the case that $\theta^0(\cdot, q)$ and $\theta^1(\cdot, q)$ are continuous except possibly at $p = p_c(q)$. In addition it may be conjectured that $\theta^0(\cdot, q)$ is left-continuous everywhere. A verification of this conjecture would include a proof that

$$\theta^0(p_c(q), q) = \lim_{p \uparrow p_c(q)} \theta^0(p, q) = 0,$$

implying in particular that $\theta(p_c(1), 1) = 0$; this last statement is one of the famous open problems of percolation theory (see [26, 32]).

Finally we record some information about the set of values of p at which there exists a unique random-cluster measure.

Theorem 5.3. *Assume that $q \geq 1$ and $d \geq 2$. Then $|\mathcal{R}_{p,q}| = 1$ if any of the following holds:*
(a) $\theta^1(p,q) = 0$,
(b) $\theta^0(p,q) = \theta^1(p,q)$,
(c) $p > p'$, *where p' $(= p'(d))$ is a certain real number satisfying $p_c(q) \leq p' < 1$.*

Part (a) was proved in [1, p. 37]. There is more information than Theorem 5.3 when $d = 2$; recall Theorem 4.3 which asserted that, when $d = 2$ and $q \geq 1$, then

$$|\mathcal{R}_{p,q}| = 1 \quad \text{if} \quad p \neq \frac{\sqrt{q}}{1+\sqrt{q}}\,.$$

The proof of Theorem 4.3 was deferred to the end of this section, and makes use of the fact that

$$p_c(q) \geq \frac{\sqrt{q}}{1+\sqrt{q}} \quad \text{if} \quad q \geq 1,\ d = 2; \tag{5.5}$$

see [60]. It is conjectured that equality is valid here, but no proof is known for general q (≥ 1). Certainly equality holds for $q = 1$, $q = 2$, and for large q ([36, 41, 43, 45, 54]).

Proof of Theorem 5.2. We shall prove (a) at the end of Section 7. Part (b) is a consequence of Proposition 4.4(b). Part (d) follows from (a)–(c), on noting that $\theta^b(\cdot, q)$ is non-decreasing for $b = 0, 1$. We turn therefore to the proof of (c). Certainly $\phi^0_{p,q} = \phi^1_{p,q}$ if $p \notin \mathcal{D}_q$ (by Theorem 4.2), whence $\theta^0(p,q) = \theta^1(p,q)$ for $p \notin \mathcal{D}_q$. Suppose conversely that

$$\theta^0(p,q) = \theta^1(p,q). \tag{5.6}$$

We shall now give the main steps in a proof that

$$h^0(p,q) = h^1(p,q); \tag{5.7}$$

this will imply that $|\mathcal{R}_{p,q}| = 1$ by Theorem 4.2.

Fix an edge $e = \langle u, v\rangle$, and let $J_e = \{\omega(e) = 1\}$ as usual. For a vertex w, let $I_w = \{w \leftrightarrow \infty\}$, and let H_w be the event that w is in an infinite open path not using e. We write A^{c} for the complement of an event A. It is a consequence of the forthcoming Theorems 7.2 and 7.3 that there exists a probability measure ψ on $(\Omega, \mathcal{F})^2$ with marginals $\phi^0_{p,q}$ and $\phi^1_{p,q}$, and assigning probability 1 to the set of pairs $(\omega_0, \omega_1) \in \Omega^2$ satisfying $\omega_0 \leq \omega_1$ [this may be proved directly also, without recourse to the theorems of Section 7]. Let $F(\omega)$ be the set of vertices which are joined to infinity by open paths of the configuration ω $(\in \Omega)$. We have that

$$0 \leq \psi\big(F(\omega_0) \neq F(\omega_1)\big) \leq \sum_{w \in \mathbb{Z}^d} \{\phi^1_{p,q}(I_w) - \phi^0_{p,q}(I_w)\} = 0, \tag{5.8}$$

by (5.6). Now $J_e \cap I_u \cap I_v$ is an increasing event, whence

$$\phi^0_{p,q}(J_e \cap I_u \cap I_v) \leq \phi^1_{p,q}(J_e \cap I_u \cap I_v). \tag{5.9}$$

Also

$$\phi^0_{p,q}(J^c_e \cap I_u \cap I_v) = \phi^0_{p,q}(J^c_e \cap H_u \cap H_v) = \phi^0_{p,q}(J^c_e \mid H_u \cap H_v)\phi^0_{p,q}(H_u \cap H_v). \tag{5.10}$$

However, $\phi^0_{p,q}(J^c_e \mid H_u \cap H_v) = \phi^1_{p,q}(J^c_e \mid H_u \cap H_v)$ by the DLR condition (Theorem 3.1(b)). In addition, $\phi^0_{p,q}(H_u \cap H_v) \le \phi^1_{p,q}(H_u \cap H_v)$ since $H_u \cap H_v$ is an increasing event. Therefore (5.10) implies

$$\phi^0_{p,q}(J^c_e \cap I_u \cap I_v) \le \phi^1_{p,q}(J^c_e \mid H_u \cap H_v)\phi^1_{p,q}(H_u \cap H_v) = \phi^1_{p,q}(J^c_e \cap H_u \cap H_v) = \phi^1_{p,q}(J^c_e \cap I_u \cap I_v). \tag{5.11}$$

Adding (5.9) and (5.11), we obtain

$$\phi^0_{p,q}(I_u \cap I_v) \le \phi^1_{p,q}(I_u \cap I_v).$$

Equality holds here by (5.8), and therefore equality holds in (5.9), which is to say that

$$\phi^0_{p,q}(J_e \cap I_u \cap I_v) = \phi^1_{p,q}(J_e \cap I_u \cap I_v). \tag{5.12}$$

It is obvious that

$$\phi^0_{p,q}(J_e \cap I^c_u \cap I_v) = \phi^1_{p,q}(J_e \cap I^c_u \cap I_v) \tag{5.13}$$

since both sides equal 0; the same equation holds with $I^c_u \cap I_v$ replaced by $I_u \cap I^c_v$.

Finally we prove that

$$\phi^0_{p,q}(J_e \cap I^c_u \cap I^c_v) = \phi^1_{p,q}(J_e \cap I^c_u \cap I^c_v) \tag{5.14}$$

which, in conjunction with (5.12) and (5.13) (together with the associated remark), implies (5.7) as required. Let $\epsilon > 0$. With $A = \{u \nleftrightarrow \partial\Lambda,\ v \nleftrightarrow \partial\Lambda\}$, we have that

$$0 \le \phi^0_{p,q}(A) - \phi^1_{p,q}(A) < \epsilon \quad \text{for all large } \Lambda,$$

and we pick Λ accordingly. This is valid since the central term above converges, as $\Lambda \to \mathbb{Z}^d$, to $\phi^0_{p,q}(I^c_u \cap I^c_v) - \phi^1_{p,q}(I^c_u \cap I^c_v)$, which equals 0 by (5.8). The events $\{u \nleftrightarrow \partial\Lambda\}$ and $\{v \nleftrightarrow \partial\Lambda\}$ are finite-dimensional, whence

$$0 \le \phi^0_{\Delta,p,q}(A) - \phi^1_{\Delta,p,q}(A) < 2\epsilon \quad \text{for all large } \Delta, \tag{5.15}$$

and we pick Δ ($\supseteq \Lambda$) accordingly. Let $S = S(\omega) = \{x \in \Delta : x \leftrightarrow \partial\Lambda\}$ and $G = G(\omega) = \Lambda \setminus S$. We now employ a coupling of $\phi^0_{\Delta,p,q}$, $\phi^1_{\Delta,p,q}$ constructed as in [52, p. 254]. Following this reference, there exists a probability measure ψ_Δ on $\Omega^0_\Delta \times \Omega^1_\Delta$, with marginals $\phi^0_{\Delta,p,q}$ and $\phi^1_{\Delta,p,q}$, which assigns probability 1 to pairs (ω_0, ω_1) satisfying $\omega_0 \le \omega_1$, and with the additional property that, conditional on

$G = G(\omega_1)$, both marginals of ψ_Δ on $\mathbb{E}_G$ equal the *free boundary condition* random-cluster measure $\phi^0_{G,p,q}$. Writing $\mathcal{G}$ for the class of all subsets of Λ which contain u and v, it follows that

$$\begin{aligned}
\phi^1_{\Delta,p,q}(J_e \cap A) &= \sum_{g\in\mathcal{G}} \phi^1_{\Delta,p,q}\big(J_e,\ G = g\big) = \sum_{g\in\mathcal{G}} \psi_\Delta\big(\omega_1 \in J_e,\ G(\omega_1) = g\big)\\
&= \sum_{g\in\mathcal{G}} \psi_\Delta\big(\omega_1 \in J_e \,\big|\, G(\omega_1) = g\big)\psi_\Delta\big(G(\omega_1) = g\big)\\
&= \sum_{g\in\mathcal{G}} \psi_\Delta\big(\omega_0 \in J_e \,\big|\, G(\omega_1) = g\big)\psi_\Delta\big(G(\omega_1) = g\big)\\
&= \psi_\Delta\big(\omega_0 \in J_e,\ \omega_1 \in A\big),
\end{aligned}$$

and in addition

$$\phi^0_{\Delta,p,q}(J_e \cap A) = \psi_\Delta\big(\omega_0 \in J_e,\ \omega_0 \in A\big).$$

Therefore

$$0 \le \phi^0_{\Delta,p,q}(J_e \cap A) - \phi^1_{\Delta,p,q}(J_e \cap A) = \psi_\Delta\big(\omega_0 \in J_e,\ \omega_0 \in A,\ \omega_1 \notin A\big),$$

which by (5.15) does not exceed 2ϵ. Take the limits as $\Delta \to \mathbb{Z}^d$, $\Lambda \to \mathbb{Z}^d$, and $\epsilon \downarrow 0$, to obtain (5.14). $\square$

Proof of Theorem 5.3. It was proved in [1, Thm. A.2] that

$$\phi^0_{p,q} = \phi^1_{p,q} \qquad \text{if} \quad \theta^1(p,q) = 0;$$

this implies $|\mathcal{R}_{p,q}| = 1$ by (3.7). We do not include the proof here, since condition (b) is implied by condition (a). Suppose that (b) holds. By Theorem 5.2(c), $p \notin \mathcal{D}_q$, whence $|\mathcal{R}_{p,q}| = 1$ by Theorem 4.2.

Next we sketch a proof that $\phi^0_{p,q} = \phi^1_{p,q}$ if p is sufficiently close to 1. There are certain topological complications in doing this, and we avoid giving all the relevant details, most of which may be found in a closely related passage of [42, Sect. 2]. We begin by defining a lattice $\mathcal{L}$, having the same vertex set as $\mathbb{L}$ but with edge-relation

$$x \sim y \quad \text{if} \quad |x_i - y_i| \le 1 \text{ for } 1 \le i \le d.$$

For $\omega \in \Omega$, we call a vertex x *white* if $\omega(e) = 1$ for all e incident with x in $\mathbb{L}$, and *black* otherwise. For any set V of vertices of $\mathcal{L}$, we define the *black cluster* $B(V)$ as the union of V together with the set of all vertices x_0 of $\mathcal{L}$ for which there exists a path $x_0, e_0, x_1, e_1, \ldots, e_{n-1}, x_n$ of alternating vertices and edges of $\mathcal{L}$ such that $x_0, x_1, \ldots, x_{n-1} \notin V$, $x_n \in V$, and $x_0, x_1, \ldots, x_{n-1}$ are all black. Note that the colours of vertices in V have no effect on $B(V)$, but that $V \subseteq B(V)$. We define

$$\|B(V)\| = \sup\left\{ \sum_{i=1}^d |x_i - y_i| : x \in V,\ y \in B(V) \right\}.$$

For any integer n and vertex x, the event $\{\|B(x)\| \geq n\}$ is a decreasing event (we confuse the singleton x with the set $\{x\}$), whence

$$\begin{aligned}(5.16)\qquad \phi^0_{p,q}(\|B(x)\| \geq n) &\leq \phi^0_{\Lambda,p,q}(\|B(x)\| \geq n)\\ &\leq \phi^0_{\Lambda,\pi,1}(\|B(x)\| \geq n) \quad \text{for any box } \Lambda,\end{aligned}$$

where $\pi = p/(p+(1-p)q)$ and we have used the comparison inequalities (see (2.5)). Using a Peierls argument (see [42, pp. 151–152]) there exists $\alpha(p)$ such that: the percolation (product) measure $\phi_{\pi,1} = \lim_{\Lambda\to\mathbb{Z}^d} \phi^0_{\Lambda,\pi,1}$ satisfies

$$(5.17)\qquad \phi_{\pi,1}(\|B(x)\| \geq n) \leq e^{-n\alpha(p)} \quad \text{for all } n,$$

and furthermore $\alpha(p) > 0$ if p is sufficiently large, say $p > p'$ for some $p' \in [p_c(q), 1)$.

Let A be an increasing event defined in terms of the edges in the finite subset E of $\mathbb{E}$, and let Λ be a box such that $E \subseteq \mathbb{E}_\Lambda$. Let Δ be a large box satisfying $\Lambda \subseteq \Delta$. For any subset S of Λ^c $(= \mathbb{Z}^d \setminus \Lambda)$ containing $\partial\Delta$, define the 'interior boundary' $D(S)$ of S to be the set of all vertices x of $\mathcal{L}$ satisfying:

(a) $x \notin S$,

(b) x is adjacent in $\mathcal{L}$ to some vertex of S,

(c) there exists a path of $\mathbb{L}$ from x to some vertex in Λ, this path using no vertex of S.

We write $\overline{S} = S \cup D(S)$. Denote by $I(S)$ the set of vertices x_0 for which there exists a path $x_0, e_0, x_1, e_1, \ldots, e_{n-1}, x_n$ of $\mathbb{L}$ with $x_n \in \Lambda$, $x_i \notin \overline{S}$ for all i. Note that every vertex of $\partial I(S)$ is adjacent to some vertex lying in $D(S)$. We shall concentrate on the case $S = B(\partial\Delta)$.

Let $\epsilon > 0$ and $p > p'$. By (5.16)–(5.17), there exists a box Δ' sufficiently large that

$$(5.18)\qquad \phi^0_{p,q}(K_{\Lambda,\Delta}) \geq 1 - \epsilon \quad \text{if } \Delta \supseteq \Delta',$$

where $K_{\Lambda,\Delta} = \big\{\overline{B(\partial\Delta)} \cap \Lambda = \varnothing\big\}$. We pick Δ' accordingly, and let $\Delta \supseteq \Delta'$.

Let us assume that $K_{\Lambda,\Delta}$ occurs, so that $I = I(B(\partial\Delta))$ satisfies $I \supseteq \Lambda$. We note three facts about $B(\partial\Delta)$ and $D(B(\partial\Delta))$:

(a) $D(B(\partial\Delta))$ is $\mathbb{L}$-connected in that, for all pairs $x, y \in D(B(\partial\Delta))$, there exists a path of $\mathbb{L}$ joining x to y using vertices of $D(B(\partial\Delta))$ only,

(b) every vertex in $D(B(\partial\Delta))$ is white,

(c) $D(B(\partial\Delta))$ is measurable with respect to the colours of vertices in $\mathbb{Z}^d \setminus I$, in the sense that the event $\{B(\partial\Delta) = h,\ D(B(\partial\Delta)) = D(h)\}$ lies in the σ-field generated by the colours of vertices in $I(h)^c$, for any given h satisfying $\overline{h} \subseteq \Lambda^c$.

Claim (a) may be proved by adapting the argument used to prove Lemma 2.23 of [42]; claim (b) is a consequence of the definition of $D(B(\partial\Delta))$; claim (c) holds since $D(B(\partial\Delta))$ is part of the ('internal') boundary of the black cluster of $\mathcal{L}$ generated by $\partial\Delta$. We do not include full proofs of (a) and (c) which would be rather long, and which would have much in common with Section 2 of [42].

Let $\mathcal{H}_\Lambda$ denote the set of all subsets of Λ^c, and let h be a subset of Λ^c satisfying $\overline{h} \in \mathcal{H}_\Lambda$. The $\phi^0_{p,q}$-probability of A, conditional on $\{B(\partial\Delta) = h\}$, is given by the

the wired measure $\phi^1_{I(h),p,q}$. This holds since: (a) every vertex in $\partial I(h)$ is adjacent to some vertex of $D(h)$, and (b) $D(h)$ is $\mathbb{L}$-connected and all vertices in $D(h)$ are white. Therefore, by conditional probability and the FKG inequality,

$$\begin{aligned}\phi^0_{p,q}(A) &\geq \phi^0_{p,q}\left(\phi^1_{I,p,q}(A)1_{K_{\Lambda,\Delta}}\right) \\ &\geq \phi^0_{p,q}\left(\phi^1_{\Delta,p,q}(A)1_{K_{\Lambda,\Delta}}\right) && \text{since } I \subseteq \Delta \\ &\geq \phi^1_{\Delta,p,q}(A) - \epsilon && \text{by (5.18).}\end{aligned} \tag{5.19}$$

Take the limits as $\Delta \to \mathbb{Z}^d$, $\epsilon \downarrow 0$, to obtain $\phi^0_{p,q} \geq \phi^1_{p,q}$, whence $\phi^0_{p,q} = \phi^1_{p,q}$. □

Proof of Theorem 4.3. This was deferred from Section 4, and uses a graphical duality that is well known (see [7, 15, 60] for example). We write $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E})$ for the square lattice. Recall that the dual G^{d} of a planar graph G is obtained by placing a vertex within each face of G, and by joining two such vertices by an edge whenever the two corresponding faces of G have a boundary edge in common. (If G is finite, its dual graph possesses a vertex in the *infinite* face of G in addition to vertices in its finite faces.) It is easy to see that the dual of $\mathbb{L}^2$ is isomorphic to $\mathbb{L}^2$.

Let $G = (V, E)$ be a finite simple plane graph, and let $G^{\mathrm{d}} = (V^{\mathrm{d}}, E^{\mathrm{d}})$ be its dual. In the following, we shall make use of Euler's formula (see [61]):

$$k(\omega) = |V| - |\eta(\omega)| + f(\omega) - 1 \quad \text{for } \omega \in \Omega_E = \{0,1\}^E, \tag{5.20}$$

where $k(\omega)$ is the number of components, and $f(\omega)$ is the number of faces of the graph $(V, \eta(\omega))$ including the infinite face. Any configuration ω gives rise to a configuration ω^{d} lying in the space $\Omega^{\mathrm{d}}_E = \{0,1\}^{E^{\mathrm{d}}}$ defined as follows. If e $(\in E)$ is crossed by the dual edge e^{d} $(\in \mathbb{E}^{\mathrm{d}})$, then $\omega^{\mathrm{d}}(e^{\mathrm{d}}) = 1 - \omega(e)$. As before, each configuration ω^{d} gives rise to a set $\eta(\omega^{\mathrm{d}}) = \{e^{\mathrm{d}} \in E^{\mathrm{d}} : \omega^{\mathrm{d}}(e^{\mathrm{d}}) = 1\}$ of 'open edges' of the dual. By drawing a picture, one may easily be convinced that every face of $(V, \eta(\omega))$ contains a unique component of $(V^{\mathrm{d}}, \eta(\omega^{\mathrm{d}}))$, and therefore

$$f(\omega) = k(\omega^{\mathrm{d}}), \tag{5.21}$$

in the obvious notation.

The random-cluster measure on G is given by

$$\phi_{G,p,q}(\omega) \propto \left(\frac{p}{1-p}\right)^{|\eta(\omega)|} q^{k(\omega)}, \quad \text{for } \omega \in \Omega_E;$$

see (1.1). Using (5.20), (5.21), and the fact that $|\eta(\omega)| + |\eta(\omega^{\mathrm{d}})| = |E|$, we find that

$$\phi_{G,p,q}(\omega) \propto \left(\frac{q(1-p)}{p}\right)^{|\eta(\omega^{\mathrm{d}})|} q^{k(\omega^{\mathrm{d}})}, \quad \text{for } \omega^{\mathrm{d}} \in \Omega^{\mathrm{d}}_E;$$

it follows that

$$\phi_{G,p,q}(\omega) = \phi_{G^{\mathrm{d}},p',q}(\omega^{\mathrm{d}}), \quad \text{for } \omega \in \Omega_E, \tag{5.22}$$

where $\phi_{G^{\mathrm{d}},p',q}$ is the random-cluster measure on G^{d}, and p' satisfies

$$\frac{p'}{1-p'} = \frac{q(1-p)}{p}, \quad 0 < p' < 1. \tag{5.23}$$

Equation (5.22) may be expressed by saying that the dual of a random-cluster measure is itself a random-cluster measure, but with a different parameter value. Of special importance is the 'self-dual' value of p, i.e., the fixed point of the mapping $p \mapsto p'$ given in (5.23); this is easily calculated to be $p = \kappa_q$ where $\kappa_q = \sqrt{q}/(1+\sqrt{q})$.

Next we apply (5.22) to the square lattice. Let $\Lambda = \Lambda(M,N) = [-M,N]^2$, and think of $\Lambda(M,N)$ as a subgraph of $\mathbb{L}^2$ in the natural way. The dual graph $\Lambda^{\mathrm{d}} = \Lambda(M,N)^{\mathrm{d}}$ may be described as the graph obtained from $\Lambda(M+1,N)+(\frac{1}{2},\frac{1}{2})$ by an identification of all vertices in the boundary $\partial\Lambda^{\mathrm{d}}$ of this graph. Applying (5.22) to the pair $(\Lambda, \Lambda^{\mathrm{d}})$, and noting that the identification of vertices in $\partial\Lambda^{\mathrm{d}}$ amounts to working with wired boundary conditions, we deduce that

$$\phi^0_{\Lambda,p,q}(\omega) = \phi^1_{\Lambda^{\mathrm{d}},p',q}(\omega^{\mathrm{d}}),$$

in the natural notation. Finally we take the limit $\Lambda \uparrow \mathbb{Z}^d$ to obtain that

$$\phi^0_{p,q}(A) = \phi^1_{p',q}(A^{\mathrm{d}}), \quad \text{for } q \geq 1, \tag{5.24}$$

for any appropriate event A; here, A^{d} contains all ω^{d} for which $\omega \in A$.

The argument of Zhang reported in [26, p. 195] may be adapted to show that

$$\theta^0(\kappa_q, q) = 0 \quad \text{if } q \geq 1; \tag{5.25}$$

this implies in turn that $p_c(q) \geq \kappa_q$, i.e., (5.5). (This inequality may be found without full proof in [60].) To see (5.25), we argue as follows. As in Theorem 3.3, any infinite cluster is $\phi^0_{p,q}$-a.s. and $\phi^1_{p,q}$-a.s. unique. Now set $p = \kappa_q$, so that $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are dual measures in the sense of (5.24). If $\phi^0_{p,q}(0 \leftrightarrow \infty) > 0$ then $\phi^1_{p,q}(0 \leftrightarrow \infty) > 0$ also, and Zhang's argument yields a contradiction, based on the a.s. uniqueness of infinite clusters. Therefore (5.25) holds. See [60] for related arguments of this type.

It follows from (5.25) and (5.3) that $\theta^1(p,q) = 0$ for $p < \kappa_q$, whence, by Theorem 5.3, $|\mathcal{R}_{p,q}| = 1$ if $p < \kappa_q$. That $|\mathcal{R}_{p,q}| = 1$ when $p > \kappa_q$ is a consequence of the duality relation (5.24), on observing that $p < \kappa_q$ if and only if $p' > \kappa_q$ in (5.23). $\square$

6. Time evolutions on finite boxes

Two of the main purposes of this paper are to construct time-evolutions of random-cluster processes, and to find useful level-set representations of such processes. Related results for other models, particularly the Ising model, may be found in [9, 35, 48]. As remarked in the introduction, we follow a route which attains both targets simultaneously, and which is based on FKG orderings of measures rather than on the general methods of [48].

An application of the level-set representation is presented in Theorem 5.2(a), which is the random-cluster equivalent of the continuity theorem of [9].

Assume $q \geq 1$. We shall construct a Markov process on the state space $X = [0,1]^{\mathbb{E}}$, and we do this via a graphical construction involving a family of doubly-stochastic Poisson processes. First we describe these processes. For each edge $e \in \mathbb{E}$:

(a) $A(e) = (A_n(e) : n \geq 1)$ and $B(e) = (B_n(e) : n \geq 1)$ are the (increasing) sequences of arrival times of two independent Poisson processes having rate 1,

(b) $C(e) = (C_n(e) : n \geq 1)$ is the (increasing) sequence of arrival times of a Poisson process having rate $q-1$, independent of $A(e)$ and $B(e)$,

(c) $\alpha(e) = (\alpha_n(e) : n \geq 1)$, $\beta(e) = (\beta_n(e) : n \geq 1)$, and $\sigma(e) = (\sigma_n(e) : n \geq 1)$ are families of independent random variables having the uniform distribution on the interval $(0,1)$, independent of $A(e)$, $B(e)$, and $C(e)$.

Furthermore, we assume that the three paired processes $(A(e), \alpha(e))$, $(B(e), \beta(e))$, and $(C(e), \sigma(e))$ are independent for different edges e. It is standard that these processes may be constructed in such a way that, for each e, only finitely many arrivals take place for $A(e)$, $B(e)$, and $C(e)$, in any finite time interval. We write P for the appropriate probability measure.

Let Λ be a box, let $\zeta \in X$, and define the subset X_Λ^ζ of X by

$$X_\Lambda^\zeta = \{\xi \in X : \xi(e) = \zeta(e) \text{ for } e \notin \mathbb{E}_\Lambda\}.$$

We let $(Z_{\Lambda,t}^\zeta : t \geq 0)$ be the Markov process on the state space X_Λ^ζ given in the following way. First we set $Z_{\Lambda,0}^\zeta = \zeta$, and we require that $Z_{\Lambda,\cdot}^\zeta$ has right-continuous sample paths. The process $Z_{\Lambda,\cdot}^\zeta$ jumps at the times $\{A_m(e), B_m(e), C_m(e) : m \geq 1,\ e \in \mathbb{E}_\Lambda\}$ and remains constant between these times. We need now to specify how the process behaves at each of these special epochs. Fix an edge $e \in \mathbb{E}_\Lambda$ and a time $t > 0$, and suppose that t is an arrival time of *exactly one* of $A(e)$, $B(e)$, $C(e)$ but of no $A(f)$, $B(f)$, $C(f)$ for $f \neq e$. Certainly the limit $\nu = Z_{\Lambda,t-}^\zeta$ exists. We define $Z_{\Lambda,t}^\zeta$ by

$$Z_{\Lambda,t}^\zeta(f) = \begin{cases} \nu(f) & \text{if } f \neq e, \\ \rho(e) & \text{if } f = e, \end{cases} \tag{6.1}$$

where $\rho(e)$ is given by

$$\rho(e) = \begin{cases} \nu(e) \vee \alpha_m(e) & \text{if } t = A_m(e), \\ \nu(e) \wedge \beta_m(e) & \text{if } t = B_m(e), \\ \nu(e) \wedge \{\sigma_m(e) \vee F(e,\nu)\} & \text{if } t = C_m(e). \end{cases} \tag{6.2}$$

(As usual, $\alpha \vee \beta = \max\{\alpha, \beta\}$ and $\alpha \wedge \beta = \min\{\alpha, \beta\}$.) The function $F : \mathbb{E} \times X \to [0,1]$ is defined by

$$F(e,\nu) = \sup_{\pi \in \mathcal{P}_e} \min_{f \in \pi} \nu(f) \tag{6.3}$$

where $\mathcal{P}_e$ is the set of all paths of $\mathbb{L}$ which do not use the edge e but which have the same endpoints as e. In (6.3), the minimum is taken over all edges f lying in the path π. The supremum in (6.3) is over the countably infinite set $\mathcal{P}_e$; however, in (6.2), we have that $e \in \mathbb{E}_\Lambda$ and $\nu \in X_\Lambda^\zeta$, so that $F(e,\nu)$ is expressible as a certain supremum over a *finite* set (depending on Λ and ζ).

There are two final details. First, if two or more of the three processes $A(e)$, $B(e)$, $C(e)$ fire at exactly the same instant t, we do not change the current value of $Z_{\Lambda,t-}^\zeta(e)$. Secondly, subject to the last sentence, if t is an arrival time of *two* Poisson processes indexed by different edges e and f, then we update the process on the edges e and f according to the usual rules. There is probability zero that such a time t ever occurs for any edge e (in either case).

To what end do we define such a random process $Z_{\Lambda,\cdot}^\zeta$? The purpose of the construction is to achieve level-set representations of evolving random-cluster processes on Λ. Let p satisfy $0 < p < 1$, and recall that $\Omega = \{0,1\}^{\mathbb{E}}$. For $\nu \in X$, we define two 'projected elements' $\pi^p \nu$ and $\pi_p \nu$ of Ω by

$$\pi^p \nu(e) = \begin{cases} 1 & \text{if } 1-p \le \nu(e), \\ 0 & \text{if } 1-p > \nu(e), \end{cases} \tag{6.4}$$

and

$$\pi_p \nu(e) = \begin{cases} 1 & \text{if } 1-p < \nu(e), \\ 0 & \text{if } 1-p \ge \nu(e), \end{cases} \tag{6.5}$$

for $e \in \mathbb{E}$. The 'projected processes' $(\pi^p Z_{\Lambda,t}^\zeta : t \ge 0)$ and $(\pi_p Z_{\Lambda,t}^\zeta : t \ge 0)$ take values in the (respective) state spaces

$$\pi^p X_\Lambda^\zeta = \big\{\omega \in \Omega : \omega(f) = \pi^p \zeta(f) \text{ for } f \notin \mathbb{E}_\Lambda\big\}, \tag{6.6}$$

$$\pi_p X_\Lambda^\zeta = \big\{\omega \in \Omega : \omega(f) = \pi_p \zeta(f) \text{ for } f \notin \mathbb{E}_\Lambda\big\}. \tag{6.7}$$

We point out that

$$\pi_p \nu \le \pi^p \nu \quad \text{for all } p,\ \nu, \tag{6.8}$$

and

$$\pi_{p_1}\nu_1 \le \pi_{p_2}\nu_2,\ \pi^{p_1}\nu_1 \le \pi^{p_2}\nu_2, \quad \text{if} \quad p_1 \le p_2 \text{ and } \nu_1 \le \nu_2. \tag{6.9}$$

In writing $\nu_1 \le \nu_2$ here, we are using the partial order '$\le$' on X given by $\nu_1 \le \nu_2$ if and only if $\nu_1(e) \le \nu_2(e)$ for all $e \in \mathbb{E}$.

We introduce one more piece of notation before stating the main result of this section. For $\nu, \zeta \in X$, and a box Λ, we denote by (ν,ζ) $[= (\nu,\zeta)_\Lambda]$ the configuration which agrees with ν on $\mathbb{E}_\Lambda$ and with ζ off $\mathbb{E}_\Lambda$. We sometimes suppress the subscript Λ when using this notation. For example, the expression $Z_{\Delta,t}^{(\nu,\zeta)}$ denotes the value of the process on the box Δ at time t, with initial value $(\nu,\zeta)_\Delta$. Finally, we denote by Υ_Λ^p the set of all ζ $(\in X)$ with the property that $\pi^p[(0,\zeta)_\Lambda]$ has at most one infinite cluster.

Theorem 6.1. (a) *The process* $(\pi_p Z^\zeta_{\Lambda,t} : t \geq 0)$ *is a Markov chain on the state space* $\pi_p X^\zeta_\Lambda$ *having unique stationary distribution* $\phi^{\pi_p\zeta}_{\Lambda,p,q}$, *and this stationary measure is reversible for the process. Furthermore*

$$\pi_{p_1} Z^\zeta_{\Lambda,t} \leq \pi_{p_2} Z^\zeta_{\Lambda,t} \quad \textit{for all } t, \textit{ if } p_1 \leq p_2. \tag{6.10}$$

(b) *Statement (a) is valid with the operator* π_p *replaced throughout by* π^p, *so long as* $\zeta \in \Upsilon^p_\Lambda$.

Note that the equilibrium measures $\phi^{\pi_p\zeta}_{\Lambda,p,q}$ and $\phi^{\pi^p\zeta}_{\Lambda,p,q}$ depend on the values of ζ *outside* $\mathbb{E}_\Lambda$ only.

In the next section we shall consider such dynamics on the whole lattice $\mathbb{L}$, rather than on finite boxes only. This will be achieved by passing to the limit as $\Lambda \uparrow \mathbb{Z}^d$, and by using certain monotonicity properties of the processes $\{Z^\zeta_{\Lambda,\cdot}\}$ for different Λ and ζ. We state these properties next.

We equip the product space $X = [0,1]^{\mathbb{E}}$ with the Borel σ-field $\mathcal{B}$. An event $A \in \mathcal{B}$ is called *increasing* if $\nu' \in A$ whenever $\nu' \geq \nu$ and $\nu \in A$; A is called *decreasing* if its complement is increasing.

Lemma 6.2. (a) *If* $\zeta \leq \nu$ *then* $Z^\zeta_{\Lambda,t} \leq Z^\nu_{\Lambda,t}$ *for all* Λ, t.
(b) *Let* E *be an increasing event in* $\mathcal{B}$, *and let* Λ *be a box. The function*

$$g^b(t) = P(Z^{(b,\zeta)}_{\Lambda,t} \in E)$$

is non-decreasing if $b = 0$ *and non-increasing if* $b = 1$.

Using this lemma together with Theorem 6.1, we shall prove the (weak) convergence of the process $Z^\zeta_{\Lambda,t}$ as $t \to \infty$.

Theorem 6.3. *For* $\zeta \in X$ *and a box* Λ, *there exists a probability measure* μ^ζ_Λ *on* $(X, \mathcal{B})$, *with* $\mu^\zeta_\Lambda(X^\zeta_\Lambda) = 1$, *such that*

$$Z^{(\nu,\zeta)}_{\Lambda,t} \Rightarrow \mu^\zeta_\Lambda \quad \textit{as } t \to \infty, \textit{ for all } \nu.$$

Whilst Lemma 6.2 expresses a stochastic monotonicity, there is a sample path monotonicity of the graphical representation which will enable us to take the limit as $\Lambda \uparrow \mathbb{Z}^d$. Furthermore, if ν and ζ are close to one another, then so are $Z^{(\nu,b)}_{\Lambda,t}$ and $Z^{(\zeta,b)}_{\Lambda,t}$, for $b \in \{0,1\}$.

Lemma 6.4. (a) *Let* Λ *and* Δ *be boxes satisfying* $\Lambda \subseteq \Delta$. *Then*

$$Z^{(\zeta,0)}_{\Lambda,t} \leq Z^{(\zeta,0)}_{\Delta,t} \quad \textit{for all } \zeta \textit{ and } t, \tag{6.11}$$

and

$$Z^{(\zeta,1)}_{\Lambda,t} \geq Z^{(\zeta,1)}_{\Delta,t} \quad \textit{for all } \zeta \textit{ and } t. \tag{6.12}$$

(b) *Let* Λ *be a box, and let* $b \in \{0, 1\}$. *For* $\nu, \zeta \in X$,

$$\left|Z_{\Lambda,t}^{(\nu,b)}(e) - Z_{\Lambda,t}^{(\zeta,b)}(e)\right| \le \max_{f \in \mathbb{E}_\Lambda} \left\{|\nu(f) - \zeta(f)|\right\} \quad \textit{for all } t \ge 0 \textit{ and all } e \in \mathbb{E}. \tag{6.13}$$

Before moving to the proofs, we make two notes concerning the value of q. First, the above construction may be extended in order to couple together random-cluster processes with different values of p *and* different values of q (satisfying $q \ge 1$); this is achieved by a suitable coupling of the processes $\{C(e) : e \in \mathbb{E}\}$ for different q. Secondly, some of the arguments of this section may be recast in the 'non-FKG' case when $q < 1$. When $q < 1$, we alter the definitions of the processes $A(e)$, $B(e)$, $C(e)$ so that $B(e)$ has rate q and $C(e)$ has rate $1 - q$. With minor changes elsewhere, this enables the construction to proceed, but unfortunately with the loss of Lemmas 6.2 and 6.4.

Proof of Theorem 6.1. The projected process $(\pi_p Z_{\Lambda,t}^\zeta : t \ge 0)$ takes values in the finite state space $\Omega_\Lambda^\zeta = \pi_p X_\Lambda^\zeta$; recall (6.7). First we perform a little calculation involving $F(e, \nu)$, defined in (6.3). Let $\gamma \in \Omega_\Lambda^\zeta$ and let ν $(\in X)$ be such that $\pi_p \nu = \gamma$. We have by (6.3) that $F(e, \nu) \le 1 - p$ if and only if, for all $\pi \in \mathcal{P}_e$, there exists $f \in \pi$ with $\pi_p \nu(f) = 0$, which is to say that $\gamma = \pi_p \nu \in D_e$, the event that the endpoints of e are in different components of $(\mathbb{Z}^d, \eta(\pi_p \nu) \setminus \{e\})$; recall that $\eta(\omega) = \{f : \omega(f) = 1\}$. We have shown that

$$F(e, \nu) \le 1 - p \quad \text{if and only if} \quad \gamma = \pi_p \nu \in D_e. \tag{6.14}$$

Clearly the projected process changes its value only if $Z_{\Lambda,\cdot}^\zeta$ changes its value. Assume that $Z_{\Lambda,t}^\zeta = \nu$ and $\pi_p Z_{\Lambda,t}^\zeta = \pi_p \nu = \gamma$. Let $\gamma' \in \Omega_\Lambda^\zeta$. Examining (6.1)–(6.3), we see that the rate at which $\pi_p Z_{\Lambda,\cdot}^\zeta$ jumps subsequently to the new state γ' depends only on the arrivals of the doubly stochastic Poisson processes (A, α), (B, β), (C, σ), at times subsequent to time t, and upon the set of edges $E = \{e \in \mathbb{E}_\Lambda : F(e, \nu) \le 1 - p\}$. By (6.14), $E = \{e \in \mathbb{E}_\Lambda : \gamma \in D_e\}$, which depends on γ only, and not further on ν. It follows that $\pi_p Z_{\Lambda,\cdot}^\zeta$ is a time-homogeneous Markov chain on Ω_Λ^ζ. This argument is expanded in the following computation of the jump rates.

For $\gamma \in \Omega$ and $e \in \mathbb{E}$, we denote by γ^e and γ_e the configurations

$$\gamma^e(f) = \begin{cases} \gamma(f) & \text{if } f \ne e, \\ 1 & \text{if } f = e, \end{cases} \qquad \gamma_e(f) = \begin{cases} \gamma(f) & \text{if } f \ne e, \\ 0 & \text{if } f = e. \end{cases} \tag{6.15}$$

Let $G_\Lambda^\zeta = (G_\Lambda^\zeta(\gamma, \omega) : \gamma, \omega \in \Omega_\Lambda^\zeta)$ denote the generator of the process $(\pi_p Z_{\Lambda,t}^\zeta : t \ge 0)$. Since $Z_{\Lambda,\cdot}^\zeta$ changes its value (a.s.) only on single edges at any time, we have that

$$G_\Lambda^\zeta(\gamma, \omega) = 0 \quad \text{if} \quad \sum_e |\gamma(e) - \omega(e)| \ge 2,$$

and it remains to calculate $G_\Lambda^\zeta(\gamma_e, \gamma^e)$ and $G_\Lambda^\zeta(\gamma^e, \gamma_e)$ for $\gamma \in \Omega_\Lambda^\zeta$ and $e \in \mathbb{E}_\Lambda$. Consider $G_\Lambda^\zeta(\gamma_e, \gamma^e)$. A calculation based on (6.1) and (6.2) shows that

$$P(\pi_p Z_{\Lambda,t+h}^\zeta = \gamma^e \mid \pi_p Z_{\Lambda,t}^\zeta = \gamma_e) = ph + \mathrm{o}(h), \quad \text{as } h \downarrow 0,$$

since such a transition during the time-interval $(t, t+h)$ requires that the Poisson process $A(e)$ fires in this interval, and that the associated value α_r satisfies $\alpha_r > 1-p$; recall that $A(e)$ has rate 1, and $P(\alpha_r > 1-p) = p$. Hence

$$G_\Lambda^\zeta(\gamma_e, \gamma^e) = p \quad \text{for} \quad \gamma \in \Omega_\Lambda^\zeta,\ e \in \mathbb{E}_\Lambda. \tag{6.16}$$

To prepare for the other case, let $\gamma \in \Omega_\Lambda^\zeta$, $e \in \mathbb{E}_\Lambda$, and suppose that ν $(\in X)$ is such that $\pi_p \nu = \gamma^e$; we shall see later that the choice of ν is otherwise immaterial. Suppose that $Z_{\Lambda,t}^\zeta = \nu$, implying $\pi_p Z_{\Lambda,t}^\zeta = \gamma^e$, and consider the intensity of the possible transition from γ^e to γ_e. Such a transition requires a diminution in the value of $Z_{\Lambda,t}^\zeta(e)$, which by (6.2) may take place in either of two ways. The first of these involves the firing of the process $B(e)$, and the corresponding value β_r must satisfy $\beta_r \le 1-p$; the intensity of such an event is $1-p$, since $B(e)$ has rate 1 and $P(\beta_r \le 1-p) = 1-p$. The second of these ways involves a firing of the process $C(e)$ and requires that the corresponding value σ_r satisfies

$$\sigma_r \vee F(e, \nu) \le 1-p.$$

This cannot occur if $F(e,\nu) > 1-p$, whilst if $F(e,\nu) \le 1-p$ it occurs with intensity $(q-1)(1-p)$, since $C(e)$ has rate $q-1$ and $P(\sigma_r \le 1-p) = 1-p$. Combining this with the previous remark, we conclude by (6.14) that

$$G_\Lambda^\zeta(\gamma^e, \gamma_e) = \begin{cases} 1-p & \text{if } \gamma^e \notin D_e \\ q(1-p) & \text{if } \gamma^e \in D_e. \end{cases}$$

We complete the calculation of the generator G_Λ^ζ by requiring that

$$\sum_{\omega \in \Omega_\Lambda^\zeta} G_\Lambda^\zeta(\gamma, \omega) = 0 \quad \text{for all} \quad \gamma \in \Omega_\Lambda^\zeta.$$

It is now straightforward to check that

$$\phi_{\Lambda,p,q}^{\pi_p\zeta}(\gamma_e) G_\Lambda^\zeta(\gamma_e, \gamma^e) = \phi_{\Lambda,p,q}^{\pi_p\zeta}(\gamma^e) G_\Lambda^\zeta(\gamma^e, \gamma_e),$$

whence the process is reversible with stationary measure $\phi_{\Lambda,p,q}^{\pi_p\zeta}$ (see [33, p. 219]). Inequality (6.10) follows from (6.9).

The proof of part (b) is essentially the same as for (a), but with one notable difference. In place of (6.14) we have now that

$$F(e, \nu) < 1-p \quad \text{if and only if} \quad \pi^p \nu \in D_e, \tag{6.17}$$

whenever $\nu \in X_\Lambda^\zeta$ and $\zeta \in \Upsilon_\Lambda^p$. To see this, we argue as follows. If $F(e,\nu) < 1-p$ then $\pi^p \nu \in D_e$, by (6.3). Conversely, suppose that $\pi^p\nu \in D_e$ where $\nu \in X_\Lambda^\zeta$ and $\zeta \in \Upsilon_\Lambda^p$. Since $\pi^p \nu \in D_e$, we have that

$$\mu(\pi) := \min_{f \in \pi} \nu(f) \quad \text{satisfies} \quad \mu(\pi) < 1-p \quad \text{for all} \quad \pi \in \mathcal{P}_e;$$

therefore $F(e,\nu) \le 1-p$. Suppose $F(e,\nu) = 1-p$. Then there exists an infinite sequence $\pi(n)$ of distinct paths ($n = 1, 2, \dots$) lying in $\mathcal{P}_e$ such that $\mu(\pi(n)) < 1-p$ but $\mu(\pi(n)) \to 1-p$ as $n \to \infty$. Let $\mathcal{E}$ be the set of edges belonging to infinitely many of the paths $\pi(n)$; for $f \in \mathcal{E}$, we have that

$$\nu(f) \ge \lim_{n\to\infty} \mu(\pi(n)) = 1-p,$$

so that $\pi^p\nu(f) = 1$.

Write $e = \langle u, v\rangle$, and let $C(u)$ (resp. $C(v)$) denote the set of vertices of $\mathbb{L}$ joined to u (resp. v) by paths comprising edges f with $\pi^p\nu(f) = 1$. By a counting argument, we have that u (resp. v) lies in some infinite path of $\mathcal{E}$, and therefore $|C(u)| = |C(v)| = \infty$. Since $\pi^p\nu$ has at most one infinite cluster, we have that $C(u) = C(v)$, whence $\pi^p\nu \notin D_e$, a contradiction. This proves that $F(e,\nu) < 1-p$, as required for (6.17). The rest of the proof of (b) follows that of (a). $\square$

Proof of Lemma 6.2. (a) This follows from the transition rules (6.1)–(6.2) together with the fact that $F(e,\nu)$ is non-decreasing in ν.
(b) We have that

$$g^b(s+t) = P\Big\{P\big(Z^{(b,\zeta)}_{\Lambda,s+t} \in E \mid Z^{(b,\zeta)}_{\Lambda,s}\big)\Big\}, \quad b = 0, 1.$$

Using the time-homogeneity of the driving processes (A,α), (B,β), (C,σ), and the fact that

$$Z^{(b,\zeta)}_{\Lambda,s} \begin{cases} \ge (0,\zeta) & \text{if } b = 0,\\ \le (1,\zeta) & \text{if } b = 1,\end{cases}$$

we deduce by part (a) that

$$g^b(s+t) \begin{cases} \ge g^b(t) & \text{if } b = 0,\\ \le g^b(t) & \text{if } b = 1.\end{cases}$$

$\square$

Proof of Theorem 6.3. We have from Lemma 6.2(a) that

$$Z^{(0,\zeta)}_{\Lambda,t} \le Z^{(\nu,\zeta)}_{\Lambda,t} \le Z^{(1,\zeta)}_{\Lambda,t} \quad \text{for all } t \text{ and } \nu.$$

Also, $Z^{(b,\zeta)}_{\Lambda,t}$ is stochastically increasing if $b = 0$, and stochastically decreasing if $b = 1$ (by Lemma 6.2(b)). It therefore suffices to show that

$$Z^{(1,\zeta)}_{\Lambda,t} - Z^{(0,\zeta)}_{\Lambda,t} \Rightarrow 0 \quad \text{as} \quad t \to \infty.$$

Let $\epsilon > 0$, and write $\mathcal{E} = \{N^{-1}, 2N^{-1}, \dots, (N-1)N^{-1}\}$ where N is a positive integer satisfying $N^{-1} < \epsilon$. Then

$$\begin{aligned} P\Big(|Z^{(1,\zeta)}_{\Lambda,t}(e) - Z^{(0,\zeta)}_{\Lambda,t}(e)| > \epsilon \text{ for some } e \in \mathbb{E}_\Lambda\Big) \\ \le \sum_{e\in\mathbb{E}_\Lambda}\sum_{p\in\mathcal{E}} P\big(Z^{(0,\zeta)}_{\Lambda,t}(e) < 1-p < Z^{(1,\zeta)}_{\Lambda,t}(e)\big).\end{aligned}$$

Now

$$P\big(Z^{(0,\zeta)}_{\Lambda,t}(e) < 1-p < Z^{(1,\zeta)}_{\Lambda,t}(e)\big) \le P\big(\pi_p Z^{(1,\zeta)}_{\Lambda,t}(e)=1\big) - P\big(\pi_p Z^{(0,\zeta)}_{\Lambda,t}(e)=1\big)$$
$$\to 0 \qquad \text{as } t\to\infty$$

by the ergodicity of the Markov chain $(\pi_p Z^{\zeta}_{\Lambda,t} : t \ge 0)$; cf. Theorem 6.1. □

Proof of Lemma 6.4. (a) We consider the case (6.11) of '0' boundary conditions; the other case is exactly analogous. Certainly

$$0 = Z^{(\zeta,0)}_{\Lambda,t}(e) \le Z^{(\zeta,0)}_{\Delta,t}(e) \quad \text{for } e \notin \mathbb{E}_\Lambda.$$

For $e \in \mathbb{E}_\Lambda$, note first that $Z^{(\zeta,0)}_{\Lambda,0}(e) = Z^{(\zeta,0)}_{\Delta,0}(e)$, since $\Lambda \subseteq \Delta$. It now suffices to check that, at each arrival time of one of the Poisson processes $A(e)$, $B(e)$, $C(e)$, the process $Z^{(\zeta,0)}_{\Lambda,\cdot}(e)$ cannot jump *above* $Z^{(\zeta,0)}_{\Delta,\cdot}(e)$. This is a consequence of the transition rules (6.1)–(6.3) on noting that $F(e,\nu)$ is non-decreasing in ν.
(b) Since the processes $Z^{(\nu,b)}_{\Lambda,t}$, $Z^{(\zeta,b)}_{\Lambda,t}$ have only finitely many transitions in any finite time-interval, it suffices to prove that, if a transition occurs at time T, then

$$(6.18)\qquad \big|Z^{(\nu,b)}_{\Lambda,T}(e) - Z^{(\zeta,b)}_{\Lambda,T}(e)\big| \le \max_{f\in\mathbb{E}_\Lambda}\Big\{\big|Z^{(\nu,b)}_{\Lambda,T-}(f) - Z^{(\zeta,b)}_{\Lambda,T-}(f)\big|\Big\} \quad \text{for all } e\in\mathbb{E}_\Lambda.$$

Clearly (6.18) holds for any edge e on which there is no transition at time T. Suppose that a transition occurs on e at time T. We have from (6.3) that

$$\big|F(e,\xi) - F(e,\xi')\big| \le \max_{f\in\mathbb{E}}\Big\{\big|\xi(f)-\xi'(f)\big|\Big\} \quad \text{for all } \xi,\xi'\in X.$$

Examining each of the cases listed in (6.2), we deduce that (6.18) holds. □

7. Dynamics in the infinite-volume limit

In this section we study certain Markov processes on the state space $X = [0,1]^{\mathbb{E}}$. We show the existence of two different transition semigroups with the same (unique) invariant measure. The first of these semigroups gives rise to a 'level-set representation' of *free boundary condition* random-cluster processes, and the second of *wired boundary condition* processes.

We arrive at such Markov processes by studying the limit of the finite-volume process $Z^{\zeta}_{\Lambda,t}$, defined in the last section, as $\Lambda \uparrow \mathbb{Z}^d$. The two 'extreme' boundary conditions ζ are $\zeta = 0, 1$, and we define accordingly the following monotone limits:

$$(7.1)\qquad Z^{(\zeta,0)}_t = \lim_{\Lambda\uparrow\mathbb{Z}^d} Z^{(\zeta,0)}_{\Lambda,t}, \quad Z^{(\zeta,1)}_t = \lim_{\Lambda\uparrow\mathbb{Z}^d} Z^{(\zeta,1)}_{\Lambda,t},$$

which limits exist by virtue of Lemma 6.4(a). In particular we write

$$(7.2)\qquad Z^0_t = Z^{(0,0)}_t, \quad Z^1_t = Z^{(1,1)}_t.$$

We shall show that the processes $(Z_t^b : t \geq 0)$, for $b = 0, 1$, are Markovian, and we shall explore their properties in the limit as $t \to \infty$.

A possible alternative to the methodology of this section might employ the 'martingale method' described in [38, 48]. For general accounts of the theory of Markov processes, consult the books [11, 48, 58].

The state space $X = [0, 1]^{\mathbb{E}}$ is a compact metric space equipped with the Borel σ-field $\mathcal{B}$. Let $D(X)$ be the set of functions $G : \mathbb{R} \to X$ which are right-continuous with left limits. For $s \in [0, \infty)$, let e_s be the evaluation mapping defined by $e_s(G) = G(s)$. Let $\mathcal{H}$ be the smallest σ-field of subsets of $D(X)$ with respect to which each e_s is measurable, and let $\mathcal{H}_t$ be the smallest such σ-field defined in terms of $\{e_s : s \leq t\}$. We write $B(X)$ for the space of bounded measurable functions from X to $\mathbb{R}$, and $C(X)$ for the space of continuous functions.

We now introduce two transition functions and semigroups, as follows. For $b \in \{0, 1\}$ and $t \geq 0$, let

$$P_t^b(\zeta, A) = P(Z_t^{(\zeta,b)} \in A), \qquad \zeta \in X, \ A \in \mathcal{B}, \tag{7.3}$$

and let $S_t^b : B(X) \to B(X)$ be given by

$$S_t^b f(\zeta) = P(f(Z_t^{(\zeta,b)})), \qquad \zeta \in X, \ f \in B(X). \tag{7.4}$$

Theorem 7.1. *Let $b \in \{0, 1\}$. The process $(Z_t^b : t \geq 0)$ is a Markov process with sample paths in $D(X)$ and Markov transition function $(P_t^b : t \geq 0)$.*

Theorem 7.2. *There exists a translation-invariant probability measure μ on $(X, \mathcal{B})$ such that*

$$Z_t^b \Rightarrow \mu \quad \textit{as} \quad t \to \infty, \quad \textit{for } b = 0, 1.$$

Note that the weak limit in the latter theorem is identical for the two processes Z_t^0 and Z_t^1. It follows by monotonicity that, as $t \to \infty$,

$$Z_t^{(\zeta,b)} \Rightarrow \mu \quad \text{for} \quad \zeta \in X \text{ and } b = 0, 1; \tag{7.5}$$

recall Lemma 6.2(a) and (7.1).

We turn attention now to the 'level-set processes' of Z_t^0 and Z_t^1. Fix $p \in (0, 1)$, and write

$$L_{p,t}^0 = \pi_p Z_t^0, \quad L_{p,t}^1 = \pi^p Z_t^1, \qquad t \geq 0; \tag{7.6}$$

here, π^p and π_p are defined in (6.4) and (6.5).

Theorem 7.3. (a) *The processes $(L_{p,t}^b : t \geq 0)$, $b = 0, 1$, are Markov processes on the state space $\Omega = \{0, 1\}^{\mathbb{E}}$, with weak limits given by*

$$L_{p,t}^b \Rightarrow \phi_{p,q}^b \quad \textit{as } t \to \infty, \tag{7.7}$$

where $\phi_{p,q}^b$ is the random-cluster measure defined in (3.5) for $b = 0, 1$. The measure $\phi_{p,q}^b$ is reversible for the process $L_{p,t}^b$.

(b) *The measures $\phi^b_{p,q}$, for $b = 0, 1$, are 'level-set' measures of μ, in that*

$$\phi^0_{p,q}(A) = \mu\big(\{\zeta : \pi_p\zeta \in A\}\big), \quad \phi^1_{p,q}(A) = \mu\big(\{\zeta : \pi^p\zeta \in A\}\big), \tag{7.8}$$

for all $A \in \mathcal{F}$.

We make several remarks before proving the above theorems. First, the two weak limits $\phi^0_{p,q}$ and $\phi^1_{p,q}$ in Theorem 7.3 are identical if and only if $p \notin \mathcal{D}_q$, where $\mathcal{D}_q$ is given in Theorem 4.1.

Next, let μ be the limit measure of Theorem 7.2, let $e \in \mathbb{E}$, and define the marginal 'atomic' function

$$J(x) = \mu\big(\{\zeta \in X : \zeta(e) = x\}\big) \quad \text{for } 0 \le x \le 1;$$

since μ is translation-invariant, J does not depend on the choice of the edge e.

Proposition 7.4. *We have that*

$$h^1(p,q) - h^0(p,q) = J(1-p)$$

where $h^b(p,q) = \phi^b_{p,q}(\omega(e) = 1)$.

In the light of Theorem 4.2, this implies that $p \in \mathcal{D}_q$ if and only if $J(1-p) \neq 0$, thereby providing a representation of $\mathcal{D}_q$ in terms of atoms of the weak limit μ of the stochastic random-cluster processes Z^0_t and Z^1_t. It is this representation that we employ at the end of this section in order to prove the left-continuity of the percolation probability $\theta^0(\cdot, q)$ (cf. Theorem 5.2(a) and [9]).

As discussed already after Theorem 4.2, it is believed that there exists $Q = Q(d)$ such that

$$\mathcal{D}_q = \begin{cases} \varnothing & \text{if } q < Q, \\ \{p_c(q)\} & \text{if } q > Q, \end{cases}$$

and it is a first rate challenge to prove this. The above results provide a probabilistic (but incomplete) justification for this claim, as follows. The set $\mathcal{D}_q$ is exactly the set of atoms of the μ-measure of the random variable $1 - \zeta(e)$, for $\zeta \in X$. These atoms presumably arise through an accumulation of edges e having the same value $Z^b_t(e)$. Such coalescences occur only at the times of firing of the processes $C(e)$; see (6.2). These Poisson processes have rate $q - 1$, indicating that coalescences are more frequent for larger q.

Next we make some remarks about uniqueness of infinite clusters. The Burton–Keane [14] result implies (see Theorem 3.3) the $\phi^b_{p,q}$-a.s. uniqueness of the infinite cluster, for $b \in \{0, 1\}$ and $0 \le p \le 1$. It is another matter to obtain such uniqueness *simultaneously for all values of* p. That is, we may ask whether or not

$$\mu(I^b_p = 1 \text{ for all } p \text{ and } b = 0, 1) = 1,$$

where $I^0_p(\zeta)$ (resp. $I^1_p(\zeta)$) is the number of infinite open clusters of $\pi_p\zeta$ (resp. $\pi^p\zeta$). Such matters have been considered by Alexander [3].

Finally, we describe the transition rules of the projected processes $L^0_{p,t}$ and $L^1_{p,t}$; it turns out that the transition mechanisms of these two chains differ in an interesting (but ultimately unimportant) regard. It is convenient to summarise the following discussion by writing down directly the infinitesimal generators of the two processes, and we do this next.

We begin with some notation. Let $e = \langle x, y\rangle \in \mathbb{E}$, and let $\mathcal{P}_e$ be (as after (6.3)) the set of all paths of $\mathbb{L}$ which join x to y but do not use the edge e. Let $\mathcal{Q}_e$ be the set of all pairs $\alpha = (\alpha_1, \alpha_2, \dots)$, $\beta = (\beta_1, \beta_2, \dots)$ of vertex-disjoint semi-infinite paths (where α_i and β_j are the vertices of these paths) with $\alpha_1 = x$ and $\beta_1 = y$; we require $\alpha_i \neq \beta_j$ for all i, j. Thus $\mathcal{Q}_e$ comprises pairs (α, β) of paths; we call an element (α, β) of $\mathcal{Q}_e$ *open* if all the edges of α and β are open.

For $b = 0, 1$, let G^b be the linear operator, with domain a suitable subset of $C(\Omega)$, given by

$$G^b f(\omega) = \sum_{e\in\mathbb{E}} \Big\{ p\big(f(\omega^e) - f(\omega)\big) + h^b(e,\omega)\big(f(\omega_e) - f(\omega)\big) \Big\}, \quad \omega \in \Omega, \tag{7.9}$$

where ω^e and ω_e are given in (6.15); here, $h^b(e,\omega)$ is defined by

$$h^b(e,\omega) = (1-p)\Big\{1 + (q-1)1_{D^b(e)}(\omega)\Big\}, \quad \omega \in \Omega, \tag{7.10}$$

where

$$D^0(e) = \{\text{no path in } \mathcal{P}_e \text{ is open}\}, \tag{7.11}$$

$$D^1(e) = \{\text{no element in } \mathcal{P}_e \cup \mathcal{Q}_e \text{ is open}\}. \tag{7.12}$$

Note that $G^b f$ is well defined for all cylinder functions f, since the infinite sum in (7.9) may then be written as a finite sum. However, $G^b f$ is not generally continuous when $q > 1$, even for cylinder functions f. For example, suppose $q > 1$, let f be the indicator function of the event that a given edge e is open, and let ω be a configuration satisfying

(a) $\omega(e) = 1$,

(b) no path in $\mathcal{P}_e$ is open, under ω,

(c) some pair (α, β) in $\mathcal{Q}_e$ is open, under ω.

Then

$$G^b f(\omega) = -h^b(e,\omega).$$

However, $h^b(e,\cdot)$ is discontinuous at ω for $b = 0, 1$, since, for $b \in \{0, 1\}$ and for every finite box Λ, there exists $\omega' \in \Omega$ agreeing with ω on $\mathbb{E}_\Lambda$ such that $h^b(e,\omega') \neq h^b(e,\omega)$. Perhaps such difficulties may be avoided by restricting the space Ω of configurations. With a little further care, one may see that the Markov transition functions of $L^0_{p,t}$ and $L^1_{p,t}$ are not Feller; see the notes at the end of this section.

In describing the transition rules of the processes $L^0_{p,t}$ and $L^1_{p,t}$, we shall make use of the following lemma, which is of use also in the proofs of Theorems 7.1 and 7.3. Recall the function $F(e, \nu)$ defined on $\mathbb{E} \times X$ by (6.3).

Lemma 7.5. *Let $e \in \mathbb{E}$, $\nu \in X$, and let $(\nu_\Lambda)_\Lambda$ be a family of elements of X indexed by finite boxes Λ.*

(a) *If* $\nu_\Lambda \uparrow \nu$ *as* $\Lambda \to \mathbb{Z}^d$, *then*

$$F(e,\nu_\Lambda) \uparrow F(e,\nu). \tag{7.13}$$

(b) *If* $\nu_\Lambda \in X^1_\Lambda$ *and* $\nu_\Lambda \downarrow \nu$ *as* $\Lambda \to \mathbb{Z}^d$, *then*

$$F(e,\nu_\Lambda) \downarrow G(e,\nu) \tag{7.14}$$

where

$$G(e,\nu) = \sup_{\pi \in \mathcal{P}_e \cup \mathcal{Q}_e} \inf_{f \in \pi} \nu(f). \tag{7.15}$$

Note that, in the definition (7.15) of $G(e,\nu)$, $\mathcal{P}_e$ contains certain paths π, and $\mathcal{Q}_e$ contains certain pairs $\pi = (\alpha,\beta)$ of paths; for $\pi = (\alpha,\beta) \in \mathcal{Q}_e$, the infimum in (7.15) is over all edges f lying either in α or in β.

Consider the process Z^0_t. Since Z^0_t is the increasing limit of $Z^{(0,0)}_{\Lambda,t}$ as $\Lambda \to \mathbb{Z}^d$, we have from the definition (6.5) of π_p that

$$L^0_{p,t} = \lim_{\Lambda \uparrow \mathbb{Z}^d} \pi_p Z^{(0,0)}_{\Lambda,t}. \tag{7.16}$$

Assume that $Z^{(0,0)}_{\Lambda,t} = \zeta_\Lambda$ for each Λ, and $\zeta = \lim_{\Lambda \uparrow \mathbb{Z}^d} \zeta_\Lambda$, so that

$$L^0_{p,t} = \pi_p \zeta = \lim_{\Lambda \uparrow \mathbb{Z}^d} \pi_p \zeta_\Lambda. \tag{7.17}$$

Fix an edge $e \in \mathbb{E}$, and assume first that ζ is such that $\pi_p\zeta(e) = 0$. At what rate does the state of e change from 0 to 1? Examining the transitions of the process $Z^0_{\Lambda,\cdot}$ (see (6.1)–(6.3)), we see that this occurs at the next firing of the process $A(e)$ that results in an associated α_m satisfying $\alpha_m > 1-p$; the intensity of this transition is p, as in the proof of Theorem 6.1. Assume next that ζ is such that $\pi_p\zeta(e) = 1$, and consider the intensity at which e assumes the state 0. Returning to (6.2), we see as in the proof of Theorem 6.1 that there are two independent sources of such a transition, namely the two processes $B(e)$ and $C(e)$. The process $B(e)$ fires at rate 1, and produces such a transition with probability $P(\beta_m \le 1-p) = 1-p$; the associated effective intensity is $1-p$. The process $C(e)$ fires at rate $q-1$, and produces such a transition with probability

$$\begin{cases} 0 & \text{if } \lim_\Lambda F(e,\nu_\Lambda) > 1-p, \\ 1-p & \text{if } \lim_\Lambda F(e,\nu_\Lambda) \le 1-p, \end{cases} \tag{7.18}$$

where $\nu_\Lambda = Z^0_{\Lambda,T-}$ and T is the time of the firing in question of $C(e)$. Now $\nu = \lim_{\Lambda \to \mathbb{Z}^d} \nu_\Lambda$ is an increasing limit, whereby $F(e,\nu_\Lambda) \uparrow F(e,\nu)$ by Lemma 7.5(a). We have therefore that $\lim_\Lambda F(e,\nu_\Lambda) \le 1-p$ if and only if $F(e,\nu) \le 1-p$, which is equivalent to the statement $\pi_p\nu \in D^0(e)$, by (6.3), (6.5), and (7.11). In conclusion, the state of e flips from 1 to 0 at rate

$$\begin{cases} (1-p) & \text{if } \nu \notin D^0(e) \\ (1-p) + (q-1)(1-p) & \text{if } \nu \in D^0(e), \end{cases} \tag{7.19}$$

in agreement with (7.9) with $b = 0$.

We turn next to the process $L^1_{p,t}$. This time, Z^1_t is the decreasing limit of $Z^1_{\Lambda,t}$ as $\Lambda \to \mathbb{Z}^d$, and

$$L^1_{p,t} = \lim_{\Lambda \to \mathbb{Z}^d} \pi^p Z^1_{\Lambda,t} \tag{7.20}$$

as in (7.16); we have used the definition (6.4) of π^p here, noting that the corresponding statement with π_p (in place of π^p) fails in general. We now follow the above argument step by step, noting that increasing limits are replaced by decreasing limits, π_p by π^p, $F(e,\nu)$ by $G(e,\nu)$ (defined in (7.15)), and $D^0(e)$ by $D^1(e)$. Our conclusion is in agreement with (7.9) with $b = 1$.

Next appear the proofs, beginning with Lemma 7.5.

Proof of Lemma 7.5. (a) Suppose $\nu_\Lambda \uparrow \nu$. Certainly $F(e,\nu_\Lambda)$ is non-decreasing in Λ, whence the limit

$$\lambda = \lim_{\Lambda \to \mathbb{Z}^d} F(e,\nu_\Lambda)$$

exists and satisfies $\lambda \le F(e,\nu)$. Now, for $x \in (0,1)$, we have that $\lambda \le x$ if and only if $F(e,\nu_\Lambda) \le x$ for all Λ. By (6.3), this occurs if and only if

$$\forall \pi \in \mathcal{P}_e,\ \forall \Lambda,\ \exists f \in \pi \text{ with } \nu_\Lambda(f) \le x.$$

Since all paths in $\mathcal{P}_e$ are finite, this implies

$$\forall \pi \in \mathcal{P}_e,\ \exists f \in \pi \text{ with } \nu(f) \le x,$$

which implies in turn that $F(e,\nu) \le x$. Therefore $F(e,\nu) \le \lambda$.
(b) Suppose $\nu_\Lambda \in X^1_\Lambda$ and $\nu_\Lambda \downarrow \nu$. First we prove that the decreasing limit $\lambda = \lim_\Lambda F(e,\nu_\Lambda)$ satisfies

$$\lambda \le G(e,\nu). \tag{7.21}$$

Let $x \in (0,1)$, and suppose $G(e,\nu) < x$; we shall deduce that $\lambda < x$, thus obtaining (7.21). Write $e = \langle u,v\rangle$, and call a finite set S of edges of $\mathbb{L}$ a *cutset* (for e) if

(i) $e \notin S$,
(ii) every path in $\mathcal{P}_e$ contains at least one edge of S,
(iii) S is minimal with the two properties above, in the sense that no strict subset of S satisfies (i) and (ii).

We write $G(e,\nu) = \max\{A,B\}$ where

$$A = \sup_{\pi \in \mathcal{P}_e} \min_{f \in \pi} \nu(f), \quad B = \sup_{\pi \in \mathcal{Q}_e} \inf_{f \in \pi} \nu(f).$$

Since $G(e,\nu) < x$, we have that $A, B < x$, which implies that there exists a cutset S with $\nu(f) < x$ for all $f \in S$. To see this, argue as follows. For $w \in \mathbb{Z}^d$, let $C_w(\nu)$ be the set of vertices of $\mathbb{L}$ that are connected to the vertex w by paths π of $\mathbb{L}$ satisfying: π does not contain the edge e, and every edge f of π satisfies $\nu(f) \ge x$. If $u \in C_v(\nu)$, then there exists $\pi \in \mathcal{P}_e$ with $\nu(f) \ge x$ for all $f \in \pi$, which contradicts the fact that $A < x$. Therefore $u \notin C_v(\nu)$. Furthermore either $C_u(\nu)$ or $C_v(\nu)$ (or both) is finite, since if both were infinite then there would exist $\pi = (\alpha, \beta) \in \mathcal{Q}_e$ with $\nu(f) \geq x$ for all f in α and β, thereby contradicting the fact that $B < x$. Suppose without loss of generality that $C_u(\nu)$ is finite, and let R be the subset of $\mathbb{E}\backslash\{e\}$ containing all edges g having exactly one vertex in $C_u(\nu)$. Certainly $\nu(g) < x$ for all $g \in R$, and additionally every path in $\mathcal{P}_e$ contains some edge of R. However R may fail to be minimal with this property, in which case we replace R by a subset S which is minimal; S is the required cutset.

We have that $\nu(f) < x$ for all $f \in S$, implying (since S is finite) that

$$\text{for all large } \Lambda,\ \nu_\Lambda(f) < x \text{ for all } f \in S,$$

and therefore (using the finiteness of S again)

$$\text{for all large } \Lambda,\ F(e, \nu_\Lambda) < x,$$

implying that $\lambda < x$ as required for (7.21).

Finally we prove that

$$\lambda \geq G(e, \nu), \tag{7.22}$$

and we achieve this by proving that $\lambda \geq A$ and $\lambda \geq B$, separately. That $\lambda \geq A$ is an immediate consequence of the fact that $\nu_\Lambda \geq \nu$, so we turn towards the inequality $\lambda \geq B$. For $\pi = (\alpha, \beta) \in \mathcal{Q}_e$, where α has endpoint u, and β has endpoint v, let α_Λ (respectively β_Λ) denote the initial segment of α (resp. β) joining u (resp. v) to the earliest vertex w_1 of α (resp. w_2, of β) lying in $\partial\Lambda$. Since $w_1, w_2 \in \partial\Lambda$ and $w_1 \neq w_2$, there exists a path γ joining w_1 to w_2 and using no other vertex of Λ. We denote by π' the path comprising α_Λ, followed by γ, followed by β_Λ taken in reverse order; note that $\pi' \in \mathcal{P}_e$, and denote by $\mathcal{P}_{e,\Lambda}$ the set of all $\pi' \in \mathcal{P}_e$ obtainable in this way from any $\pi = (\alpha, \beta) \in \mathcal{Q}_e$. Now

$$\begin{aligned}
F(e, \nu_\Lambda) &\geq \sup_{\pi' \in \mathcal{P}_{e,\Lambda}} \min_{f \in \pi'} \nu_\Lambda(f) && \text{since } \mathcal{P}_{e,\Lambda} \subseteq \mathcal{P}_e \\
&= \sup_{\pi' \in \mathcal{P}_{e,\Lambda}} \min_{f \in \pi' \cap \mathbb{E}_\Lambda} \nu_\Lambda(f) && \text{since } \nu_\Lambda(f) = 1 \text{ for } f \notin \mathbb{E}_\Lambda \\
&\geq \sup_{\pi' \in \mathcal{P}_{e,\Lambda}} \min_{f \in \pi' \cap \mathbb{E}_\Lambda} \nu(f) && \text{since } \nu_\Lambda \geq \nu \\
&= \sup_{\pi \in \mathcal{Q}_e} \min_{f \in \pi \cap \mathbb{E}_\Lambda} \nu(f) \\
&\geq \sup_{\pi \in \mathcal{Q}_e} \inf_{f \in \pi} \nu(f) = B,
\end{aligned}$$

where we have used the fact that every $\pi' \in \mathcal{P}_{e,\Lambda}$ arises from some $\pi \in \mathcal{Q}_e$. Inequality (7.22) follows. $\square$

Proof of Theorem 7.1. The transitions of the process $(Z_t^b : t \geq 0)$ are given in terms of families of independent doubly-stochastic Poisson processes. In order that Z_t^b be a Markov process, it suffices therefore to prove the following:

(a) sample paths lie in $D(X)$,

(b) the distribution of $(Z^b_{s+t} : t \geq 0)$, given $(Z^b_u : 0 \leq u \leq s)$, depends only on Z^b_s.

First we prove (a). Let F be a finite subset of $\mathbb{E}$, let $t > 0$, and let

$$S = \sup\{A_m(e), B_r(e), C_s(e) < t : e \in F,\ m, r, s \geq 1\},$$
$$T = \inf\{A_m(e), B_r(e), C_s(e) \geq t : e \in F,\ m, r, s \geq 1\}.$$

Since, for each edge e, the processes $A(e)$, $B(e)$, $C(e)$ have only finitely many arrivals in any finite time-interval, we have that $S < t \leq T$. Now

$$Z^b_{\Lambda,s}(e) = Z^b_{\Lambda,S}(e) \quad \text{for} \quad S \leq s < T,\ e \in F. \tag{7.23}$$

Therefore $Z^b_s(e) = Z^b_S(e)$ for $s \in [S, t)$, whence the limit $Z^b_{t-}(e)$ exists for $e \in F$.

If $T > t$, then $Z^b_t(e) = Z^b_{t+}(e)$ for $e \in F$ by (7.23), whence $Z^b_\cdot$ is right-continuous at t. If $T = t$, then $Z^b_{\Lambda,s}(e) = Z^b_{\Lambda,t}(e)$ for $e \in F$ and $t \leq s < U$ where

$$U = \inf\{A_m(e), B_r(e), C_s(e) > t : e \in F,\ m, r, s \geq 1\},$$

implying right-continuity as before.

Next we prove (b). We have that $Z^b_{s+t} = \lim_{\Lambda\to\mathbb{Z}^d} Z^b_{\Lambda,s+t}$, where the processes $Z^b_{\Lambda,s+t}$ are given in terms of a graphical representation of compound Poisson processes. Therefore, conditional on $(Z^b_{\Lambda,u},\ Z^b_u : 0 \leq u \leq s,\ \Lambda \subseteq \mathbb{Z}^d)$, the process $(Z^b_{s+t} : t \geq 0)$ has law which depends only on the family $(Z^b_{\Lambda,s} : \Lambda \subseteq \mathbb{Z}^d)$ indexed by finite boxes Λ. Write $\zeta_\Lambda = Z^b_{\Lambda,s}$ and $\zeta = \lim_{\Lambda\to\mathbb{Z}^d} \zeta_\Lambda = Z^b_s$. We need to show that the (conditional) law of Z^b_{s+t} does not depend on the family (ζ_Λ) but only on its limit ζ. To achieve this, we shall use Lemma 6.4(b).

First we introduce one more piece of notation. Let $s, t \geq 0$ and $\nu \in X$. Denote by $Y^{(\nu,b)}_{\Lambda,s+t}$ the state (in X^b_Λ) at time $s+t$ obtained from the evolution rules (6.1)–(6.3), starting at time s in state $(\nu, b) = (\nu, b)_\Lambda$.

Suppose that $b = 0$, so that $\zeta_\Lambda \uparrow \zeta$ as $\Lambda \to \mathbb{Z}^d$. Let $\epsilon > 0$ and let Δ be a finite box. There exists a box Λ' such that $\Lambda' \supseteq \Delta$ and

$$\zeta(e) - \epsilon \leq \zeta_\Lambda(e) \leq \zeta(e) \quad \text{for all } e \in \mathbb{E}_\Delta, \text{ if } \Lambda \supseteq \Lambda'.$$

It follows by Lemma 6.4(b) that

$$Y^{(\zeta,b)}_{\Delta,s+t} - \epsilon \leq Y^{(\zeta_\Lambda,b)}_{\Delta,s+t} \leq Y^{(\zeta_\Lambda,b)}_{\Lambda,s+t} \leq Y^{(\zeta,b)}_{\Lambda,s+t}, \quad \text{if } \Lambda \supseteq \Lambda'.$$

Use the fact that $Y^{(\zeta_\Lambda,b)}_{\Lambda,s+t} = Z^b_{\Lambda,s+t}$, and pass to the limits as $\Lambda \to \mathbb{Z}^d$, $\Delta \to \mathbb{Z}^d$, $\epsilon \downarrow 0$, to obtain that

$$\lim_{\Lambda\to\mathbb{Z}^d} Y^{(\zeta,b)}_{\Lambda,s+t} = Z^b_{s+t}, \tag{7.24}$$

implying as required that Z^b_{s+t} depends on ζ but not further on the family (ζ_Λ). The same argument is valid when $b = 1$, with the above inequalities reversed and the sign of ϵ changed.

The Markov transition function associated with the process Z_t^b is the family $(Q_{s,t}^b : 0 \le s \le t)$ given by

$$Q_{s,t}^b(\zeta, A) = P\big(Z_{s+t}^b \in A \mid Z_s^b = \zeta\big), \quad \zeta \in X,\ A \in \mathcal{B}.$$

In the light of the remarks above and particularly (7.24), we have that

$$Q_{s,t}^b(\zeta, A) = Q_{0,t-s}^b(\zeta, A),$$

and that

$$Q_{0,t-s}^b(\zeta, A) = P(Z_{t-s}^{(\zeta,b)} \in A) = P_{t-s}^b(\zeta, A)$$

as required. □

Proof of Theorem 7.2. We have from Lemma 6.2 that the limits ψ^b, given by

$$\psi^b(A) = \lim_{t\to\infty} P(Z_t^b \in A), \quad b = 0, 1,$$

exist for any increasing event A. Therefore Z_t^0 and Z_t^1 converge weakly as $t \to \infty$. It therefore suffices to show that

$$Z_t^1 - Z_t^0 \Rightarrow 0 \quad \text{as} \quad t \to \infty.$$

Since we are working with the product topology on X, it will be enough to show that, for all $\epsilon > 0$ and all finite subsets F of $\mathbb{E}$,

$$P\Big(|Z_t^1(f) - Z_t^0(f)| > \epsilon \text{ for some } f \in F\Big) \to 0 \quad \text{as } t \to \infty. \tag{7.25}$$

Let $\mathcal{D} = \mathcal{D}_q$ be as in Theorem 4.1, and let $\epsilon > 0$. Pick a finite subset $\mathcal{E}$ of $\mathcal{D}^c = (0,1) \setminus \mathcal{D}$ such that every interval of the form $(x, x+\epsilon)$ contains some point of $\mathcal{E}$, as x ranges over $[0, 1-\epsilon)$; recall from Theorem 4.2 that

$$\phi_{p,q}^0 = \phi_{p,q}^1 \quad \text{if} \quad p \in \mathcal{E}. \tag{7.26}$$

We have that, for $f \in \mathbb{E}$,

$$\begin{aligned}
P(|Z_t^1(f) - Z_t^0(f)| > \epsilon) &\le \sum_{p\in\mathcal{E}} P\big(Z_t^0(f) < 1-p \le Z_t^1(f)\big) \\
&\le \sum_{p\in\mathcal{E}} P\big(Z_{\Lambda,t}^0(f) < 1-p \le Z_{\Lambda,t}^1(f)\big) && \text{for all boxes } \Lambda \\
&\to \sum_{p\in\mathcal{E}} \big\{\phi_{\Lambda,p,q}^1(J_f) - \phi_{\Lambda,p,q}^0(J_f)\big\} && \text{as } t \to \infty \\
&\to \sum_{p\in\mathcal{E}} \big\{\phi_{p,q}^1(J_f) - \phi_{p,q}^0(J_f)\big\} && \text{as } \Lambda \to \mathbb{Z}^d \\
&= 0 && \text{by (7.26)},
\end{aligned}$$

where $J_f = \{\omega(f) = 1\}$. Equation (7.25) follows since F is finite.

That the limit measure μ is translation-invariant is a consequence of (for example) Theorem 7.3 and the fact that $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are translation-invariant (see Theorem 3.1). $\square$

Proof of Theorem 7.3. (a) That the projected processes $(L^b_{p,t} : t \geq 0)$, $b = 0, 1$, are Markovian follows from Theorem 7.1 and the discussion after Lemma 7.5.

Let A be an increasing event in $\mathcal{F}$. Using Lemma 6.2, we have that the limits

$$\psi^b_p(A) = \lim_{t\to\infty} P(L^b_{p,t} \in A)$$

exist for $b = 0, 1$. Since $L^0_{p,t} \leq L^1_{p,t}$, it follows that

$$\psi^0_p(A) \leq \psi^1_p(A) \quad \text{for increasing } A \in \mathcal{F}. \tag{7.27}$$

Assume now that A is an increasing event defined in terms of the edges in the finite subset F of $\mathbb{E}$. Then

$$\begin{aligned} \psi^0_p(A) &= \lim_{t\to\infty} P(L^0_{p,t} \in A) \\ &\geq \lim_{t\to\infty} P(\pi_p Z^0_{\Lambda,t} \in A) && \text{since } L^0_{p,t} \geq \pi_p Z^0_{\Lambda,t} \\ &= \phi^0_{\Lambda,p,q}(A) && \text{by Theorem 6.1} \\ &\to \phi^0_{p,q}(A) && \text{as } \Lambda \to \mathbb{Z}^d, \end{aligned} \tag{7.28}$$

and similarly

$$\psi^1_p(A) \leq \phi^1_{p,q}(A). \tag{7.29}$$

Combining (7.27)–(7.29), we deduce that

$$\phi^0_{p,q}(A) = \psi^0_p(A) = \psi^1_p(A) = \phi^1_{p,q}(A) \quad \text{if } p \notin \mathcal{D}_q,$$

where $\mathcal{D}_q$ is given in Theorem 4.1 (see also Theorem 4.2). This proves (7.7) whenever $p \notin \mathcal{D}_q$, since $\mathcal{F}$ is generated by the increasing finite-dimensional cylinders.

In order to show that

$$\phi^0_{p,q}(A) = \psi^0_p(A), \quad \phi^1_{p,q}(A) = \psi^1_p(A),$$

for *all* p and any such event A, it suffices to show that $\psi^0_p(A)$ is left-continuous in p, and $\psi^1_p(A)$ is right-continuous (the conclusion will then follow by Proposition 4.4). We confine ourselves here to the case of $\psi^0_p(A)$, since the other case is exactly similar.

Fix $p \in (0, 1)$, and let A be an increasing finite-dimensional event of $\mathcal{F}$, defined in terms of the edges in the finite set F. Let

$$B_p = \{\zeta \in X : \pi_p \zeta \in A\}, \quad C_p = \{\zeta \in X : \pi^p \zeta \in A\}$$

be the corresponding events in $\mathcal{B}$, and note, from (6.4)–(6.5), that B_p is increasing and open, and that C_p is increasing and closed. Furthermore, $C_{p-\epsilon} \subseteq B_p$ if

$\epsilon > 0$, and $B_p \setminus C_{p-\epsilon} \to \varnothing$ as $\epsilon \downarrow 0$. We have by stochastic monotonicity that $\lim_{t\to\infty} P(Z_t^0 \in B_p)$ exists, and by weak convergence (see Theorem 7.2) that

$$\lim_{t\to\infty} P(Z_t^0 \in B_p) \geq \mu(B_p).$$

We claim further that $P(Z_t^0 \in B_p) \leq \mu(B_p)$ for all t, whence

$$P(Z_t^0 \in B_p) \to \mu(B_p) \quad \text{as } t \to \infty. \tag{7.30}$$

To see the claim, suppose $P(Z_T^0 \in B_p) > \mu(B_p) + \eta$ for some T and $\eta > 0$. Then $P(Z_t^0 \in C_{p-\epsilon}) > \mu(C_{p-\epsilon}) + \frac{1}{2}\eta$ for some $\epsilon > 0$ and for all $t \geq T$. This contradicts the fact that $Z_t^0 \Rightarrow \mu$, since $C_{p-\epsilon}$ is closed.

Now, for $h > 0$,

$$\begin{aligned}\psi_p^0(A) - \psi_{p-h}^0(A) &= \lim_{t\to\infty} \Big\{ P(Z_t^0 \in B_p) - P(Z_t^0 \in B_{p-h}) \Big\} \\ &= \mu(B_p \backslash B_{p-h}) \qquad \text{by (7.30)}.\end{aligned}$$

However $B_p \backslash B_{p-h} \to \varnothing$ as $h \downarrow 0$ since B_p and B_{p-h} are open; hence $\psi_{p-h}^0(A) \to \psi_p^0(A)$ as $h \downarrow 0$.

In the corresponding argument for $\psi_p^1(A)$, the set B_p is replaced by the increasing closed event C_p, and the difference $B_p \backslash B_{p-h}$ is replaced by $C_{p+h} \backslash C_p$.

Finally we prove that $L_{p,t}^0$ is reversible; the argument is similar for $L_{p,t}^1$. Let f and g be increasing cylinder functions mapping Ω to $\mathbb{R}$, and let $U_{\Lambda,t}^0$ (resp. U_t^0) be the transition semigroup of the process $\pi_p Z_{\Lambda,t}^0$ (resp. $L_t^0 = \pi_p Z_t^0$). If $\Lambda \subseteq \Delta$ then

$$f(\eta) U_{\Lambda,t}^0 g(\eta) \leq f(\eta) U_{\Delta,t}^0 g(\eta) \leq f(\eta) U_t^0 g(\eta), \quad \eta \in \Omega,$$

by Lemmas 6.2 and 6.4. Therefore

$$\phi_{\Delta,p,q}^0\Big(f(\eta)U_{\Lambda,t}^0 g(\eta)\Big) \leq \phi_{\Delta,p,q}^0\Big(f(\eta)U_{\Delta,t}^0 g(\eta)\Big) \leq \phi_{p,q}^0\Big(f(\eta)U_t^0 g(\eta)\Big) \quad \text{if } \Lambda \subseteq \Delta,$$

since $\phi_{\Delta,p,q}^0 \leq \phi_{p,q}^0$. Take the limits as $\Delta \to \mathbb{Z}^d$ and $\Lambda \to \mathbb{Z}^d$, and use the monotone convergence theorem to deduce that

$$\phi_{\Delta,p,q}^0\Big(f(\eta)U_{\Delta,t}^0 g(\eta)\Big) \to \phi_{p,q}^0\big(f(\eta)U_t^0 g(\eta)\big) \quad \text{as } \Delta \to \mathbb{Z}^d. \tag{7.31}$$

The left side of (7.31) is unchanged when f and g are exchanged, by the reversibility of $\pi_p Z_{\Delta,t}^0$ (see Theorem 6.1). Therefore the right side of (7.31) is unchanged by this exchange, implying the required reversibility (see [48, p. 91]).

(b) It suffices to prove (7.8) for increasing finite-dimensional events A, since such events generate $\mathcal{F}$. For such A, (7.8) follows from (7.30) in the case of $\phi_{p,q}^0$, and similarly for $\phi_{p,q}^1$. □

Proof of Proposition 7.4. This is an immediate consequence of Theorem 7.3(b). □

Proof of Theorem 5.2(a). This was deferred from Section 5. We follow the argument of [9] as reported in [26]. For $p \in (0,1)$ and $\zeta \in X$, we call an edge e *p-open* if $\pi_p\zeta(e) = 1$, which is to say that $\zeta(e) > 1-p$. Let $C_p = C_p(\zeta)$ be the p-open cluster of $\mathbb{L}$ containing the origin, and note that $C_{p'} \subseteq C_p$ if $p' \le p$.

The function θ^0 is defined by (5.1) in terms of the measure $\phi^0_{p,q}$. In the light of Theorem 7.3(b), we have that

$$\theta^0(p,q) = \mu(|C_p| = \infty),$$

where μ is given in Theorem 7.2. Therefore

(7.32)
$$\begin{aligned}\theta^0(p,q) - \theta^0(p-,q) &= \lim_{p'\uparrow p} \mu\big(|C_p| = \infty,\ |C_{p'}| < \infty\big)\\ &= \mu\big(|C_p| = \infty,\ |C_{p'}| < \infty \text{ for all } p' < p\big).\end{aligned}$$

Let $p > p_c(q)$ and suppose $|C_p| = \infty$. If $p_c(q) < \alpha < p$, there exists a.s. an α-open infinite cluster I_α, and furthermore I_α is a.s. a subgraph of C_p, since otherwise there would exist at least two infinite p-open clusters (an event having probability 0, by Theorem 3.3). It follows that there exists a p-open path π joining the origin to some vertex of I_α. Such a path π has finite length and each edge e in π satisfies $\zeta(e) > 1-p$; therefore $\beta = \min\{\zeta(e) : e \in \pi\}$ satisfies $\beta > 1-p$. If p' satisfies $p' \ge \alpha$ and $1-\beta < p' < p$ then there exists a p'-open path joining the origin to some vertex of I_α, so that $|C_{p'}| = \infty$. However $p' < p$, implying that the event on the right-hand side of (7.32) has probability zero, as required. $\square$

Proof of non-Feller property. Finally we show (as promised before Lemma 7.5) that the processes $L^b_{p,t}$ are not Feller. For simplicity we take $d = 2$ and $b = 0$; a similar argument is valid for $d > 2$ and/or $b = 1$. Take e to be the edge with endpoints $(0,0)$ and $(1,0)$, and let f be the indicator function of the event that the edge e is open. We shall show that the function $U^0_s f$ is not continuous for sufficiently small positive values of s, where U^0_s is the transition semigroup associated with $L^0_{p,t}$. Let V be the set of vertices $x = (x_1, x_2)$ satisfying

$$\text{either } x_1 \ge |x_2| + 1, \quad \text{or } -x_1 \ge |x_2|,$$

and let $\mathbb{E}_V$ be the set of edges having both endpoints in V; note that $e \in \mathbb{E}_V$. Fix a positive integer n, and let Δ be the box $[-n,n]^2$. Let ω^0, ω^1 ($\in \Omega$) be the configurations given by

$$\omega^b(f) = \begin{cases} 1 & \text{if } f \in \mathbb{E}_\Delta \cap \mathbb{E}_V,\\ 0 & \text{if } f \in \mathbb{E}_\Delta \setminus \mathbb{E}_V,\\ b & \text{otherwise,}\end{cases} \qquad \text{where } b = 0,1.$$

Note that ω^0 and ω^1 depend on n, and also that $\omega^1 \notin D^0(e)$ but $\omega^0 \in D^0(e)$. Taking ω^0 and ω^1 as initial configurations, we claim that this property persists with strictly positive probability for a non-zero time-interval, under the evolution according to the appropriate semigroup U^0_s.

For $b = 0, 1$, let $K^b_{\Lambda,t}$ be the process $\pi_p Z^{(\zeta^b,0)}_{\Lambda,t}$ for some ζ^b satisfying $\omega^b = \pi_p \zeta^b$; the value of ζ^b is otherwise immaterial. We write $K^b_t = \lim_{\Lambda\to\mathbb{Z}^d} K^b_{\Lambda,t}$, which limit exists by the usual monotonicity. We claim that there exist ϵ, η (> 0), not depending on the value of n, such that

$$P\big(K^1_\eta(e) = 1,\ K^0_\eta(e) = 0\big) > \epsilon. \tag{7.33}$$

This implies that

$$P(K^1_\eta(e) = 1) - P(K^0_\eta(e) = 1) > \epsilon,$$

irrespective of the value of n, and therefore that the semigroup U^0_s is not Feller. In order to prove (7.33), we use a percolation argument. Let $\eta > 0$. For each edge f, we set $X_f = 0$ if none of the processes $A(f)$, $B(f)$, $C(f)$ have fired during the time-interval $[0, \eta]$, and $X_f = 1$ otherwise. Since the sum of the intensities of these three processes is $q + 1$, we have that $\{X_f : f \in \mathbb{E}\}$ is a family of independent Bernoulli variables with common parameter $1 - e^{-(q+1)\eta}$. Choose η sufficiently small such that

$$1 - e^{-(q+1)\eta} < \tfrac{1}{4},$$

noting that $\frac{1}{4}$ is less than the critical probability of bond percolation on the square lattice (see [26]). Routine percolation arguments may now be used to obtain that there exists $\epsilon' > 0$ such that

$$P\Big(K^1_{\Lambda,\eta} \notin D^0(e),\ K^0_{\Lambda,\eta} \in D^0(e) \text{ for all } t \in [0, \eta] \,\Big|\, X_e = 0\Big) > \epsilon',$$

for all Λ containing $[-2n, 2n]^2$. Suppose that $A(e)$ and $B(e)$ do not fire during $[0, \eta]$, but that $C(e)$ does indeed fire once, with an associated value σ satisfying $\sigma < 1 - p$. At this time T of firing, the edge e is removed from the lower process $K^0_{\Lambda,T}$ but not from the upper process $K^1_{\Lambda,T}$, for all large Λ. Therefore

$$P\Big(K^1_{\Lambda,\eta}(e) = 1,\ K^0_{\Lambda,\eta}(e) = 0\Big) > \epsilon, \quad \text{for all } \Lambda \text{ containing } [-2n, 2n]^2,$$

with $\epsilon = \epsilon'(1 - p)e^{-2\eta}\{(q - 1)\eta e^{-(q-1)\eta}\}$. Now take the limit as $\Lambda \to \mathbb{Z}^d$ to obtain (7.33). $\square$

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