Entanglement in the quantum Ising model

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Abstract

We study the asymptotic scaling of the entanglement of a block of spins for the ground state of the one-dimensional quantum Ising model with transverse field. When the field is sufficiently strong, the entanglement grows at most logarithmically in the number of spins. The proof utilises a transformation to a model of classical probability called the continuum random-cluster model, and is based on a property of the latter model termed ratio weak-mixing. Our proof applies equally to a large class of disordered interactions.

1 The quantum Ising model

The quantum Ising model in a transverse magnetic field is one of the most famous examples of exactly solvable one-dimensional quantum models. The solution was first given by Pfeuty in [26], based on earlier works by Lieb, Schultz, and Mattis [18] and by McCoy [21]. The diagonalisation of the Hamiltonian and the determination of the energy eigenstates is based on methods developed by Jordan and Wigner [16] in the theory of second quantisation of fermion fields, and by Bogoliubov [7] in the theory of superconductivity. This model exhibits a second-order phase transition in the ground state when the temperature of the system is zero. The existence of the phase transition and

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the computation of the spin–spin correlation functions were studied in [26]; rigorous results for the correlation functions in the presence of disorder are provided in [1, 9].

Quantum systems, unlike classical systems, can support composite pure states for which it is impossible to assign a definite state to two or more subsystems. States with this property are known as entangled states and have attracted a great deal of interest recently due to their resource-like properties. The investigation of the entanglement properties of strongly interacting quantum spin systems, with a view toward quantum phase transitions, was initiated by Osterloh et al. [25] and by Osborne and Nielsen [24] (see, for example, [4] and the references therein for further studies). It is now understood that the strength of quantum entanglement is related to the number of parameters required to describe a quantum state classically. Thus, for 1D systems, the scaling of the *geometric entropy* — the degree of entanglement of a distinguished subsystem with respect to the rest — has emerged as the crucial parameter which quantifies whether the state is hard or easy to simulate [28]. It has been conjectured that the entropy of entanglement obeys an area law, scaling as the boundary area in the subcritical phase, with a possible logarithmic correction for the critical phase. There is a paucity of rigorous results concerning the scaling of the entanglement of a block for the quantum Ising model; the above results are typically obtained by numerical calculations, or conformal field theory methods [4]. There are some rigorous derivations of the scaling of the entropy function for certain 1D spin models (specialised essentially to the XY model), see [4] for further references.

In this paper, we utilise a new method for studying the entanglement properties of the quantum Ising model. This is based on a representation formulated by Aizenman, Klein, and Newman [1] of the model in terms of a continuum random-cluster model on a certain space–time graph. (See also the earlier paper [9].) Using a technique termed ratio weak-mixing, developed by Alexander [2, 3] for random-cluster and Potts models on discrete lattices, we prove a bound on the entanglement entropy in the subcritical regime, when the magnetic field intensity is strong compared to the spin coupling.

The quantum Ising model is defined as follows. Let $L \geq 0$. For $m \geq 0$, let $\Delta_m = \{-m, -m+1, \dots, m+L\}$ be a subset of the one-dimensional lattice $\mathbb{Z}$, and attach to each vertex $x \in \Delta_m$ a quantum spin-$\frac{1}{2}$ with local Hilbert space $\mathbb{C}^2$. The Hilbert space $\mathcal{H}$ for the system is $\mathcal{H} = \bigotimes_{x=-m}^{m+L} \mathbb{C}^2$. A convenient basis for each spin is provided by the two eigenstates $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, of the Pauli operator

$$\sigma_x^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

at the site x, corresponding to the eigenvalues ± 1. The other two Pauli operators with respect to this basis are represented by the matrices

$$\sigma_x^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_x^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1.1}$$

A complete basis for $\mathcal{H}$ is given by the tensor products (over x) of the eigenstates of $\sigma_x^{(3)}$. In the following, $|\phi\rangle$ denotes a vector and $\langle\phi|$ its adjoint. As a notational convenience in this paper, we shall represent sub-intervals of $\mathbb{Z}$ as real intervals, writing for example $\Delta_m = [-m, m+L]$.

The spins in Δ_m interact via the quantum Ising Hamiltonian

$$H_m = -\frac{1}{2}\sum_{\langle x,y\rangle} \lambda_{x,y}\sigma_x^{(3)}\sigma_y^{(3)} - \sum_x \delta_x \sigma_x^{(1)}, \tag{1.2}$$

generating the operator $e^{-\beta H_m}$ where β denotes inverse temperature. Here, $\lambda_{x,y} \ge 0$ and $\delta_x \ge 0$ are the spin-coupling and external-field intensities, respectively, and $\sum_{\langle x,y\rangle}$ denotes a sum over all (distinct) unordered pairs of spins. We concentrate here on the case of interactions between neighbouring spins: $\lambda_{x,y} = 0$ for $|x-y| \ge 2$. While we shall phrase our results for the translation-invariant case $\lambda_{x,x+1} = \lambda$ and $\delta_x = \delta$, our approach can be extended to random couplings satisfying the condition

$$\mathbb{P}(\lambda_{x,y} < \lambda) = \mathbb{P}(\delta_x > \delta) = 1, \tag{1.3}$$

with $\theta \equiv \lambda/\delta$ a sufficiently small constant (see Section 8). The ensuing Hamiltonian has a unique pure ground state $|\psi_m\rangle$ defined at $T = 0$ ($\beta \to \infty$) as the eigenvector corresponding to the lowest eigenvalue of H_m. In the translation-invariant case the ground state $|\psi_m\rangle$ depends only on the ratio θ.

For definiteness, we shall work here with a free boundary condition on Δ_m, but we note that the same methods are valid with a periodic (or wired) boundary condition, in which Δ_m is embedded on a circle. One difference worthy of note is that the correlation functions of the *critical* model are expected to depend on the choice of boundary conditions, see [26].

We write $\rho_m(\beta) = e^{-\beta H_m}/\operatorname{tr}(e^{-\beta H_m})$, and

$$\rho_m = \lim_{\beta\to\infty} \rho_m(\beta) = |\psi_m\rangle\langle\psi_m|$$

for the density operator corresponding to the ground state of the system. The existence of the limit follows by random-cluster methods, see [1], and we return to this in Section 3. The ground-state entanglement of $|\psi_m\rangle$ is quantified by partitioning the spin chain Δ_m into two disjoint sets $[0, L]$ and $\Delta_m \setminus [0, L]$ and by considering the entropy of the *reduced density operator*

$$\rho_m^L = \mathrm{tr}_{\Delta_m \setminus [0,L]}(|\psi_m\rangle\langle\psi_m|). \tag{1.4}$$

One may similarly define, for finite β, the reduced operator $\rho_m^L(\beta)$. In both cases, the trace is performed over the Hilbert space $(\bigotimes_{x=-m}^{-1} \mathbb{C}^2) \otimes (\bigotimes_{x=L+1}^{m+L} \mathbb{C}^2)$ of the spins belonging to $\Delta_m \setminus [0, L]$. Note that ρ_m^L is a positive semi-definite operator on the Hilbert space $\mathcal{H}_L$ of dimension $d = 2^{L+1}$ of spins indexed by the interval $[0, L]$. By the spectral theorem for normal matrices [6], this operator may be diagonalised and has real, non-negative eigenvalues, which we denote $\lambda_j^{\downarrow}(\rho_m^L)$. The arrow indicates that the eigenvalues are arranged in decreasing order.

Definition 1.5. *The* entanglement *of the interval* $[0, L]$ *relative to its complement* $\Delta_m \setminus [0, L]$ *is given by*

$$S(\rho_m^L) = -\,\mathrm{tr}(\rho_m^L \log_2 \rho_m^L). \tag{1.6}$$

This quantity may be expressed thus in terms of the eigenvalues of ρ_m^L:

$$S(\rho_m^L) = -\sum_{j=1}^{2^{L+1}} \lambda_j^{\downarrow}(\rho_m^L) \log_2 \lambda_j^{\downarrow}(\rho_m^L), \tag{1.7}$$

where $0 \log_2 0$ is interpreted as 0.

In Section 2, we prove our main theorem: the order of the entanglement scaling is at most $\log_2 L$ for the ground state in the subcritical regime. This result follows as a corollary of the main estimate, given by Theorem 6.5, in Section 6. In Sections 3–4, we describe the mapping of the density operator of the quantum Ising model to a stochastic integral in terms of a Poisson measure (as in [1]). The mapping begins by considering states with $\beta < \infty$ and deriving the ground state in the limit $\beta \to \infty$. This allows us to express the matrix elements of the ground state in terms of a continuous percolation model on a two-dimensional space–time graph, with one continuous axis describing time. In this setting, the elements of the reduced state are related to a random-cluster model on the same graph, but with the addition of a 'slit' along the interval $[0, L]$ at time 0. The continuum random-cluster model is presented in detail in Section 5. Section 6 contains the main result, which allows us to establish the scaling of the entanglement entropy, while in Section 7 we explain the technique of ratio-weak mixing on which the proof is based. The extension of our results to disordered systems is discussed in Section 8.

2 Entropy of the reduced state

In this section, we study the behaviour of the entropy of the reduced state ρ_m^L in the subcritical regime (with $\theta = \lambda/\delta$ small). In order to derive an adequate upper bound on the entropy, we shall analyze the influence on the spectrum of the reduced density operator produced by imposing a change in the boundary conditions of the spin chain. Specifically, we consider the distance between the largest eigenvalues of two states defined on $[0, L]$ with respect to two different lattices, Δ_m, Δ_n, with $m \le n$. The entropy will be estimated by studying the operator norm

$$\|\rho_m^L - \rho_n^L\| \equiv \sup_{\|\psi\|=1} \bigl|\langle\psi|\rho_m^L - \rho_n^L|\psi\rangle\bigr|, \tag{2.1}$$

where the supremum is taken over all vectors $|\psi\rangle \in \mathcal{H}_L$ with unit L^2-norm belonging to the Hilbert space $\mathcal{H}_L$ of spins in $[0, L]$. We shall see in Section 4 that $\|\rho_m^L - \rho_n^L\|$ may be expressed in terms of a certain random-cluster representation of the quantum Ising model. In Sections 6 and 7 we shall use a coupling of random-cluster measures and the method of 'ratio weak-mixing' to prove the following.

Theorem 2.2. *Let $\lambda, \delta \in (0, \infty)$ and write $\theta = \lambda/\delta$. There exist constants $\alpha, C \in (0, \infty)$ depending on θ only, and a constant $\gamma = \gamma(\theta)$ satisfying $0 < \gamma < \infty$ if $\theta < 1$, such that, for all $L \ge 1$,*

$$\|\rho_m^L - \rho_n^L\| \le \min\{2, CL^\alpha e^{-\gamma m}\}, \qquad 2 \le m \le n. \tag{2.3}$$

Furthermore, we may find such γ satisfying $\gamma \to \infty$ as $\theta \downarrow 0$.

Proof. That $\|\rho_m^L - \rho_n^L\| \le 2$ is a consequence of the fact that the ρ_m^L are density operators. An upper bound of the form $C'L^\alpha e^{-\gamma m}$ holds by Theorem 6.5 and the preceding discussion whenever $m \ge M$ for suitable $M = M(\theta)$. Inequality (2.3) follows on replacing C' by $C = e^{\gamma M} \max\{C', 2\}$. □

We shall apply (2.3) iteratively in order to obtain an upper bound for the decay of the vector of eigenvalues $\{\lambda_j^\downarrow(\rho_m^L) : j = 1, 2, \dots\}$, valid for all large m. The proof makes use of the following decomposition property, valid for any pure state of a bipartite system, see [23].

Theorem 2.4 (Schmidt decomposition). *Let $|\psi_m\rangle$ be the pure ground state of the composite system $[0, L] \cup (\Delta_m \setminus [0, L])$. There exist orthonormal bases $\{|u_j\rangle_{[0,L]}, |v_k\rangle_{\Delta_m\setminus[0,L]}\}$ for the states of $[0, L], \Delta_m \setminus [0, L]$ respectively, such that*

$$|\psi_m\rangle = \sum_{j=1}^{s} \sqrt{\lambda_j^\downarrow(\rho_m^L)}\, |u_j\rangle_{[0,L]}|v_j\rangle_{\Delta_m\setminus[0,L]}, \tag{2.5}$$

where s, the Schmidt rank, *is given by $s = \min\{2^{L+1}, 2^{2m}\}$.*

Proof. We begin by writing $|\psi_m\rangle$ in terms of an orthonormal basis $|\alpha\rangle_{[0,L]}|\beta\rangle_{\Delta_m\setminus[0,L]}$ where $|\alpha\rangle_{[0,L]}$ (respectively, $|\beta\rangle_{\Delta_m\setminus[0,L]}$) is an orthonormal basis for the spins in $[0, L]$ (respectively, $\Delta_m \setminus [0, L]$):

$$|\psi_m\rangle = \sum_{\alpha=1}^{2^{L+1}} \sum_{\beta=1}^{2^{2m}} \psi^{[m]}_{\alpha\beta} \, |\alpha\rangle_{[0,L]}|\beta\rangle_{\Delta_m\setminus[0,L]},$$

where

$$\sum_{\alpha=1}^{2^{L+1}} \sum_{\beta=1}^{2^{2m}} |\psi^{[m]}_{\alpha\beta}|^2 = 1.$$

The coefficients $\psi^{[m]}_{\alpha\beta}$ constitute a $2^{L+1} \times 2^{2m}$ matrix and, as such, we can apply the singular-value decomposition [6] to write

$$\psi^{[m]}_{\alpha\beta} = \sum_{j=1}^{s} U_{\alpha j} d_j V_{j\beta},$$

where $s = \min\{2^{L+1}, 2^{2m}\}$, $U_{\alpha j}$ is a $2^{L+1} \times s$-sized isometry, $d_j \geq 0$ for all $j = 1, 2, \dots, 2^{L+1}$, and $V_{j\beta}$ is an $s \times 2^{2m}$-sized isometry. Defining

$$|u_j\rangle_{[0,L]} = \sum_{\alpha=1}^{s} U_{\alpha j}|\alpha\rangle_{[0,L]}, \quad |v_j\rangle_{\Delta_m\setminus[0,L]} = \sum_{\beta=1}^{s} V_{j\beta}|\beta\rangle_{\Delta_m\setminus[0,L]},$$

we see, because U and V are isometries, that $\{|u_j\rangle_{[0,L]}\}$ and $\{|v_j\rangle_{\Delta_m\setminus[0,L]}\}$ are orthonormal sets of vectors for the spins in $[0, L]$ and $\Delta_m \setminus [0, L]$, respectively.

A simple computation shows that the reduced density operator ρ^L_m for the spins in $[0, L]$ is given by

$$\rho^L_m = \sum_{j=1}^{s} d_j^2 |u_j\rangle_{[0,L]}\langle u_j| \tag{2.6}$$

and so we identify $d_j = \sqrt{\lambda^\downarrow_j(\rho^L_m)}$, after re-ordering the index j if necessary. Note that the rank of ρ^L_m is less than or equal to the Schmidt rank of $|\psi_m\rangle$. $\square$

We compute the entanglement of $[0, L]$ with respect to the rest of the system as in (1.7),

$$S(\rho^L_m) = -\sum_{j=1}^{s} \lambda^\downarrow_j(\rho^L_m) \log_2 \lambda^\downarrow_j(\rho^L_m). \tag{2.7}$$

Here is our main theorem. With the exception of the natural logarithm function ln, all logarithms in the remainder of this section are taken to base 2.

Theorem 2.8. *Consider the quantum Ising model* (1.2) *on* $n = 2m + L + 1$ *spins, with parameters* λ, δ, *and let* γ, α, C *be as in Theorem 2.2. If* $\gamma > 4 \ln 2$, *there exist constants* c_1 *and* c_2 *depending on* γ *only such that*

$$S(\rho_m^L) \le c_1 \log_2 L + c_2, \qquad m \ge 0. \tag{2.9}$$

In summary, the entanglement entropy $S(\rho_m^L)$ is at most logarithmic in L if the field strength δ is sufficiently large. The bound $4 \ln 2$ is sufficient but not necessary, and may be improved with more care in the proof. We do not know how to replace this condition by $\gamma > 0$.

We believe that the upper bound (2.9) is, in many cases, not tight. For the translation-invariant subcritical case $\theta = \lambda/\delta < 2$ it is expected, on physical grounds, that the upper bound can be improved to a constant. (See [4] and the references therein for an extensive review of the physical arguments for entanglement scaling in non-critical and critical quantum spin models.) Renormalisation group arguments and conformal field theory methods suggest that, at a critical point, the upper bound should scale with $\log L$. For $\theta > 2$ the system enters the supercritical regime where the system has two degenerate ground states, and the ground state is no longer a pure state. Nonetheless, it is expected that the entropy of a block is again bounded by a constant. For higher dimensions $d \ge 2$ our argument breaks down because the number of non-zero Schmidt coefficients for a distinguished region grows too quickly for our perturbation argument.

The proof of Theorem 2.8 follows an iterative inductive procedure, where at each step the distance k from the boundary of $[0, L]$ is increased and the spectrum of the relative density operator ρ_k^L is estimated. We illustrate the procedure by the following simple case: consider the ground state $|\psi_0\rangle$ for the Ising model defined on only $L + 1$ spins. In this case the reduced density operator ρ_0^L for $[0, L]$ is exactly $\rho_0^L = |\psi_0\rangle\langle\psi_0|$, i.e., a pure state, with entropy $S(\rho_0^L) = 0$. When $m = 1$, the reduced density operator ρ_1^L for the region $[0, L]$ is mixed, but it has at most 2^2 non-zero eigenvalues. This follows from the Schmidt decomposition applied to the ground state $|\psi_1\rangle$ across the bipartition $[0, L] \cup (\Delta_1 \setminus [0, L])$. Thus, the entropy of the block $[0, L]$ is bounded above by $S(\rho_1^L) \le 2$. Consider now the reduced density operator ρ_k^L. By the Schmidt decomposition, the operator ρ_k^L has at most 2^{2k} non-zero eigenvalues. Assume that $2k < L+1$, and consider the addition of a single spin at either boundary. The new reduced density operator ρ_{k+1}^L has at most four times as many non-zero eigenvalues as ρ_k^L. However, by (2.3),

$$\|\rho_k^L - \rho_{k+1}^L\| \le \min\{2, CL^\alpha e^{-\gamma k}\}, \tag{2.10}$$

so that the eigenvalues of ρ_k^L remain close to those of ρ_{k+1}^L.

Proof of Theorem 2.8. Let $K = \lceil \gamma^{-1} \ln(CL^\alpha) \rceil$, with C, α, γ as in Theorem 2.2. We shall assume that $m, K \geq 2$, $\gamma > 4 \ln 2$. There are two cases, depending on whether $m \leq K$ or $m > K$. Assume first that $2 \leq m \leq K$. The rank of ρ_m^L equals the Schmidt rank 2^{2m} of $|\psi_m\rangle$. Therefore,

$$S(\rho_m^L) \leq \sup_\rho \left\{ -\sum_{j=1}^{2^{2m}} \rho_j \log \rho_j \right\},$$

where the supremum is over all non-negative sequences $\rho = (\rho_j : 1 \leq j \leq 2^{2m})$ with sum 1. Hence,

$$S(\rho_m^L) \leq \log 2^{2m} = 2m \leq 2K, \qquad m \leq K. \tag{2.11}$$

Assume next that $m \geq K$. We shall apply the following theorem, see [6].

Theorem 2.12 (Weyl perturbation theorem). *For Hermitian operators A and B on a Hilbert space of dimension n,*

$$\max_j \left| \lambda_j^\downarrow(A) - \lambda_j^\downarrow(B) \right| \leq \|A - B\|. \tag{2.13}$$

Let $\epsilon(r) = CL^\alpha e^{-\gamma(K+r)}$, and note by the definition of K that

$$\epsilon(r) \leq e^{-\gamma r}, \qquad r \geq 0. \tag{2.14}$$

Setting $A = \rho_K^L$, $B = \rho_{K+1}^L$ in Theorem 2.12, we deduce by (2.3) that

$$\max_j \left| \lambda_j^\downarrow(\rho_K^L) - \lambda_j^\downarrow(\rho_{K+1}^L) \right| \leq \epsilon(0). \tag{2.15}$$

Therefore,

$$\begin{aligned} &|\lambda_j^\downarrow(\rho_{K+1}^L)| \leq \lambda_j^\downarrow(\rho_K^L) + \epsilon(0), && j = 1, 2, \ldots, 2^{2K}, \\ &|\lambda_j^\downarrow(\rho_{K+1}^L)| \leq \epsilon(0), && j = 2^{2K} + 1, 2^{2K} + 2, \ldots, 2^{2(K+1)}. \end{aligned} \tag{2.16}$$

We shall now iterate this process in order to obtain a bound on the eigenvalues of ρ_{K+r}^L, for $r \geq 1$. There are three cases:

(i) $j \leq 2^{2K}$, in which case

$$\lambda_j^\downarrow(\rho_{K+r}^L) \leq \lambda_j^\downarrow(\rho_K^L) + \sum_{l=0}^{r-1} \epsilon(l); \tag{2.17}$$

(ii) $2^{2K} \le 2^{2(K+s)} < j \le 2^{2(K+s+1)} \le 2^{2(K+r)}$, in which case

$$\lambda_j^{\downarrow}(\rho_{K+r}^L) \le \sum_{l=s}^{r-1} \epsilon(l); \tag{2.18}$$

(iii) $2^{2(K+r)} < j$, in which case

$$\lambda_j^{\downarrow}(\rho_{K+r}^L) = 0. \tag{2.19}$$

Let $s = \lfloor \frac{1}{2} \log j \rfloor - K$, so that, by (2.14),

$$\begin{aligned} \lambda_j^{\downarrow}(\rho_m^L) &\le \lambda_j^{\downarrow}(\rho_K^L) + \sum_{l=0}^{\infty} e^{-\gamma l}, && j \le 2^{2K}, \\ \lambda_j^{\downarrow}(\rho_m^L) &\le \sum_{l=s}^{\infty} e^{-\gamma l}, && 2^{2K} < j, \end{aligned}$$

which is to say that

$$\begin{aligned} \lambda_j^{\downarrow}(\rho_m^L) &\le \lambda_j^{\downarrow}(\rho_K^L) + c_0, && j \le 2^{2K}, \\ \lambda_j^{\downarrow}(\rho_m^L) &\le c_0 e^{-\gamma s}, && 2^{2K} < j, \end{aligned} \tag{2.20}$$

where

$$c_0 = \frac{1}{1 - e^{-\gamma}} \le \frac{4}{3}. \tag{2.21}$$

By (2.20),

$$\lambda_j^{\downarrow}(\rho_m^L) \le c_0' j^{-\xi}, \qquad 2^{2K} < j, \tag{2.22}$$

where $\xi = \gamma/(2 \ln 2) > 2$ and $c_0' = c_0'(L) = c_0 e^{\gamma(K+1)}$.

By (1.7),

$$S(\rho_m^L) = S_1 + S_2, \tag{2.23}$$

where

$$S_1 = -\sum_{j=1}^{\nu} \lambda_j^{\downarrow}(\rho_m^L) \log \lambda_j^{\downarrow}(\rho_m^L), \quad S_2 = -\sum_{j=\nu+1}^{2^{L+1}} \lambda_j^{\downarrow}(\rho_m^L) \log \lambda_j^{\downarrow}(\rho_m^L),$$

where ν ($\ge 2^{2(K+2)}$) is an integer to be chosen later. We shall bound S_1 and S_2 separately. Since the $\lambda_j^{\downarrow}(\rho_m^L)$, $1 \le j \le \nu$, are non-negative with sum Q satisfying $Q \le 1$,

$$S_1 \le \log \nu. \tag{2.24}$$

We shall use the tail estimate (2.22) to bound S_2, making use of the fact that the function $f(x) = -x \log x$ satisfies: $f(0) = 0$, and $f(x) < f(y)$ whenever $0 < x < y < e^{-1}$.

By (2.21), (2.22), and the definition of ξ,

$$\lambda_j^{\downarrow}(\rho_m^L) \le \frac{c_0'}{j^\xi} < e^{-1}, \qquad j \ge 2^{2(K+2)},$$

and so, recalling that $\nu \ge 2^{2(K+2)}$ and $\xi > 2$,

$$\begin{aligned} S_2 &\le -\sum_{j=\nu+1}^{2^{L+1}} \frac{c_0'}{j^\xi} \log\left(\frac{c_0'}{j^\xi}\right) \le -\sum_{j=\nu+1}^{\infty} \frac{c_0'}{j^\xi} \log\left(\frac{c_0'}{j^\xi}\right) \\ &\le -[c_0' \log c_0'] \sum_{j=\nu+1}^{\infty} \frac{1}{j^\xi} + \frac{\xi c_0'}{\ln 2} \sum_{j=\nu+1}^{\infty} \frac{\ln j}{j^\xi} \\ &\le |c_0' \log c_0'| \int_\nu^\infty \frac{1}{x^\xi}\, dx + \frac{\xi c_0'}{\ln 2} \int_\nu^\infty \frac{1}{x^\xi} \ln x \, dx \\ &\le \frac{c_0' \nu^{1-\xi}}{\xi - 1} \left(|\log c_0'| + \xi \log \nu + \frac{\xi}{\xi - 1} \right). \end{aligned}$$

We now set $\nu = \lceil e^{\gamma(K+1)} \rceil$ to obtain

$$S_2 \le c_1 K + c_2, \tag{2.25}$$

for suitable constants c_1, c_2 depending on γ only. By (2.23)–(2.25),

$$S(\rho_m^L) \le c_1' K + c_2', \qquad m \ge K, \tag{2.26}$$

which may be combined with (2.11) to obtain (2.9) with adjusted constants. $\square$

3 Percolation representation of the ground state

Aizenman, Klein, and Newman [1] derived a random-cluster representation for the thermal state of the quantum Ising Hamiltonian (1.2), thereby relating spin-correlation properties to graph-connectivity properties. In this representation, the thermal density operator, defined as

$$\rho_m(\beta) = \frac{e^{-\beta H_m}}{\operatorname{tr}(e^{-\beta H_m})}, \qquad \beta = T^{-1} > 0, \tag{3.1}$$

is described by a stochastic integral with respect to a Poisson measure. This Poisson measure is defined on the space–time graph $\Lambda_{m,\beta} = \Delta_m \times [0, \beta]$, generated by associating a continuous (imaginary) time variable $t \in [0, \beta]$ to each site $x \in \Delta_m$. We refer to a line of the form $\{x\} \times [0, \beta]$ as the *time-line* at the site x.

For completeness, we reproduce here the derivation of the random-cluster representation of the ground state, and we derive the corresponding representation for the reduced state on $[0, L]$. Note that the derivations are valid with the line Δ_m replaced by any finite graph G. By (1.2) with $\nu = \frac{1}{2}\sum_{\langle x,y\rangle} \lambda\mathbb{I}$ and $\mathbb{I}$ the identity operator,

$$e^{-\beta(H_m+\nu)} = e^{-\beta(U+V)}, \tag{3.2}$$

where

$$U = -\delta \sum_x \sigma_x^{(1)}, \quad V = -\frac{1}{2}\sum_{\langle x,y\rangle} \lambda(\sigma_x^{(3)}\sigma_y^{(3)} - \mathbb{I}),$$

and the second summation is over all neighbouring pairs in Δ_m. Although these two terms do not commute, we may use the so-called Lie–Trotter formula (see, for example, [27]) to factorize the exponential in (1.2) into *single-site* and *two-site* contributions due to U and V, respectively. By the Lie–Trotter formula,

$$e^{-(U+V)\Delta t} = e^{-U\Delta t}e^{-V\Delta t} + \mathrm{O}(\Delta t^2).$$

We divide the interval $[0, \beta]$ into N parts each of length $\Delta t = 1/N$, and deduce that

$$e^{-\beta(U+V)} = \lim_{\Delta t\to 0} \left(e^{-U\Delta t}e^{-V\Delta t}\right)^{\beta/\Delta t}. \tag{3.3}$$

We then expand the exponential, neglecting all terms of order $\mathrm{o}(\Delta t)$, to obtain

$$e^{-\beta(H_m+\nu)} = \lim_{\Delta t\to 0}\left(\prod_x \left[(1-\delta\Delta t)\mathbb{I} + \delta\Delta t P_x^1\right] \prod_{\langle x,y\rangle}\left[(1-\lambda\Delta t)\mathbb{I} + \lambda\Delta t P_{x,y}^3\right]\right)^{\beta/\Delta t}, \tag{3.4}$$

where $P_x^1 = \sigma_{(x)}^1 + \mathbb{I}$ and $P_{x,y}^3 = \frac{1}{2}(\sigma_x^{(3)}\sigma_y^{(3)} + \mathbb{I})$.

Let B be the set of basis vectors $|\sigma\rangle$ for $\mathcal{H}$ of the form $|\sigma\rangle = \bigotimes_x |\pm\rangle_x$. There is a natural one–one correspondence between B and the space $P = \prod_{x=-m}^{m+L}\{-1, +1\}$. We shall sometimes speak of members of P as basis vectors, and of $\mathcal{H}$ as the Hilbert space generated by P. Similarly, the space $\mathcal{H}_L$ of spins indexed by the interval $[0, L]$ may be viewed as being generated by $P_L = \prod_{x=0}^{L}\{-1, +1\}$.

The stochastic-integral representation may be obtained from (3.4) by inserting the resolution of the identity

$$\sum_{\sigma \in P} |\sigma\rangle\langle\sigma| = \mathbb{I} \tag{3.5}$$

between any two factors of the products. The product (3.4) contains a collection of operators acting on sites x and on neighbouring pairs $\langle x, y\rangle$. By labelling the time-segments as $\Delta t_1, \Delta t_2, \ldots, \Delta t_N$ in $[0, \beta]$, and neglecting terms of order $o(\Delta t)$, we may see that each given time-segment arising in (3.4) contains one of: the identity $\mathbb{I}$; an operator of the form P_x^1; an operator of the form $P_{x,y}^3$. Each such operator occurs in the time-segment with a certain weight.

Let us consider the action of these operators on the states $|\sigma\rangle$ for each infinitesimal time interval Δt_i, $i \in \{1, 2, \ldots, N\}$. The matrix elements of each of the single-site operators are given by

$$\langle\sigma'|\sigma_x^{(1)} + \mathbb{I}|\sigma\rangle = \delta_{\sigma'_x, \sigma_x} + \delta_{\sigma'_x, \overline{\sigma}_x} = 1, \tag{3.6}$$

where σ_x is the value of the spin at x in the (product) basis vector $|\sigma\rangle$, and $\overline{\sigma}_x$ is the opposite spin to σ_x. When it occurs in some time-segment Δt_i, we place a mark in the interval $\{x\} \times \Delta t_i$, and we call this mark a *death*. Such a death has a corresponding weight $\delta\Delta t + o(\Delta t)$.

The matrix elements involving neighbouring pairs $\langle x, y\rangle$ yield

$$\tfrac{1}{2}\langle\sigma'_x\sigma'_y|\sigma_x^{(3)}\sigma_y^{(3)} + \mathbb{I}|\sigma_x\sigma_y\rangle = \delta_{\sigma_x, \sigma'_x}\delta_{\sigma_y, \sigma'_y}\delta_{\sigma_x, \sigma_y}. \tag{3.7}$$

When this occurs in some time-segment Δt_i, we place a connection, called a *bridge*, between the intervals $\{x\} \times \Delta t_i$ and $\{y\} \times \Delta t_i$. Such a bridge has a corresponding weight $\lambda\Delta t + o(\Delta t)$.

In the limit $\Delta t \to 0$, the spin operators generate thus a Poisson process with intensity δ of deaths in each time-line $\{x\} \times [0, \beta]$, and a Poisson process with intensity λ of bridges between each pair $\{x\} \times [0, \beta]$, $\{y\} \times [0, \beta]$ of time-lines, for neighbouring x and y. This is an independent family of Poisson processes. We write D_x for the set of deaths at the site x, and $B_{x,y}$ for the set of bridges between neighbouring sites x and y. The configuration space is the set $\Omega_{m,\beta}$ containing all finite sets of deaths and bridges, and we may assume without loss of generality that no death is the endpoint of any bridge.

For two point $(x, s), (y, t) \in \Lambda_{m,\beta}$, we write $(x, s) \leftrightarrow (y, t)$ if there exists a path from the first to the second that traverses time-lines and bridges but crosses no death. A *cluster* is a maximal subset C of $\Lambda_{m,\beta}$ such that $(x, s) \leftrightarrow (y, t)$ for all $(x, s), (y, t) \in C$. Thus the connection relation $\leftrightarrow$ generates a percolation process on $\Lambda = \Lambda_{m,\beta}$, and we write $\mathbb{P}_{\Lambda,\lambda,\delta}$ for the probability measure corresponding to the weight function on the configuration space $\Omega_{m,\beta}$. That is, $\mathbb{P}_{\Lambda,\lambda,\delta}$ is the measure governing a family of independent Poisson processes of deaths (with intensity δ) and of bridges (with intensity λ). The ensuing percolation process has been studied in [5].

We shall later need to count the number of clusters of a configuration $\omega \in \Omega_{m,\beta}$ subject to any of four possible boundary conditions, of which we specify two next (the other two appear in the next section). The meaning of *periodic boundary condition* is that any clusters containing two points of the form $(x, 0)$ and (x, β), for some $x \in \Delta_m$, are deemed to be the same cluster, and they contribute only 1 to the total cluster count. The meaning of *wired boundary condition* is that any clusters containing two points of the form $(x, 0)$ and (y, β), for $x, y \in \Delta_m$, are deemed to be the same cluster and contribute only 1 to the total count. We write $k^{\mathrm{p}}(\omega)$ (respectively, $k^{\mathrm{w}}(\omega)$) for the number of clusters of ω subject to the periodic (respectively, wired) boundary condition. Note that $k^{\mathrm{w}}(\omega) - 1$ is the number of clusters of ω (with free boundary conditions) that do not intersect $[-m, m + L] \times \{0, \beta\}$.

Equations (3.6)–(3.7) are to be interpreted as saying the following. In calculating the operator $e^{-\beta(H_m+\nu)}$, one averages over contributions from realizations of the Poisson processes, on the basis that the quantum spins are constant on every cluster of the corresponding percolation process, and each such spin-function is equiprobable.

More explicitly,

$$e^{-\beta(H_m+\nu)} = \int d\mathbb{P}_{\Lambda,\lambda,\delta}(\omega) \left(\mathcal{T} \prod_{(x,t)\in D} \prod_{(\langle x,y\rangle,t')\in B} P_x^1(t) P_{x,y}^3(t') \right), \tag{3.8}$$

where $\mathcal{T}$ denotes the time-ordering of the terms in the products, and B (respectively, D) is the set of all bridges (respectively, deaths) of the configuration $\omega \in \Omega_{m,\beta}$. The $P_x^1(t)$ and $P_{x,y}^3(t)$ are to be interpreted as the relevant operators encountered at the deaths and bridges of ω.

Let $\omega \in \Omega_{m,\beta}$. Let $\Sigma(\omega) = \Sigma_{m,L}(\omega)$ be the space of all functions $s : \Delta_m \times [0, \beta] \to \{-1, +1\}$ that are constant on the clusters of ω, and let μ_ω be the counting measure on $\Sigma(\omega)$. Let $K(\omega)$ be the time-ordered product of operators in (3.8). We may evaluate the matrix elements of $K(\omega)$ by inserting the resolution of the identity between any two factors in the product, obtaining thus that

$$\langle \sigma' | K(\omega) | \sigma \rangle = \sum_{s\in\Sigma(\omega)} 1\{s(\cdot, 0) = \sigma\} 1\{s(\cdot, \beta) = \sigma'\}, \qquad \sigma, \sigma' \in P, \tag{3.9}$$

where $1\{A\}$, and later 1_A, denotes the indicator function of A. This is the number of spin-allocations to the clusters of ω with given spin-vectors at times 0 and β.

The matrix elements of the density operator $\rho_m(\beta)$ are therefore given by

$$\langle\sigma'|\rho_m(\beta)|\sigma\rangle = \frac{1}{Z_m}\int 1\{s(\cdot,0)=\sigma\}1\{s(\cdot,\beta)=\sigma'\}\, d\mu_\omega(s)\, d\mathbb{P}_{\Lambda,\lambda,\delta}(\omega), \quad (3.10)$$

for $\sigma,\sigma'\in P$, where

$$Z_m = Z_m(\beta) = \operatorname{tr}(e^{-\beta(H_m+\nu)}) \quad (3.11)$$

is the *partition function*. Thus,

$$\begin{aligned}\langle\sigma'|\rho_m(\beta)|\sigma\rangle &= \frac{1}{Z_m}\int d\mathbb{P}_{\Lambda,\lambda,\delta}(\omega)\sum_{s\in\Sigma(\omega)} 1\{s(\cdot,0)=\sigma\}1\{\sigma(\cdot,\beta)=\sigma'\}\\ &= \frac{1}{Z_m}\int d\mathbb{P}_{\Lambda,\lambda,\delta}(\omega)\, 2^{k^{\mathrm{w}}(\omega)-1}1_{E(\sigma,\sigma')}(\omega), \qquad \sigma,\sigma'\in P, \quad (3.12)\end{aligned}$$

where the final term in the integrand is the indicator function of the event $E(\sigma,\sigma')$ containing all $\omega\in\Omega_{m,\beta}$ such that: for all $x,y\in[-m,m+L]$:

$$\begin{aligned}(x,0)&\nleftrightarrow(y,0) &&\text{whenever} && \sigma_x\neq\sigma_y,\\ (x,\beta)&\nleftrightarrow(y,\beta) &&\text{whenever} && \sigma'_x\neq\sigma'_y,\\ (x,0)&\nleftrightarrow(y,\beta) &&\text{whenever} && \sigma_x\neq\sigma'_y.\end{aligned}$$

See Figure 1 for an illustration of the space–time configurations contributing to the Poisson integral (3.12) for the matrix elements of $\rho_m(\beta)$.

On setting $\sigma=\sigma'$ in (3.12) and summing over $\sigma\in P$, we find that

$$Z_m = \operatorname{tr}(e^{-\beta(H_m+\nu)}) = \int 2^{k^{\mathrm{p}}(\omega)}\, d\mathbb{P}_{\Lambda,\lambda,\delta}(\omega). \quad (3.13)$$

A method was developed in [1] (as amplified in the next section) to represent $\langle\sigma'|\rho_m(\beta)|\sigma\rangle$ as a certain probability, and to prove that it converges as $\beta\to\infty$. In particular, it was shown in [1] that the ground state $\rho_m=|\psi_m\rangle\langle\psi_m|$ satisfies

$$\rho_m = \lim_{\beta\to\infty}\frac{1}{Z_m}e^{-\beta(H_m+\nu)}. \quad (3.14)$$

4 Percolation representation of the reduced state

The analysis of the last section may be repeated for the reduced density operator $\rho_m^L(\beta)$ by tracing (3.8) over a complete set of states of the spins indexed by

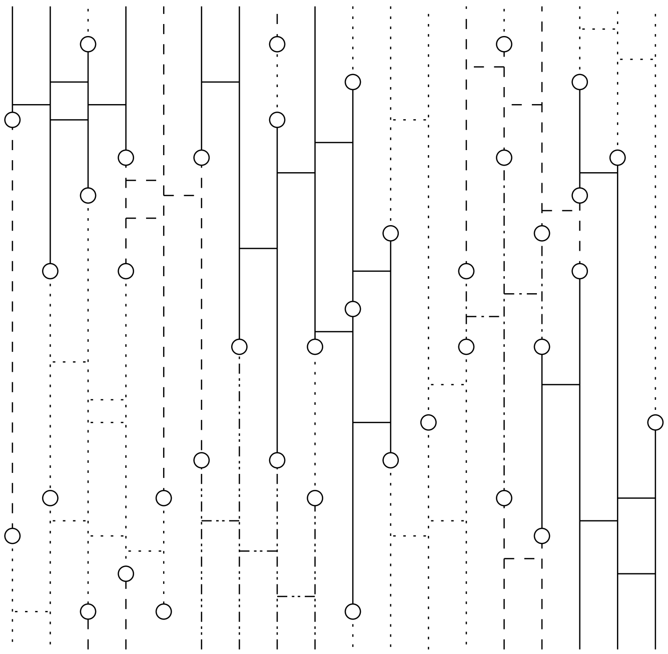

Figure 1: An example of a space–time configuration contributing to the Poisson integral (3.12). The cuts are shown as circles and the distinct connected clusters (each of which contributes a factor 2 to the term $2^{k^{\mathrm{w}}(\omega)}$) are indicated with different line-types.

$\Delta_m \setminus [0, L]$. The corresponding boundary condition for the configuration $\omega \in \Omega_{m,\beta}$ turns out to be *partially periodic*, in that any two clusters of ω containing points of the form $(x, 0)$ and (x, β), for some $x \in [-m, -1] \cup [m+1, m+L]$, are deemed to be the same cluster and contribute only 1 to the total cluster count. No such assumption is made for sites $x \in [0, L]$, and we refer to the boundary condition on $[0, L]$ as *free*. Let $k^{\mathrm{pp}}(\omega)$ be the number of clusters of ω subject to the partially periodic boundary condition. We shall need a fourth way to count clusters also, as follows. The *periodic/wired* boundary condition is that derived from the partially periodic condition by the additional assumption of a wired condition on $[0, L]$: any two clusters of ω containing points of the form $(x, 0)$ and (y, β), for some $x, y \in [0, L]$, are deemed to be the same cluster and contribute only 1 to the total cluster count. We write $k^{\mathrm{pw}}(\omega)$ for the number of clusters with the periodic/wired boundary condition. Note that $k^{\mathrm{pw}}(\omega) - 1$ is the number of clusters of ω (with the partially periodic boundary condition) that do not intersect $[0, L] \times \{0, \beta\}$.

As in (3.10)–(3.12), the matrix elements of the reduced state $\rho_m^L(\beta)$ are

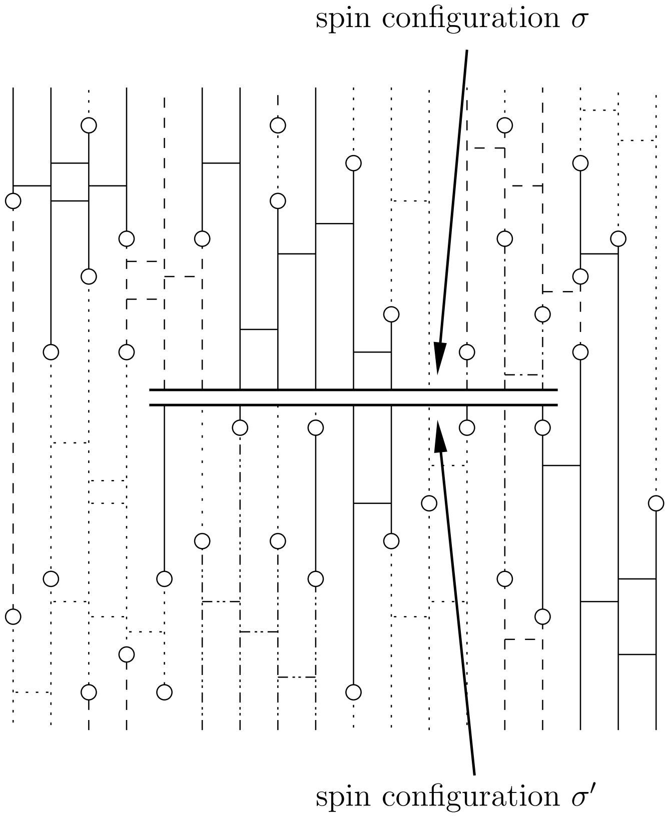

Figure 2: An example of a space–time configuration contributing to the matrix elements for the reduced density operator $\rho_m^L(\beta)$. The box has partially periodic boundary conditions and is drawn in such a way that the slit S_L is at the centre. The spin configurations on the top and the bottom of the cut, and the connected clusters for this new cut geometry are indicated.

given by

$$\langle\sigma'_L|\rho_m^L(\beta)|\sigma_L\rangle = \frac{1}{Z_m}\int d\mathbb{P}_{\Lambda,\lambda,\delta}(\omega)\, 2^{k^{\mathrm{pw}}(\omega)-1} 1_{E(\sigma_L,\sigma'_L)}(\omega), \quad \sigma_L, \sigma'_L \in P_L, \tag{4.1}$$

where $E(\sigma_L, \sigma'_L)$ is the event that: if $x, y \in [0, L]$ are such that $\sigma_{L,x} \neq \sigma'_{L,y}$ then $(x, 0) \not\leftrightarrow^{\mathrm{pp}} (y, \beta)$. Here, $\leftrightarrow^{\mathrm{pp}}$ denotes the connectivity relation subject to the partially periodic boundary condition. See Figure 2 for an illustration of the slit space–time, and of the connected clusters contributing to the matrix elements of $\rho_m^L(\beta)$.

We shall study the entropy of the reduced state via the operator norm of

(2.1). Let $|\psi\rangle \in \mathcal{H}_L$ have unit L^2-norm, so that

$$|\psi\rangle = \sum_{\sigma_L \in P_L} c(\sigma_L)\sigma_L$$

for some function $c : P_L \to \mathbb{C}$ with $\sum_{\sigma_L \in P_L} c(\sigma_L)\overline{c(\sigma_L)} = 1$. Then

$$\langle\psi|\rho_m^L(\beta)|\psi\rangle = \frac{1}{a_{m,\beta}} \sum_{\sigma_L, \sigma'_L \in P_L} c(\sigma_L)\overline{c(\sigma'_L)}\phi_{m,\beta}(\sigma_L, \sigma'_L) \tag{4.2}$$

where

$$\phi_{m,\beta}(\sigma_L, \sigma'_L) = \frac{1}{N_m}\langle\sigma'_L|\rho_m^L(\beta)|\sigma_L\rangle, \qquad \sigma_L, \sigma'_L \in P_L, \tag{4.3}$$

$$N_m = \sum_{\sigma_L, \sigma'_L \in P_L} \langle\sigma'_L|\rho_m^L(\beta)|\sigma_L\rangle, \tag{4.4}$$

and

$$a_{m,\beta} = Z_m/N_m. \tag{4.5}$$

We shall see in the next sections that (4.2)–(4.4) may be written in terms of a certain probability measure on $\Omega_{m,\beta}$ called the *random-cluster measure*.

5 The continuum random-cluster model

Perhaps the best way to express the percolation representations of the ground and reduced states is in terms of the so-called random-cluster model on $\mathbb{Z}\times\mathbb{R}$. We summarise the definition and basic properties of this model in this section, using the language of probability theory. The remaining part of the paper is a self-contained account of the model, and includes the proof of Theorem 2.2, see Theorem 6.5. Of special interest will be the property of so-called ratio weak-mixing, studied earlier for the lattice case in [2, 3].

We shall consider the (two-dimensional) continuum random-cluster model on the 'space–time' subset $\mathbb{Z}\times\mathbb{R}$ of the plane. The underlying space is $\{(x, t) : x \in \mathbb{Z},\ t \in \mathbb{R}\}$, and we refer to $\mathbb{Z}$ as the space-line and $\mathbb{R}$ as the time-line. Everything proved here has a counterpart, subject to minor changes, in the more general setting of $\mathbb{Z}^d \times \mathbb{R}$ with $d \geq 2$, but we shall restrict ourselves to the case $d = 1$.

We shall construct a family of probabilistic models on $\mathbb{Z} \times \mathbb{R}$. Let $\lambda, \delta \in (0, \infty)$. In the simplest such model, we construct 'deaths' and 'bridges' as follows. For each $x \in \mathbb{Z}$, we select a Poisson process D_x of points in $\{x\} \times \mathbb{R}$

with intensity δ; the processes $\{D_x : x \in \mathbb{Z}\}$ are independent, and the points in the D_x are termed 'deaths'. For each $x \in \mathbb{Z}$, we select a Poisson process B_x of points in $\{x + \frac{1}{2}\} \times \mathbb{R}$ with intensity λ; the processes $\{B_x : x \in \mathbb{Z}\}$ are independent of each other and of the D_y. For each $x \in \mathbb{Z}$ and each $(x + \frac{1}{2}, t) \in B_x$, we draw a unit line-segment in $\mathbb{R}^2$ with endpoints (x, t) and $(x + 1, t)$, and we refer to this as a 'bridge' joining its two endpoints. For $(x, s), (y, t) \in \mathbb{Z} \times \mathbb{R}$, we write $(x, s) \leftrightarrow (y, t)$ if there exists a path π in $\mathbb{R}^2$ with endpoints (x, s), (y, t) such that: π comprises sub-intervals of $\mathbb{Z} \times \mathbb{R}$ containing no deaths, together possibly with bridges. For $\Lambda, \Delta \subseteq \mathbb{Z} \times \mathbb{R}$, we write $\Lambda \leftrightarrow \Delta$ if there exist $a \in \Lambda$ and $b \in \Delta$ such that $a \leftrightarrow b$.

For $(x, s) \in \mathbb{Z} \times \mathbb{R}$, let $C_{x,s}$ be the set of all points (y, t) such that $(x, s) \leftrightarrow (y, t)$. The clusters $C_{x,s}$ have been studied in [5], where it was shown in particular that

$$\mathbb{P}_{\lambda,\delta}(|C_0| < \infty) \begin{cases} = 1 & \text{if } \theta \leq 1, \\ < 1 & \text{if } \theta > 1, \end{cases} \tag{5.1}$$

where $0 = (0, 0)$ is the origin of $\mathbb{Z} \times \mathbb{R}$, $\theta = \lambda/\delta$, and $|C|$ denotes the (one-dimensional) Lebesgue measure of the cluster C. The process thus constructed is a continuum percolation model in two dimensions. As noted in [5], it differs from the contact model on $\mathbb{Z}$ only in that two points may be joined in the direction of either increasing or decreasing time. See [19, 20] for details of the contact model.

Just as the percolation model on a lattice may be generalised to the so-called random-cluster model (see [12]), so may the continuum percolation model be extended to a continuum random-cluster model. We shall work here mostly on a bounded box rather than the whole space $\mathbb{Z} \times \mathbb{R}$. Let $a, b \in \mathbb{Z}$, $s, t \in \mathbb{R}$ satisfy $a \leq b$, $s \leq t$, and write $\Lambda = [a, b] \times [s, t]$ for the box $\{a, a + 1, \ldots, b\} \times [s, t]$ of $\mathbb{Z} \times \mathbb{R}$. Its boundary $\partial\Lambda$ is the set of all points $(x, y) \in \Lambda$ such that: either $x \in \{a, b\}$, or $y \in \{s, t\}$, or both. As sample space we take the set Ω_Λ comprising all finite subsets (of Λ) of deaths and bridges, and we assume that no death is the endpoint of any bridge. For $\omega \in \Omega_\Lambda$, we write $B(\omega)$ and $D(\omega)$ for the sets of bridges and deaths, respectively, of ω. We take as σ-field $\mathcal{F}_\Lambda$ that generated by the open sets in the associated Skorohod topology, see [5, 10].

In order to maintain the link to the quantum Ising model, we choose to impose a top/bottom periodic boundary condition on Λ; that is, for every $x \in [a, b]$, we identify the two points (x, s) and (x, t). The remaining boundary of Λ, denoted $\partial^{\mathrm{h}}\Lambda$, is the set of all points of the form $(x, u) \in \Lambda$ with $x \in \{a, b\}$. The theory developed here is valid for more general boundary conditions.

Let $\mathbb{P}_{\Lambda,\lambda,\delta}$ denote the probability measure associated with the above continuum percolation model on Λ. For a given configuration ω of deaths and bridges on Λ, let $k(\omega)$ be the number of its clusters (subject to the top/bottom periodic boundary condition). Let $q \in (0, \infty)$, and define the 'continuum random-cluster' probability measure $\mathbb{P}_{\Lambda,\lambda,\delta,q}$ by

$$d\mathbb{P}_{\Lambda,\lambda,\delta,q}(\omega) = \frac{1}{Z} q^{k(\omega)} d\mathbb{P}_{\Lambda,\lambda,\delta}(\omega), \qquad \omega \in \Omega_\Lambda, \tag{5.2}$$

for an appropriate 'partition function' Z.

The theory of the continuum random-cluster model may be developed in very much the same way as that for the random-cluster model on a discrete lattice, see [12]. We shall assume the basic theory without labouring the calculations necessary for full rigorous proof. The details may be obtained by following minor variants of the relevant strategy for the discrete case.

If μ is a probability measure and f a function on some measurable space, we denote by $\mu(f)$ the expectation of f under μ.

The space Ω_Λ is a partially ordered space with order relation given by: $\omega_1 \le \omega_2$ if $B(\omega_1) \subseteq B(\omega_2)$ and $D(\omega_1) \supseteq D(\omega_2)$. A random variable $X : \Omega_\Lambda \to \mathbb{R}$ is said to be *increasing* if $X(\omega) \le X(\omega')$ whenever $\omega \le \omega'$. An event $A \in \mathcal{F}_\Lambda$ is said to be *increasing* if its indicator function 1_A is increasing. Given two probability measures μ_1, μ_2 on the measurable pair $(\Omega_\Lambda, \mathcal{F}_\Lambda)$, we write $\mu_1 \le_{\mathrm{st}} \mu_2$ if $\mu_1(X) \le \mu_2(X)$ for all bounded increasing continuous random variables $X : \Omega_\Lambda \to \mathbb{R}$.

The measures $\mathbb{P}_{\Lambda,\lambda,\delta,q}$ have certain properties of stochastic ordering as the parameters Λ, λ, δ, q vary. There are two approaches to such stochastic inequalities, either by working on discrete graphs and passing to a spatial limit to obtain the continuum measures, or by working directly in the continuum. We shall not pursue this here, but refer the reader to [5] for a discussion of the case $q = 1$. The following two facts will be useful later. First, $\mathbb{P}_{\Lambda,\lambda,\delta,q}$ satisfies a positive-association (FKG) inequality when $q \ge 1$, and secondly,

$$\mathbb{P}_{\Lambda,\lambda,\delta,q} \le_{\mathrm{st}} \mathbb{P}_{\Lambda,\lambda,\delta} \qquad \text{when } q \ge 1. \tag{5.3}$$

In the current paper we shall work mostly with finite-volume measures, that is, with measures defined on boxes of the form of $\Lambda = [a, b] \times [s, t]$. We assume henceforth that $q \ge 1$. Having established the necessary estimates on such boxes, we will pass to the *vertical* infinite-volume limit as $s \to -\infty$, $t \to \infty$. The existence of such a limit is not explored in detail here, but we note the following (see [1]). If we work on Λ with top/bottom *wired* or *free* boundary conditions, then the limit measures exist as a consequence of positive association (very much as in the lattice case, see [12]). Furthermore, the weak limit with top/bottom periodic boundary conditions exists and agrees with the first two limit measures whenever the latter are equal. A sufficient condition for this is that the wired limit measure does not percolate. Since the limit of Λ as $t - s \to \infty$ is a strip of bounded width, this condition is satisfied for all $\lambda, \delta \in (0, \infty)$, and therefore the limit measures exist and do not depend on the choice of boundary condition.

The situation is slightly less clear in the doubly-infinite-volume limit, as $\Lambda \uparrow \mathbb{Z} \times \mathbb{R}$. The self-dual point for the continuum random-cluster measure on $\mathbb{Z} \times \mathbb{R}$ is given by $\lambda/\delta = q$, and thus one expects the free and wired limit measures to be equal at least whenever $\lambda/\delta \neq q$. It may be shown using duality that there is no percolation when $\lambda/\delta < q$, and it follows that the weak limits

$$\mathbb{P}_{\lambda,\delta,q} = \lim_{\Lambda \uparrow \mathbb{Z} \times \mathbb{R}} \mathbb{P}_{\Lambda,\lambda,\delta,q}, \qquad q \geq 1,$$

exist if $\lambda/\delta < q$. We shall make no reference to this later.

Just as the q-state Potts model may be coupled with a random-cluster model on a given graph, so may we consider a continuum Potts model on a box $\Lambda = [a, b] \times [s, t]$. Let $q \in \{2, 3, \dots\}$. We sample ω according to $\mathbb{P}_{\Lambda,\lambda,\delta,q}$, and we allocate a randomly chosen spin from the set $\{1, 2, \dots, q\}$ to each cluster of ω; the points of each cluster receive a given spin-state chosen uniformly at random from the q possible local states, and different clusters receive independent spin-states. We call the ensuing spin-configuration a q-state *continuum Potts model*, and a *continuum Ising model* when $q = 2$. When $q = 2$, by convention we take the local spin-space to be $\{-1, +1\}$ rather than $\{1, 2\}$, and this is the case of interest in the current paper. We note in passing that the q-state continuum random-cluster model corresponds to a certain q-state quantum Potts model constructed in a manner similar to that of the quantum Ising model.

The set of spin-configurations of the continuum q-state Potts model is the space Σ_Λ given as follows. Let $\mathcal{F}$ be the set of finite subsets of Λ. For $D \in \mathcal{F}$, let $J(D)$ be the set of maximal intervals of the time-lines that contain no point in D (subject to the top/bottom boundary condition on Λ). The space Σ_Λ is defined as the union over D of the set of functions $\sigma : J(D) \to \{1, 2, \dots, q\}$ with the property that $\sigma_{(x,u-)} \neq \sigma_{(x,u+)}$ for all $(x, u) \in D$. The corresponding probability measure on Σ_Λ is found by integrating over ω in the above recipe, as in the following summary. For $\sigma \in \Sigma_\Lambda$, write D_σ for the set of points $(x, u) \in \Lambda$ such that $\sigma_{(x,u-)} \neq \sigma_{(x,u+)}$. The probability measure $\overline{\mathbb{P}}$ associated with the continuum q-state Potts model on Λ is given by

$$d\overline{\mathbb{P}}(\sigma) = \frac{1}{Z'} e^{\lambda L(\sigma)} d\mathbb{P}_\delta(D_\sigma), \qquad \sigma \in \Sigma_\Lambda,$$

where $\mathbb{P}_\delta$ is the law of an independent family of Poisson processes with intensity

δ on the time-lines indexed by $[a,b]$, and

$$L(\sigma) = \sum_{x \sim y} \int_s^t \delta_{\sigma(x,u),\sigma(y,u)}\, du \tag{5.4}$$

is the total length of neighbouring time-lines where the spins are equal. Here, the summation is over all unordered pairs x, y of neighbours. We shall not develop the theory of such measures here, save for noting that $\overline{\mathbb{P}}$ has the spatial Markov property (see [8, 11] for accounts of the spatial Markov property for a lattice model). For $\sigma \in \Sigma_\Lambda$ and a measurable subset S of Λ, we write σ_S for the value of σ restricted to S, and $\mathcal{G}_S$ for the σ-field generated by σ_S.

The above definition of the continuum random-cluster model is based on an assumption of free boundary conditions on left/right sides of the region Λ (we shall always assume top/bottom periodic conditions in this paper). More general boundary conditions may be introduced as follows. Let τ be an admissible configuration of deaths and bridges off the box Λ. That is, τ comprises a set $D(\tau)$ of deaths and a set $B(\tau)$ of bridges of $(\mathbb{Z} \times \mathbb{R}) \setminus \Lambda$ such that: the intersection of $D(\tau)$ and $B(\tau)$ with any bounded sub-interval of $\mathbb{Z} \times \mathbb{R}$ is finite, and no death is the endpoint of any bridge. For $\omega \in \Omega_\Lambda$, we denote by (ω, τ) the composite configuration comprising ω on Λ and τ on its complement. We write $\mathbb{P}^\tau_{\Lambda,\lambda,\delta,q}$ for the continuum random-cluster measure on Ω_Λ with the difference that the number $k(\omega)$ of clusters in (5.2) is replaced by the number $k(\omega,\tau)$ of clusters of (ω,τ) that intersect Λ (subject, as usual, to the top/bottom periodic boundary condition). As in the lattice case, $\mathbb{P}^\tau_{\Lambda,\lambda,\delta,q}$ is stochastically increasing in τ. One may consider also periodic boundary conditions.

We extend this discussion now to boundary conditions defined in terms of spins rather than deaths/bridges. Let $q \geq 2$ be an integer. Let τ be a boundary condition as above, and let η be a mapping from its clusters to the set $\{1, 2, \ldots, q\}$; that is, η allocates a spin to each cluster of τ, viewed as a configuration on $(\mathbb{Z} \times \mathbb{R}) \setminus \Lambda$. Let the measure $\mathbb{P}^\eta_{\Lambda,\lambda,\delta,q}$ be given as $\mathbb{P}_{\Lambda,\lambda,\delta,q}$, conditioned on the event that no two points $x, y \in \partial^{\mathrm{h}}\Lambda$ with $\eta(x) \neq \eta(y)$ are connected. We now allocate spins to the clusters of the composite configuration (ω, τ) by: if a cluster C contains a vertex y that is already labelled, the entire cluster of y takes that label, and if no such vertex exists, the spin of C is chosen uniformly at random from $\{1, 2, \ldots, q\}$, independently of the spins on other clusters.

6 Basic estimate for the slit box

We consider next a variant of the above model in which the box Λ possesses a 'slit' at its centre. Let $L \geq 0$ and $S_L = [0, L] \times \{0\}$. We think of S_L as a collection of $L+1$ vertices labelled in the obvious way as $x = 0, 1, 2, \ldots, L$. For $m \geq 2$, $\beta > 0$, let $\Lambda_{m,\beta}$ be the box $[-m, m+L] \times [-\frac{1}{2}\beta, \frac{1}{2}\beta]$ subject to a 'slit' along S_L. That is, $\Lambda_{m,\beta}$ is the usual box except in that each vertex $x \in S_L$ is replaced by two distinct vertices x^+ and x^-. The vertex x^+ (respectively, x^-) is attached to the half-line $\{x\}\times(0,\infty)$ (respectively, the half-line $\{x\}\times(-\infty, 0)$); there is no direct connection between x^+ and x^-. Write $S_L^{\pm} = \{x^{\pm} : x \in S_L\}$ for the upper and lower sections of the slit S_L. We now construct the continuum random-cluster measure $\phi_{m,\beta}$ on $\Lambda_{m,\beta}$ with top/bottom periodic boundary condition and parameters λ, δ, $q = 2$. We shall abuse notation by using $\phi_{m,\beta}$ to denote also the coupling of the continuum random-cluster measure and the spin-configuration on $\Lambda_{m,\beta}$ obtained as above. An illustration of the slit box is presented in Figure 2.

Let $\Omega_{m,\beta}$ be the sample space of the continuum random-cluster model on $\Lambda_{m,\beta}$, and $\Sigma_{m,\beta}$ the set of all possible spin-configurations. That is, $\Sigma_{m,\beta}$ comprises all admissible allocations of spins to the clusters of configurations in $\Omega_{m,\beta}$. For $\sigma \in \Sigma_{m,\beta}$ and $x \in S_L$, write $\sigma_x^{\pm}$ for the spin-state of $x^{\pm}$. Let $\Sigma_L = \{-1, +1\}^{L+1}$ be the set of spin-configurations of the vectors $\{x^+ : x \in S_L\}$ and $\{x^- : x \in S_L\}$, and write $\sigma_L^+ = (\sigma_x^+ : x \in S_L)$ and $\sigma_L^- = (\sigma_x^- : x \in S_L)$.

It may be checked from (4.1) that

$$\phi_{m,\beta}(\sigma_L^- = \epsilon^-, \, \sigma_L^+ = \epsilon^+) \propto \langle \epsilon^- | \rho_m^L(\beta) | \epsilon^+ \rangle, \qquad \epsilon^-, \epsilon^+ \in \Sigma_L,$$

whence $f(\epsilon^+, \epsilon^-) = \phi_{m,\beta}(\sigma_L^- = \epsilon^-, \, \sigma_L^+ = \epsilon^+)$ is the function defined in (4.3). It is easily seen that $a_{m,\beta}$, given in (4.5), may be expressed as

$$a_{m,\beta} = \phi_{m,\beta}(\sigma_L^+ = \sigma_L^-). \tag{6.1}$$

On recalling (2.1), by (4.2),

$$\langle \psi | \rho_m^L(\beta) - \rho_n^L(\beta) | \psi \rangle = \frac{\phi_{m,\beta}(c(\sigma_L^+)\overline{c(\sigma_L^-)})}{a_{m,\beta}} - \frac{\phi_{n,\beta}(c(\sigma_L^+)\overline{c(\sigma_L^-)})}{a_{n,\beta}} \tag{6.2}$$

where $c : \Sigma_L \to \mathbb{C}$ and

$$\psi = \sum_{\sigma_L \in \Sigma_L} c(\sigma_L)\sigma_L \in \mathcal{H}_L.$$

The reduced ground state ρ_m^L is obtained from $\rho_m^L(\beta)$ by taking the limit as $\beta \to \infty$. By the remarks in Section 5, there exists a probability measure ϕ_m such that

$$\phi_{m,\beta} \Rightarrow \phi_m \qquad \text{as } \beta \to \infty.$$

Furthermore, the $\sigma_L^\pm$ are cylinder functions, and therefore, as $\beta \to \infty$,

$$\phi_{m,\beta}(c(\sigma_L^+)\overline{c(\sigma_L^-)}) \to \phi_m(c(\sigma_L^+)\overline{c(\sigma_L^-)}), \tag{6.3}$$

and

$$a_{m,\beta} \to a_m = \phi_m(\sigma_L^+ = \sigma_L^-). \tag{6.4}$$

In order to prove Theorem 2.2, we seek the function $c : \Sigma_L \to \mathbb{C}$, with

$$\|c\| = \sqrt{\sum_{\epsilon\in\Sigma_L} |c(\epsilon)|^2} = 1,$$

that maximises the modulus of (6.2). By splitting (6.2) into its real and imaginary parts, and applying the triangle inequality, we see that it suffices to consider functions c taking only non-negative real values.

Here is the main estimate of this section, of which Theorem 2.2 is an immediate corollary with adapted values of the constants.

Theorem 6.5. *Let $\lambda, \delta \in (0,\infty)$ and write $\theta = \lambda/\delta$. If $\theta < 1$, there exist $\alpha, C, M \in (0,\infty)$, depending on θ only, such that the following holds. There exists $\gamma = \gamma(\theta)$ satisfying $\gamma > 0$ when $\theta < 1$ such that, for all $L \geq 1$ and $M \leq m \leq n < \infty$,*

$$\sup_{\|c\|=1} \left| \frac{\phi_m(c(\sigma_L^+)c(\sigma_L^-))}{a_m} - \frac{\phi_n(c(\sigma_L^+)c(\sigma_L^-))}{a_n} \right| \leq CL^\alpha e^{-\gamma m}, \tag{6.6}$$

where the supremum is over all functions $c : \Sigma_L \to \mathbb{R}$ with $\|c\| = 1$. The function γ may be chosen to satisfy $\gamma(\theta) \to \infty$ as $\theta \downarrow 0$.

The condition $\theta < 1$ is important in that it permits a comparison of the $q = 2$ continuum random-cluster model on $\mathbb{Z} \times \mathbb{R}$ with the continuum percolation model. The claim of the theorem is presumably valid for $\theta < \theta_c$ where θ_c is the critical point of the former model. [It may be shown that $\theta_c \geq 2$, and we conjecture that $\theta_c = 2$, the self-dual point.] Similarly, Theorem 6.5 has a counterpart in $d \geq 2$ dimensions.

We shall require for the purposes of comparison the following exponential-decay theorem for continuum percolation. Let Λ_m denote the box $[-m, m]^2$, and let $I = \{0\} \times [-\frac{1}{2}, \frac{1}{2}]$ be a unit 'time-segment' centred at the origin.

Theorem 6.7. *Let $\lambda, \delta \in (0,\infty)$. There exist $C = C(\lambda,\delta) \in (0,\infty)$ and $\gamma = \gamma(\lambda,\delta)$ satisfying $\gamma > 0$ when $\lambda/\delta < 1$, such that:*

$$\mathbb{P}_{\lambda,\delta}\big(I \leftrightarrow \partial\Lambda_m\big) \leq Ce^{-\gamma m}, \qquad m \geq 0.$$

The function $\gamma(\lambda,\delta)$ may be chosen to satisfy $\gamma \to \infty$ as $\delta \to \infty$ for fixed λ.

Proof. Consider the continuum percolation process with parameters λ, δ. The existence of such γ is proved in [5]. That $\gamma \to \infty$ as $\delta \downarrow 0$ (with λ fixed) may be proved by bounding the cluster at the origin by a branching process. Consider an age-dependent branching process in which each particle lives for a length of time having the distribution of the sum of two independent exponentially-distributed random variables with parameter δ. During its lifetime, it has children in the manner of a Poisson process with parameter 2λ, so that a typical family-size N has generating function

$$G_N(s) = E(s^N) = \left(\frac{\delta}{\delta - 2\lambda(s-1)}\right)^2, \qquad |s| \le 1.$$

The process is subcritical if $E(N) < 1$, which is to say that $G_N'(1) = 4\lambda/\delta < 1$. When this holds, the tail of the total number M of particles decays exponentially, and similarly the aggregate lifetime U of the particles has an exponentially-decaying tail. See [13, 14] for accounts of the theory of branching processes.

The branching process dominates C in the following sense. Identify the progenitor of the branching process and the origin 0 of $\mathbb{Z} \times \mathbb{R}$. The length of the maximal death-free time-interval containing the origin has the distribution of the lifetime of 0. The number of bridges with an endpoint in this interval has the distribution of N. Each such bridge has endpoints of the form $(0, s)$ and (x, s) where $x = \pm 1$. When we iterate this, we find that the number of bridges in the maximal death-free interval containing (x, s) is dominated (stochastically) by N. Arguing inductively, the number of bridges in the cluster C is dominated stochastically by the total size M of the branching process.

The horizontal displacement of C is thus smaller (in distribution) than the total size M of the branching process. It is standard that the tail of M satisfies $P(M > m) \le Ce^{-\nu m}$ for some $C, \nu > 0$ depending on λ, δ, and furthermore that $\nu \to \infty$ if $\delta \downarrow 0$ with λ held fixed. The behaviour of ν may be calculated exactly by elementary means, as follows. One may consider a variant of the branching process in which each particle has a lifetime with the exponential distribution, parameter δ, and has *pairs* of children at rate 2λ while alive. The probability generating function of the total progeny may be found in closed form in the usual way (see [13], Problem 5.12.11), and one obtains thus a sharp estimate for ν via Markov's inequality.

Similarly, the vertical displacement of C is smaller (in distribution) than the aggregate lifetime U of the particles in the branching process. Just as above, U has exponentially-decaying tail when $E(N) < 1$, and the constant in the exponent tends to infinity as $\delta \downarrow 0$ for fixed λ.

Now,

$$\mathbb{P}_{\lambda,\delta}(0 \leftrightarrow \partial\Lambda_m) \le P(M \ge m) + P(U \ge m).$$

A little more is needed for the theorem. The interval I is connected to a number of bridges having the Poisson distribution with parameter 2λ. The clusters generated by the ends of these bridges have sizes dominated (stochastically) as above, and the claim follows. $\square$

In the proof of Theorem 6.5, we make use of the following two lemmas, which are proved in the next section using the method of 'ratio weak-mixing'.

Lemma 6.8. *Let $\lambda, \delta \in (0,\infty)$ satisfy $\lambda/\delta < 1$. There exist constants $\alpha, C_1, C_2 \in (0,\infty)$ such that: for all $L \ge 0$, $m \ge 1$, $\beta > 2m + L$, and all $\epsilon^+, \epsilon^- \in \Sigma_L$,*

$$C_1 L^{-\alpha} \le \frac{\phi_{m,\beta}(\sigma_L^+ = \epsilon^+,\, \sigma_L^- = \epsilon^-)}{\phi_{m,\beta}(\sigma_L^+ = \epsilon^+)\phi_{m,\beta}(\sigma_L^- = \epsilon^-)} \le C_2 L^{\alpha}.$$

In the second lemma we allow a general boundary condition on $\Lambda_{m,\beta}$.

Lemma 6.9. *Let $\lambda, \delta \in (0,\infty)$. There exist constants $C, \gamma \in (0,\infty)$ satisfying $0 < \gamma < 1$ when $\lambda/\delta < 1$ such that: for all $L \ge 0$, $m \ge 1$, $\beta \ge 4(m+L+1)$, all events $A \subseteq \Sigma_L \times \Sigma_L$, and all admissible random-cluster boundary-conditions τ and spin boundary-conditions η of $\Lambda_{m,\beta}$,*

$$\left| \frac{\phi^{\alpha}_{m,\beta}((\sigma_L^+, \sigma_L^-) \in A)}{\phi_{m,\beta}((\sigma_L^+, \sigma_L^-) \in A)} - 1 \right| \le C e^{-\frac{2}{7}\gamma m}, \qquad \text{for } \alpha = \tau, \eta,$$

whenever the right side of the inequality is less than or equal to 1. The function γ may be taken as that of Theorem 6.7.

The above two lemmas are stated in terms of the box $\Lambda_{m,\beta}$ with top/bottom periodic boundary conditions. Their proofs are valid under other boundary conditions also, including free boundary conditions. We make use of this observation during the proofs that follow.

The supremum in Theorem 6.5 may be handled by way of the next lemma.

Lemma 6.10. *Let μ be a probability measure on the finite set S. Let $\mathcal{C}$ be the class of functions $c : S \to [0,\infty)$ such that $\sum_{s\in S} c(s)^2 = 1$. Then*

$$\sum_{s\in S} c(s)\mu(s) \le \sqrt{\sum_{s\in S} \mu(s)^2}, \qquad c \in \mathcal{C},$$

with equality if and only if

$$c(s) = \frac{\mu(s)}{\sqrt{\sum_{t\in S}\mu(t)^2}}, \qquad s \in S.$$

Proof of Lemma 6.10. This is easily proved using a Lagrange multiplier. □

Proof of Theorem 6.5. Let $0 < \lambda < \delta$, and let γ be as in Theorem 6.7. Let $2 \le m \le n < \infty$ and take $\beta > 4(m + L + 1)$. Later we shall let $\beta \to \infty$. Since $\phi_{m,\beta} \le_{\mathrm{st}} \phi_{n,\beta}$, we may couple $\phi_{m,\beta}$ and $\phi_{n,\beta}$ via a probability measure ν on pairs (ω_1, ω_2) of configurations on $\Lambda_{n,\beta}$ in such a way that $\nu(\omega_1 \le \omega_2) = 1$. It is standard (as in [12, 22]) that we may find ν such that ω_1 and ω_2 are identical configurations within the region of $\Lambda_{m,\beta}$ that is not connected to $\partial^{\mathrm{h}}\Lambda_{m,\beta}$ in the upper configuration ω_2. Let D be the set of all pairs $(\omega_1, \omega_2) \in \Omega_{n,\beta} \times \Omega_{n,\beta}$ such that: ω_2 contains no path joining ∂B to $\partial^{\mathrm{h}}\Lambda_{m,\beta}$, where $B = [-r, r + L] \times [-2(r + L + 1), 2(r + L + 1)]$ and r $(< \frac{1}{2}m)$ will be chosen later. We take free boundary conditions on B. The relevant regions are illustrated in Figure 3.

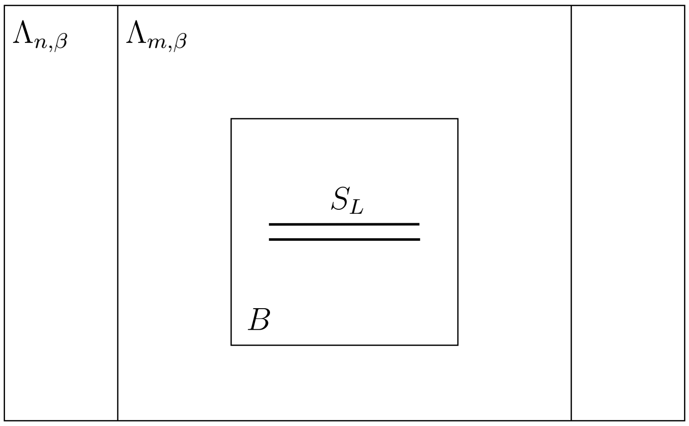

Figure 3: The boxes $\Lambda_{n,\beta}$, $\Lambda_{m,\beta}$, and B.

Having constructed the measure ν accordingly, we may now allocate spins to the clusters of ω_1 and ω_2 in the manner described earlier. This may be done in such a way that, on the event D, the spin-configurations associated with ω_1 and ω_2 within B are identical. We write σ_1 (respectively, σ_2) for the spin-configuration on the clusters of ω_1 (respectively, ω_2), and $\sigma^{\pm}_{i,L}$ for the spins of σ_i on the slit S_L.

For $c : \Sigma_L \to [0, \infty)$ with $\|c\| = 1$, let

$$S_c = \frac{c(\sigma^+_{1,L})c(\sigma^-_{1,L})}{a_{m,\beta}} - \frac{c(\sigma^+_{2,L})c(\sigma^-_{2,L})}{a_{n,\beta}}, \tag{6.11}$$

so that

$$\frac{\phi_{m,\beta}(c(\sigma^+_L)c(\sigma^-_L))}{a_{m,\beta}} - \frac{\phi_{n,\beta}(c(\sigma^+_L)c(\sigma^-_L))}{a_{n,\beta}} = \nu(S_c; D) + \nu(S_c; \overline{D}). \tag{6.12}$$

Here, $\overline{D}$ is the complement of D, and $\nu(f; D)$ denotes $\nu(f 1_D)$.

We consider first the term $\nu(S_c; D)$ in (6.12). On the event D, we have that $\sigma^{\pm}_{1,L} = \sigma^{\pm}_{2,L}$, so that

$$|\nu(S_c; D)| \le \left|1 - \frac{a_{m,\beta}}{a_{n,\beta}}\right| \frac{\phi_{m,\beta}(c(\sigma^+_L)c(\sigma^-_L))}{a_{m,\beta}}. \tag{6.13}$$

By Lemmas 6.8 and 6.10,

$$\begin{aligned}
\phi_{m,\beta}(c(\sigma^+_L)c(\sigma^-_L)) &= \sum_{\epsilon^{\pm}\in\Sigma_L} c(\epsilon^+)c(\epsilon^-)\phi_{m,\beta}(\sigma^+_L = \epsilon^+,\ \sigma^-_L = \epsilon^-) \\
&\le C_2 L^{\alpha} \phi_{m,\beta}(c(\sigma^+_L))\phi_{m,\beta}(c(\sigma^-_L)) \\
&= C_2 L^{\alpha} \left(\sum_{\epsilon\in\Sigma_L} c(\epsilon)\phi_{m,\beta}(\sigma^+_L = \epsilon) \right)^2 \\
&\le C_2 L^{\alpha} \sum_{\epsilon\in\Sigma_L} \phi_{m,\beta}(\sigma^+_L = \epsilon)^2,
\end{aligned} \tag{6.14}$$

where we have used reflection-symmetry in the horizontal axis at the intermediate step. By Lemma 6.8 and reflection-symmetry again,

$$\begin{aligned}
a_{m,\beta} &= \sum_{\epsilon\in\Sigma_L} \phi_{m,\beta}(\sigma^+_L = \sigma^-_L = \epsilon) \\
&\ge C_1 L^{-\alpha} \sum_{\epsilon\in\Sigma_L} \phi_{m,\beta}(\sigma^+_L = \epsilon)^2.
\end{aligned}$$

Therefore,

$$\frac{\phi_{m,\beta}(c(\sigma^+_L)c(\sigma^-_L))}{a_{m,\beta}} \le C_3 L^{2\alpha}, \tag{6.15}$$

where $C_3 = C_2/C_1$.

We set $A = \{\sigma^+_L = \sigma^-_L\}$ in Lemma 6.9 to find that, for sufficiently large $m \ge M'(\lambda, \delta)$,

$$\left| \frac{\phi^{\eta}_{m,\beta}(\sigma^+_L = \sigma^-_L)}{\phi_{m,\beta}(\sigma^+_L = \sigma^-_L)} - 1 \right| \le C e^{-\frac{2}{7}\gamma m} < \frac{1}{2}.$$

By averaging over η, sampled according to $\phi_{n,\beta}$, we deduce that

$$\left| \frac{\phi_{n,\beta}(\sigma^+_L = \sigma^-_L)}{\phi_{m,\beta}(\sigma^+_L = \sigma^-_L)} - 1 \right| \le C e^{-\frac{2}{7}\gamma m} < \frac{1}{2},$$

which is to say that

$$\left| \frac{a_{n,\beta}}{a_{m,\beta}} - 1 \right| \le C e^{-\frac{2}{7}\gamma m} < \frac{1}{2}. \tag{6.16}$$

We make a note for later use. By the remark after Lemma 6.9, inequality (6.15) holds also with $\phi_{m,\beta}$ replaced by the continuum random-cluster measure ϕ_B on the box B with free boundary conditions. Similarly, we may take C and M' above such that

$$\left|\frac{a_{n,\beta}}{a_B} - 1\right| \le Ce^{-\frac{2}{7}\gamma r} < \frac{1}{2}, \qquad r \ge M'(\lambda, \delta), \tag{6.17}$$

where $a_B = \phi_B(\sigma_L^+ = \sigma_L^-)$.

Inequalities (6.15) and (6.16) may be combined as in (6.13) to obtain

$$|\nu(S_c; D)| \le C_4 L^{2\alpha} e^{-\frac{2}{7}\gamma m} \tag{6.18}$$

for an appropriate constant $C_4 = C_4(\lambda, \delta)$ and all $m \ge M'(\lambda, \delta)$.

We turn to the term $\nu(S_c; \overline{D})$ in (6.12). Evidently,

$$|\nu(S_c; \overline{D})| \le A_m + B_n, \tag{6.19}$$

where

$$A_m = \frac{\nu(c(\sigma_{1,L}^+)c(\sigma_{1,L}^-); \overline{D})}{a_{m,\beta}}, \quad B_n = \frac{\nu(c(\sigma_{2,L}^+)c(\sigma_{2,L}^-); \overline{D})}{a_{n,\beta}}.$$

There exist constants C_5, M'' depending on λ, δ, such that, for $m > r \ge M''$,

$$\begin{aligned} B_n &= \frac{\nu(\overline{D})}{a_{n,\beta}} \nu(c(\sigma_{2,L}^+)c(\sigma_{2,L}^-) \mid \overline{D}) \\ &= \frac{\nu(\overline{D})}{a_{n,\beta}} \phi_{n,\beta}\big(\phi_B^\tau(c(\sigma_{2,L}^+)c(\sigma_{2,L}^-)) \mid \overline{D}\big) \\ &\le \frac{\nu(\overline{D})}{a_B} C_5 \phi_B(c(\sigma_{2,L}^+)c(\sigma_{2,L}^-)) \end{aligned} \tag{6.20}$$

by Lemma 6.9 with $\phi_{m,\beta}$ replaced by ϕ_B, and (6.17). At the middle step, we have used conditional expectation given the configuration τ on $\Lambda_{m,\beta} \setminus B$. By (6.15) applied to the measure ϕ_B, there exists $C_6 = C_6(\lambda, \delta)$ such that

$$\frac{1}{a_B} \phi_B(c(\sigma_{2,L}^+)c(\sigma_{2,L}^-)) \le C_6 L^{2\alpha}. \tag{6.21}$$

Inequalities (6.20)–(6.21) imply an upper bound for B_n.

A similar upper bound is valid for A_m, on noting that the conditioning on $\overline{D}$ imparts certain information about the configuration ω_1 outside B but nothing further about ω_1 within B. Combining this with (6.19)–(6.21), we find that, for $r \ge M'''(\lambda, \delta)$ and some $C_7 = C_7(\lambda, \delta)$,

$$|\nu(S_c; \overline{D})| \le \nu(\overline{D}) C_7 L^{2\alpha}. \tag{6.22}$$

Let $r = M'''$ to obtain by (5.3) and Theorem 6.7 that

$$\nu(\overline{D}) \leq C_8(r+L)e^{-\frac{1}{2}\gamma m} \leq C_9 L e^{-\frac{1}{2}\gamma m}, \qquad m \geq 2M''', \tag{6.23}$$

for some C_8, C_9. We combine (6.18), (6.22), (6.23) as in (6.12), and let $\beta \to \infty$ to obtain (6.6) from (6.3)–(6.4), for $m \geq \max\{M', M'', 2M'''\}$. The constants C, γ may be amended to obtain the required inequality.

Finally, we remark that α, C, and M depend on λ and δ. The left side of (6.6) is invariant under re-scalings of the time-axes, that is, under the transformations $(\lambda, \delta) \mapsto (\lambda\eta, \delta\eta)$ for $\eta \in (0, \infty)$. We may therefore work with the new values $\lambda' = \theta$, $\delta' = 1$, with appropriate constants $\alpha(\theta, 1)$, $C(\theta, 1)$, $M(\theta, 1)$. $\square$

7 Ratio weak-mixing

Our proofs of Lemmas 6.8 and 6.9 make use of various couplings of random-cluster measures. Such couplings are fairly standard (see [12, 22] for example) and have been utilised in [2, 3] in a study of ratio weak-mixing for random-cluster and spin models on discrete lattices. We follow in part the arguments of [2, 3], but we are not concerned here with the level of generality of those papers.

Here is some notation. Let Λ be a box in $\mathbb{Z} \times \mathbb{R}$ (we shall later consider a box Λ with a slit S_L, for which the same definitions and results are valid). A *path* π of Λ is an alternating sequence of disjoint intervals (contained in Λ) and unit line-segments of the form $[z_0, z_1]$, b_{12}, $[z_2, z_3]$, b_{34}, $\dots$, $b_{2k-1,2k}$, $[z_{2k}, z_{2k+1}]$, where: each pair z_{2i}, z_{2i+1} is on the same 'time-line' of Λ, and $b_{2i-1,2i}$ is a unit line-segment with endpoints z_{2i-1} and z_{2i}, perpendicular to the time-lines. Note that the equality $z_{2i} = z_{2i+1}$ is permitted. The path π is said to join z_0 and z_{2k+1}. The *length* of π is its one-dimensional Lebesgue measure, with π viewed as a union of line-segments of $\mathbb{R}^2$; note that each bridge of π contributes 1 to its length. A *circuit* D of Λ is a path except inasmuch as $z_0 = z_{2k+1}$. A set D is called *linear* if it is a disjoint union of paths and/or circuits. Let Δ, Γ be disjoint subsets of Λ. The linear set D is said to *separate* Δ and Γ if every path of Λ from Δ to Γ passes through D, and D is minimal with this property in that no strict subset of D has the property.

Let $\omega \in \Omega_\Lambda$. An *open path* π of ω is a path of Λ such that, in the notation above, the intervals $[z_{2i}, z_{2i+1}]$ contain no death of ω, and the line-segments $b_{2i-1,2i}$ are bridges of ω.

The (one-dimensional) Lebesgue measure of a measurable subset S of $\mathbb{Z}\times\mathbb{R}$ is denoted $|S|$. Let S and T be measurable subsets of Λ. The distance $d(S, T)$

from S to T is defined to be the infimum of the lengths of paths having one endpoint in S and one in T. Note that the distance function d depends on the choice of Λ (and, in particular, on the boundary conditions and the presence/absence of a slit).

Let ϕ_Λ denote the random-cluster measure on Ω_Λ with parameters λ, δ, $q = 2$ (with top/bottom periodic boundary condition). Let Γ be a measurable subset and Δ a finite subset of Λ such that $\Delta \cap \Gamma = \varnothing$. We shall prove a 'ratio weak-mixing property' of the spin-configurations in Δ and Γ. In order to introduce the necessary couplings, we consider next a certain 'wired' boundary condition on Λ. Let $\overline{\phi}$ denote the continuum random-cluster measure on Λ with parameters λ, δ, $q = 2$, but subject to the difference that the set of clusters that intersect $\Delta \cup \Gamma$ count only 1 in all towards the cluster count $k(\omega)$ in (5.2). We call $\overline{\phi}$ a 'wired random-cluster measure'. It is standard, just as in the discrete case, that $\overline{\phi}$ may be used to generate a random spin-configuration on Λ corresponding to a continuum Ising model *conditioned* on having the same spin at all points in $\Delta \cup \Gamma$: let ω be sampled according to $\overline{\phi}$, and allocate a randomly chosen spin from the spin set $\{-1, +1\}$ to each cluster of ω, these spins being independent between clusters.

Just as in the lattice case, one may use $\overline{\phi}$ to obtain random-cluster measures with other boundary conditions. Let $\tau \in \Sigma_\Gamma$, and let $T_i = \{x \in \Gamma : \tau(x) = i\}$ for $i = \pm 1$. The corresponding random-cluster measure, denoted ϕ_Λ^τ (as in Section 5), is that obtained by: (i) the set of clusters intersecting Γ counts only 1 in all towards the cluster count in (5.2), and (ii) we condition on the event that there exists no path joining T_1 and T_2. Since $\overline{\phi} \geq_{\mathrm{st}} \phi_\Lambda^\tau$, there exists a coupling κ of the two measures with the property that $\kappa((\omega_1, \omega_2) : \omega_1 \geq \omega_2) = 1$. It is natural to allocate spins to the clusters of ω_1 and ω_2 in such a way that, whenever a cluster C of ω_2 is also a cluster of ω_1, and $C \cap \Gamma = \varnothing$, then these two clusters have the same spin.

One may carry out the above construction simultaneously for two (or more) τ. Let $\tau, \tau' \in \Sigma_\Gamma$. We may find a coupling of $\overline{\phi}$, ϕ_Λ^τ, $\phi_\Lambda^{\tau'}$ such that the first component is greater than each of the other two. That is, there exists a measure κ on $\Omega_\Lambda^3 = \{(\omega, \omega_1, \omega_2)\}$ such that: ω (respectively, ω_1, ω_2) has law $\overline{\phi}$ (respectively, ϕ_Λ^τ, $\phi_\Lambda^{\tau'}$), and $\kappa(\omega \geq \omega_1, \omega_2) = 1$.

Theorem 7.1 (Ratio weak-mixing). *Let $\Gamma \subseteq \Lambda$ be measurable, let $\Delta \subseteq \Lambda$ be finite such that $\Delta \cap \Gamma = \varnothing$, and let D be a linear subset of Λ that separates Δ and Γ. Let $\lambda, \delta \in (0, \infty)$. For $\tau, \tau' \in \Sigma_\Gamma$ and $\alpha \in \Sigma_\Delta$,*

$$\left| \frac{\phi_\Lambda^\tau(\sigma_\Delta = \alpha)}{\phi_\Lambda^{\tau'}(\sigma_\Delta = \alpha)} - 1 \right| \leq 2 \left(t_1 + 2t_2 + \frac{t_1 + t_2}{1 - t_1 - 2t_2} \right), \tag{7.2}$$

whenever the right side is less than or equal to 1*, and where*

$$t_1 = \overline{\phi}(\Delta \leftrightarrow D), \qquad t_2 = \sqrt{\overline{\phi}(D \leftrightarrow \Gamma)}. \tag{7.3}$$

The corresponding conclusion is valid when Λ is taken as the slit box $\Lambda_{m,\beta}$. Note in this case that the t_i are given in terms of connection probabilities in the slit box.

Proof. We adapt the methods of [2]. Let I (respectively, E) be the region of Λ reachable from Δ (respectively, Γ) along paths of Λ not intersecting D.

Let $\tau, \tau' \in \Sigma_\Gamma$ and $\alpha \in \Sigma_\Delta$. We construct a coupling as follows, using the approach summarised prior to the statement of the theorem. Let $\overline{\omega}$ have law $\overline{\phi}$. Let $\omega = \omega^\tau$ and $\omega' = \omega^{\tau'}$ have laws ϕ_Λ^τ and $\phi_\Lambda^{\tau'}$, respectively, and be such that $\omega, \omega' \le \overline{\omega}$. Furthermore, we construct ω and ω' in such a way that, if $\overline{\omega} \in E_2 = \{D \nleftrightarrow \Gamma\}$, then $\overline{\omega}$, ω, and ω' are identical on $D \cup I$.

To the clusters of $\overline{\omega}$, ω, ω' we assign spins in the usual manner, denoted $\overline{\sigma}$, σ, σ', respectively, such that: on the event E_2, the functions $\overline{\sigma}$, σ, σ' are equal on $D \cup I$. For a reason that will be clearer later, we shall not work with the pair σ, σ' of configurations but instead with a pair ρ, ρ' defined as follows. First, we set

$$\rho_x = \sigma_x,\ \rho'_x = \sigma'_x \qquad \text{for } x \in D \cup E.$$

On the event $F = \{\rho_D = \rho'_D\}$, we sample from the measure ϕ_Λ given F to obtain a (random) configuration $\zeta \in \Sigma_I$, and we set

$$\rho_x = \rho'_x = \zeta_x \qquad \text{for } x \in I.$$

On the complement of F, we sample ρ (respectively, ρ') according to the conditional law ϕ_Λ^τ given $(\rho_x : x \in D \cup E)$ (respectively, $\phi_\Lambda^{\tau'}$ given $(\rho'_x : x \in D \cup E)$). By the spatial Markov property of the continuum Ising model alluded to after (5.4), ρ (respectively, ρ') has law ϕ_Λ^τ (respectively, $\phi_\Lambda^{\tau'}$), and furthermore:

$$\rho_I = \rho'_I \quad \text{on the event} \quad \{\rho_D = \rho'_D\}, \tag{7.4}$$

and

$$\kappa(\rho_D = \rho'_D) = \kappa(\sigma_D = \sigma'_D) \ge \kappa(E_2) = 1 - t_2^2, \tag{7.5}$$

where κ is the appropriate probability measure, and t_2 is as in (7.3).

Let H be an event satisfying

$$H \subseteq \{\rho_\Delta = \rho'_\Delta\}. \tag{7.6}$$

As in [2], if $\kappa(H) > 0$,

$$\begin{aligned}
\frac{\phi_\Lambda^\tau(\sigma_\Delta = \alpha)}{\phi_\Lambda^{\tau'}(\sigma_\Delta = \alpha)} &= \frac{\kappa(\rho_\Delta = \alpha)}{\kappa(\rho'_\Delta = \alpha)} \\
&= \frac{\kappa(H \cap \{\rho_\Delta = \alpha\})}{\kappa(H \mid \rho_\Delta = \alpha)} \cdot \frac{\kappa(H \mid \rho'_\Delta = \alpha)}{\kappa(H \cap \{\rho'_\Delta = \alpha\})} \\
&= \frac{\kappa(H \mid \rho'_\Delta = \alpha)}{\kappa(H \mid \rho_\Delta = \alpha)}. \qquad (7.7)
\end{aligned}$$

It thus suffices, by an elementary argument, to prove that

$$\kappa(\overline{H} \mid \rho_\Delta = \alpha),\ \kappa(\overline{H} \mid \rho'_\Delta = \alpha) \le t \qquad (7.8)$$

where

$$t = t_1 + 2t_2 + \frac{t_1 + t_2}{1 - t_1 - 2t_2}. \qquad (7.9)$$

To see this, assume (7.8) with $t \le \frac{1}{2}$. By (7.7),

$$1 - t \le \frac{\phi_\Lambda^\tau(\sigma_\Delta = \alpha)}{\phi_\Lambda^{\tau'}(\sigma_\Delta = \alpha)} \le \frac{1}{1-t}.$$

Now, $1/(1-t) \le 1 + 2t$ since $t \le \frac{1}{2}$, and (7.2) follows.

There are four steps in proving (7.8). Let $\mathcal{G}_D$ (respectively, $\mathcal{G}'_D$) be the σ-field generated by ρ_D (respectively, ρ'_D). Firstly, given that $\overline{\omega} \in E_1 = \{\Delta \not\leftrightarrow D\}$, the spin-vector σ_D is (conditionally) independent of σ_Δ, whence

$$\bigl|\kappa(\sigma_D \in A \mid \sigma_\Delta = \alpha) - \kappa(\sigma_D \in A \mid \sigma_\Delta = \alpha')\bigr| \le t_1, \quad A \in \mathcal{G}_D,\ \alpha' \in \Sigma_\Delta,$$

with t_1 as in (7.3). Averaging over α', we obtain

$$\bigl|\kappa(\sigma_D \in A \mid \sigma_\Delta = \alpha) - \kappa(\sigma_D \in A)\bigr| \le t_1,$$

and hence, by the equidistribution of σ and ρ,

$$\bigl|\kappa(\rho_D \in A \mid \rho_\Delta = \alpha) - \kappa(\rho_D \in A)\bigr| \le t_1, \qquad A \in \mathcal{G}_D. \qquad (7.10)$$

Secondly, let

$$g = \kappa(\rho_D \ne \rho'_D \mid \mathcal{G}_D), \qquad g' = \kappa(\rho_D \ne \rho'_D \mid \mathcal{G}'_D),$$

and, for $a > 0$, let $H = H_a$ be given as

$$H_a = \{\rho_D = \rho'_D\} \cap \{g \le a\} \cap \{g' \le a\},$$

where a will be chosen later. It is easily seen by (7.4) that H_a satisfies (7.6). By Markov's inequality and (7.5),

$$\kappa(g > a) \le \frac{1}{a}\kappa(g) \le \frac{1}{a}t_2^2,$$

and therefore, since $\{g > a\} \in \mathcal{G}_D$,

$$\begin{aligned}\kappa(g > a \mid \rho_\Delta = \alpha) &\le \kappa(g > a) + t_1 \quad \text{by (7.10)}\\ &\le \frac{1}{a}t_2^2 + t_1. \end{aligned} \tag{7.11}$$

By a similar argument,

$$\kappa(g' > a \mid \rho'_\Delta = \alpha) \le \frac{1}{a}t_2^2 + t_1. \tag{7.12}$$

Thirdly,

$$\begin{aligned}\kappa(\rho_D \ne \rho'_D,\ g \le a \mid \rho_\Delta = \alpha) &\le \operatorname{ess\,sup}\big\{\kappa(\rho_D \ne \rho'_D \mid \mathcal{G}_D)1_{\{g\le a\}}\big\}\\ &= \operatorname{ess\,sup}\{g1_{\{g\le a\}}\} \le a,\end{aligned} \tag{7.13}$$

and similarly,

$$\kappa(\rho_D \ne \rho'_D,\ g' \le a \mid \rho'_\Delta = \alpha) \le a. \tag{7.14}$$

Finally, by (7.4),

$$\{\rho_D = \rho'_D\} \cap \{\rho_\Delta = \alpha\} = \{\rho_D = \rho'_D\} \cap \{\rho'_\Delta = \alpha\}, \tag{7.15}$$

[this is where we use ρ, ρ' in place of σ, σ'], and, by (7.12) and (7.14)–(7.15),

$$\begin{aligned}\kappa(\rho_D = \rho'_D,\ g' > a \mid \rho_\Delta = \alpha) &\le \kappa(g' > a \mid \rho_D = \rho'_D,\ \rho'_\Delta = \alpha)\\ &\le \frac{\kappa(g' > a \mid \rho'_\Delta = \alpha)}{\kappa(\rho_D = \rho'_D \mid \rho'_\Delta = \alpha)}\\ &\le \frac{t_1 + t_2^2/a}{1 - a - t_1 - t_2^2/a}.\end{aligned} \tag{7.16}$$

On combining (7.11), (7.13), (7.16), and setting $a = t_2$, we obtain the first inequality of (7.8) with $H = H_a$, and the second inequality holds similarly. □

Let Δ and Γ be disjoint finite subsets of Λ that are disjoint from $\partial^{\mathrm{h}}\Lambda$. Let D be an linear subset of Λ that separates Δ and $\Gamma \cup \partial^{\mathrm{h}}\Lambda$. Let $\alpha \in \Sigma_\Delta$, $\beta, \beta' \in \Sigma_\Gamma$, and $\eta \in \Sigma_{\partial^{\mathrm{h}}\Lambda}$. By (7.2) applied to the sets Δ and $\Gamma \cup \partial^{\mathrm{h}}\Lambda$,

$$\big|\phi_\Lambda^{\beta,\eta}(\sigma_\Delta = \alpha) - \phi_\Lambda^{\beta',\eta}(\sigma_\Delta = \alpha)\big| \le 2t\phi_\Lambda^{\beta',\eta}(\sigma_\Delta = \alpha), \tag{7.17}$$

whenever $t \le \frac{1}{2}$ where

$$t = t_1 + 2t_2 + \frac{t_1 + t_2}{1 - t_1 - 2t_2}, \tag{7.18}$$

and

$$t_1 = \overline{\phi}(\Delta \leftrightarrow D), \qquad t_2 = \sqrt{\overline{\phi}(D \leftrightarrow \Gamma \cup \partial^{\mathrm{h}}\Lambda)}. \tag{7.19}$$

The suffix β, η in (7.17) indicates the composite boundary condition taking the values β on Γ and η on $\partial^{\mathrm{h}}\Lambda$. We average (7.17) over β' to obtain

$$\left|\phi_\Lambda^{\beta,\eta}(\sigma_\Delta = \alpha) - \phi_\Lambda^\eta(\sigma_\Delta = \alpha)\right| \le 2t\phi_\Lambda^\eta(\sigma_\Delta = \alpha). \tag{7.20}$$

Now,

$$\phi_\Lambda^{\beta,\eta}(\sigma_\Delta = \alpha) = \frac{\phi_\Lambda^\eta(\sigma_\Delta = \alpha,\, \sigma_\Gamma = \beta)}{\phi_\Lambda^\eta(\sigma_\Gamma = \beta)},$$

Let $A \in \mathcal{G}_\Delta$, $B \in \mathcal{G}_\Gamma$ be events with strictly positive probabilities. We 'multiply up' in (7.20) and sum over $\alpha \in A$ and $\beta \in B$ to find that

$$\left|\frac{\phi_\Lambda^\eta(A \cap B)}{\phi_\Lambda^\eta(A)\phi_\Lambda^\eta(B)} - 1\right| \le 2t, \qquad \eta \in \Sigma_{\partial^{\mathrm{h}}\Lambda}, \tag{7.21}$$

whenever $t \le \frac{1}{2}$. Upper bounds on t follow from the observation that $\overline{\phi}$ is stochastically dominated by the continuum percolation measure with parameters λ, δ (cf. (5.3)). Equation (7.21) is a general statement of so-called ratio weak-mixing.

By the same argument without the reference to the boundary $\partial^{\mathrm{h}}\Lambda$,

$$\left|\frac{\phi_\Lambda(A \cap B)}{\phi_\Lambda(A)\phi_\Lambda(B)} - 1\right| \le 2t, \qquad A \in \mathcal{G}_\Delta,\ B \in \mathcal{G}_\Gamma, \tag{7.22}$$

whenever $t \le \frac{1}{2}$, where t is in (7.18) with

$$t_1 = \overline{\phi}(\Delta \leftrightarrow D), \qquad t_2 = \sqrt{\overline{\phi}(D \leftrightarrow \Gamma)}, \tag{7.23}$$

and D is a linear set that separates Δ and Γ.

The above ideas may be used to prove Lemmas 6.8 and 6.9, for the first of which we argue as follows. Consider the box $\Lambda_{m,\beta}$ with slit S_L. Let K be an integer satisfying $0 < K < \frac{1}{2}L$, and let $\Delta = \{x^+ : x \in S_L,\ K \le x \le L - K\}$ and $\Gamma = \{x^- : x \in S_L,\ K \le x \le L - K\}$.

Lemma 7.24. *Let $\lambda, \delta \in (0, \infty)$. There exists $C = C(\lambda, \delta) \in (0, \infty)$ such that, if $\beta > 2m + L$,*

$$\left| \frac{\phi_{m,\beta}(\sigma_\Delta = \epsilon_K^+,\, \sigma_\Gamma = \epsilon_K^-)}{\phi_{m,\beta}(\sigma_\Delta = \epsilon_K^+)\phi_{m,\beta}(\sigma_\Gamma = \epsilon_K^-)} - 1 \right| \le Ce^{-\frac{1}{2}\gamma K}, \qquad \epsilon_K^+ \in \Sigma_\Delta,\ \epsilon_K^- \in \Sigma_\Gamma,$$

whenever the right side is less than or equal to 1. The function $\gamma(\lambda, \delta)$ may be taken as that in Theorem 6.7.

The proofs are preceded by a type of 'finite-energy' inequality (see [2, 12]).

Lemma 7.25. *Let S be a finite subset of Λ. For $x \in \Lambda \setminus S$, $\epsilon \in \Sigma_S = \{-1, +1\}^S$, and $\alpha \in \{-1, +1\}$,*

$$\phi_\Lambda(\sigma_S = \epsilon,\, \sigma_x = \alpha) \ge \tfrac{1}{2}\phi_\Lambda(\sigma_S = \epsilon)\mathbb{P}_{\Lambda,\lambda,\delta}(x \nleftrightarrow S). \tag{7.26}$$

Proof. Let $x \in S$, $\epsilon \in \Sigma_S$, and $\alpha \in \{-1, +1\}$. Let $E(\epsilon)$ be the decreasing event containing all $\omega \in \Omega_\Lambda$ such that: for all $s, t \in S$, $s \nleftrightarrow t$ whenever $\epsilon_s \ne \epsilon_t$. Recalling the manner in which spins are associated with clusters,

$$\phi_\Lambda(\sigma_S = \epsilon) = \phi_\Lambda(2^{-k(S)}1_{E(\epsilon)}), \qquad \epsilon \in \Sigma_S, \tag{7.27}$$

where $k(S)$ is the number of clusters intersecting S. Similarly,

$$\phi_\Lambda(\sigma_S = \epsilon,\, \sigma_x = \alpha) \ge \phi_\Lambda(2^{-k(S^+)}1_{E(\epsilon)}1_{x \nleftrightarrow S}), \tag{7.28}$$

where $S^+ = S \cup \{x\}$. Note that $k(S^+) = k(S) + 1$ when $x \nleftrightarrow S$.

For any event A,

$$\phi_\Lambda(2^{-k(S^+)}1_A) = \phi_\Lambda(2^{-k(S^+)})\widehat{\phi}(A) = K\widehat{\phi}(A), \tag{7.29}$$

where $K = \phi_\Lambda(2^{-k(S^+)})$ and $\widehat{\phi}$ is the continuum random-cluster measure on Λ with a wired boundary condition on S^+, that is, all clusters intersecting S^+ are counted as one. By (7.29) and the FKG inequality applied to $\widehat{\phi}$,

$$\begin{aligned} \phi_\Lambda(2^{-k(S^+)}1_{E(\epsilon)}1_{x \nleftrightarrow S}) &= K\widehat{\phi}(E(\epsilon) \cap \{x \nleftrightarrow S\}) \\ &\ge K\widehat{\phi}(E(\epsilon))\widehat{\phi}(x \nleftrightarrow S) \\ &= \phi_\Lambda(2^{-k(S^+)}1_{E(\epsilon)})\widehat{\phi}(x \nleftrightarrow S). \end{aligned}$$

Now $k(S) \le k(S^+) \le k(S) + 1$, so that, by (7.27)–(7.28),

$$\phi_\Lambda(\sigma_S = \epsilon,\, \sigma_x = \alpha) \ge \tfrac{1}{2}\phi_\Lambda(\sigma_S = \epsilon)\widehat{\phi}(x \nleftrightarrow S)$$

and the claim follows by the stochastic inequality (5.3). ∎

Proof of Lemma 7.24. Take $D = \{(x,0) : x \in [-m,0) \cup (L, L+m]\}$, the union of the two horizontal line-segments that, when taken with the slit S_L, complete the 'equator' of $\Lambda_{m,\beta}$. Thus, D is a linear subset of $\Lambda_{m,\beta}$ separating Δ and Γ. Since $\overline{\phi} \le_{\mathrm{st}} \mathbb{P}_{\Lambda,\lambda,\delta}$, by Theorem 6.7 there exist constants C, C' depending on λ and δ alone, such that

$$t_1 = \overline{\phi}(\Delta \leftrightarrow D) \le 2 \sum_{i=K}^{\lfloor L/2 \rfloor} Ce^{-\gamma i} \le C'e^{-\gamma K},$$

and furthermore $t_2^2 = t_1$. The claim now follows by (7.22). □

Proof of Lemma 6.8. Let γ be given as in Theorem 6.7. With $K = \lceil \ln L \rceil$, let $\sigma^{\pm}_{L,K} = (\sigma^{\pm}_x : K \le x \le L-K)$. We may apply Lemma 7.25 as follows in order to compare the laws of the spin-vector $\sigma^{\pm}_L$ and that of the reduced vector $\sigma^{\pm}_{L,K}$. First, let $x = (L,0)$, and let $\epsilon^+, \epsilon^- \in \{-1,+1\}^{L+1}$ be possible spin-vectors of the sets S^+_L and S^-_L, respectively. By Lemma 7.25 with $S = S^+_L \cup S^-_L \setminus \{x^+\}$,

$$\begin{aligned}\phi_{m,\beta}&(\sigma^+_L = \epsilon^+,\ \sigma^-_L = \epsilon^-)\\ &\ge \tfrac{1}{2}\phi_{m,\beta}(\sigma^+_y = \epsilon^+_y \text{ for } y \in S^+_L \setminus \{x^+\},\ \sigma^-_L = \epsilon^-)\mathbb{P}_{\Lambda_{m,\beta},\lambda,\delta}(x^+ \nleftrightarrow S).\end{aligned}$$

Now, $\mathbb{P}_{\Lambda_{m,\beta},\lambda,\delta}(x \nleftrightarrow S)$ is at least as large as the probability that the first event (death or bridge) encountered on moving northwards from x is a death. That is,

$$\mathbb{P}_{\Lambda_{m,\beta},\lambda,\delta}(x \nleftrightarrow S) \ge \frac{\delta}{2\lambda+\delta}.$$

On iterating the above argument, we obtain that

$$\phi_{m,\beta}(\sigma^+_L = \epsilon^+,\ \sigma^-_L = \epsilon^-) \ge \left(\frac{\delta}{2(2\lambda+\delta)}\right)^{4K} \phi_{m,\beta}(\sigma^+_{L,K} = \epsilon^+_K,\ \sigma^-_{L,K} = \epsilon^-_K), \tag{7.30}$$

where $\epsilon^{\pm}_K$ is the vector obtained from $\epsilon^{\pm}$ by removing the entries labelled by vertices x satisfying $0 \le x < K$ and $L-K < x \le L$. In summary, there exist $C, \alpha \in (0,\infty)$ depending on λ, δ such that, for $\epsilon^{\pm} \in \Sigma_L$,

$$\begin{aligned}CL^{-2\alpha}\phi_{m,\beta}(\sigma^+_{L,K} = \epsilon^+_K,\ \sigma^-_{L,K} = \epsilon^-_K) &\le \phi_{m,\beta}(\sigma^+_L = \epsilon^+,\ \sigma^-_L = \epsilon^-)\\ &\le \phi_{m,\beta}(\sigma^+_{L,K} = \epsilon^+_K,\ \sigma^-_{L,K} = \epsilon^-_K).\end{aligned}$$

Set $\Delta = \{x^+ : x \in S_L,\ K \le x \le L-K\}$, $\Gamma = \{x^- : x \in S_L,\ K \le x \le L-K\}$, and apply Lemma 7.24 to obtain that there exists $C = C(\lambda,\delta) < \infty$ such that

$$\left| \frac{\phi_{m,\beta}(\sigma^+_{L,K} = \epsilon^+_K,\ \sigma^-_{L,K} = \epsilon^-_K)}{\phi_{m,\beta}(\sigma^+_{L,K} = \epsilon^+_K)\phi_{m,\beta}(\sigma^-_{L,K} = \epsilon^-_K)} - 1 \right| \le Ce^{-\frac{1}{2}\gamma K} \le CL^{-\frac{1}{2}\gamma},$$

whenever (say) the right side is less than or equal to $\frac{1}{2}$, say for $L \geq L_0(\lambda, \delta)$.

By Lemma 7.25 again, for suitable C', α,

$$C' L^{-\alpha} \phi_{m,\beta}(\sigma^{\pm}_{L,K} = \epsilon^{\pm}_K) \leq \phi_{m,\beta}(\sigma^{\pm}_L = \epsilon^{\pm}) \leq \phi_{m,\beta}(\sigma^{\pm}_{L,K} = \epsilon^{\pm}_K).$$

The claim now follows for $L \geq L_0$, with suitable values of C_1, C_2, α. We may adjust the constants to obtain the required inequality for all $L \geq 0$. $\square$

Proof of Lemma 6.9. Let $\Delta = S^+_L \cup S^-_L$ and $\Gamma = \partial^{\rm h} \Lambda_{m,\beta}$. Let $k = \frac{3}{7}m$ and assume for simplicity that k is an integer. [If either m is small or k is non-integral, the constant C may be adjusted accordingly.] Let D be the circuit illustrated in Figure 4, comprising a path in the upper half-plane from $(-k, 0)$ to $(L + k, 0)$ together with its reflection in the x-axis.

By Theorem 7.1,

$$\left| \frac{\phi^{\alpha}_{m,\beta}((\sigma^+_L, \sigma^-_L) = (\epsilon^+, \epsilon^-))}{\phi_{m,\beta}((\sigma^+_L, \sigma^-_L) = (\epsilon^+, \epsilon^-))} - 1 \right| \leq 2t, \qquad \alpha = \eta, \tau,\ \epsilon^{\pm} \in \Sigma_L,$$

whenever $t \leq \frac{1}{2}$, with t as in (7.18). We 'multiply up' and sum over $(\epsilon^+, \epsilon^-) \in A$ to obtain

$$\left| \frac{\phi^{\alpha}_{m,\beta}(\sigma_\Delta \in A)}{\phi_{m,\beta}(\sigma_\Delta \in A)} - 1 \right| \leq 2t, \tag{7.31}$$

whenever $t \leq \frac{1}{2}$.

By (5.3), $\overline{\phi} \leq_{\rm st} \mathbb{P}_{\Lambda,\lambda,\delta}$. Let $\beta \geq 4(m + L + 1)$. It is a straightforward consequence of Theorem 6.7 that there exist $C, C', c' > 0$, depending on λ, δ only, such that

$$t_1 \leq 4 \sum_{i=0}^{\lfloor L/2 \rfloor} \mathbb{P}_{\lambda,\delta}((i, 0) \leftrightarrow D) \leq 4 \sum_{i=0}^{\lfloor L/2 \rfloor} C e^{-\gamma \frac{2}{3}(k+i)} \leq C' e^{-\frac{2}{7}\gamma m}, \tag{7.32}$$

and similarly,

$$t_2^2 \leq 8 \sum_{i=0}^{\lceil k+L/2 \rceil} C e^{-\gamma(\frac{4}{7}m + c'i)} \leq C' e^{-\frac{4}{7}\gamma m}, \tag{7.33}$$

with γ given as in Theorem 6.7. The claim of the lemma follows. $\square$

8 Disordered interactions

We have so far assumed that the spin-couplings $\lambda_{x,x+1}$ and the field-strengths δ_x appearing in the Hamiltonian (1.2) are constant. The situation is more

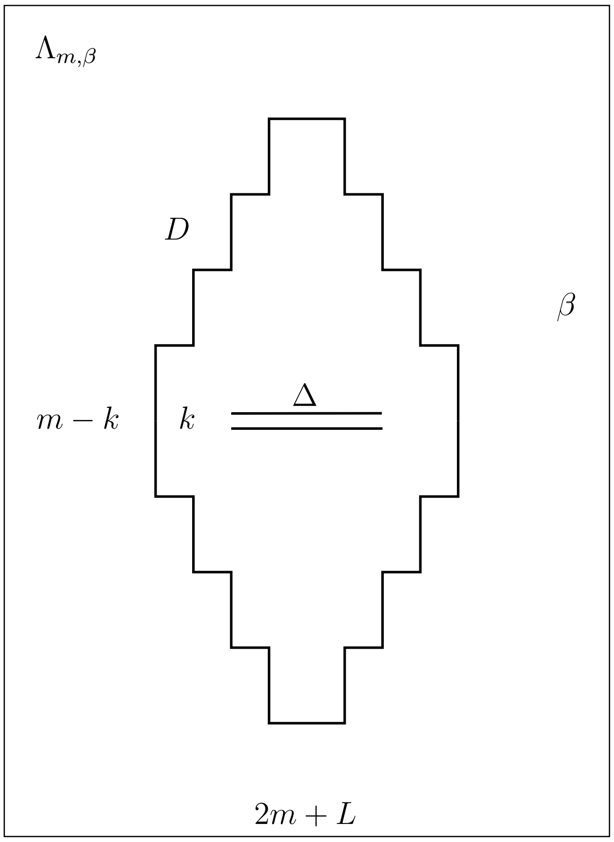

Figure 4: The circuit D is approximately a parallelogram with Δ at its centre. The sides comprise vertical steps of height 2 followed by horizontal steps of length 1. The horizontal and vertical diagonals have lengths $2k+L$ and (order) $4k+2L$ respectively, where $k = \frac{3}{7}m$.

complicated if: either the environment of couplings and strengths vary about the space $\mathbb{Z}$, or they are random (in which case the model is said to be disordered). The arguments of this paper may be applied in each case, and the outcomes are summarised in this section.

Suppose first that the $\lambda_{x,x+1}$ and δ_x are non-constant. The fundamental bound of Theorem 6.5 depends only on the ratio $\theta = \lambda/\delta$, and the connection probabilities of the continuum random-cluster model are increasing in the $\lambda_{x,x+1}$ and decreasing in the δ_x. One may therefore check that the conclusions of the paper are valid with $\gamma = \gamma(\lambda, \delta)$ whenever

$$\lambda_{x,y}/\delta_x \le \lambda/\delta, \qquad y = x-1, x+1,\ x \in \mathbb{Z}. \tag{8.1}$$

Hence, in the disordered case where (8.1) holds with probability one, the corresponding conclusion is valid.

We turn to the situation in which (8.1) does not hold with probability one. Suppose that the $\lambda_{x,x+1}$, $x \in \mathbb{Z}$, are independent, identically distributed random variables, and similarly the δ_z, $z \in \mathbb{Z}$, and assume that the $\lambda_{x,y}$ are independent of the δ_z. We write P for the corresponding probability measure, viewed as the measure governing the 'random environment', and Λ, Δ for a typical spin-correlation and field-strength, respectively. Conditional on $\lambda = (\lambda_{x,x+1} : x \in \mathbb{Z})$ and $\delta = (\delta_z : z \in \mathbb{Z})$, we write $\mathbb{P}_{\lambda,\delta}$ for the probability measure of the associated continuum percolation process. In applying the methods of this paper within the random environment, one needs to deal with sub-domains of $\mathbb{Z}$ where the environment is not propitious for the bound of Theorem 6.5. As before, we perform a comparison of the continuum random-cluster model and continuum percolation in a random environment, and we shall appeal to the following theorem of [17] (see also Theorem 1.6 of [1]).

For $(x,s),(y,t) \in \mathbb{Z}\times\mathbb{R}$ and $q \ge 1$, let

$$d_q(x,s;y,t) = \max\bigl\{|x-y|, (\ln^+|s-t|)^q\bigr\},$$

where $\ln^+ x = \max\{\ln x, 0\}$.

Theorem 8.2. [17] *Consider continuum percolation on $\mathbb{Z}\times\mathbb{R}$ in a random environment satisfying*

$$\Gamma = \max\bigl\{P\bigl([\ln(1+\Lambda)]^\beta\bigr), P\bigl([\ln(1+\Delta^{-1})]^\beta\bigr)\bigr\} < \infty,$$

for some

$$\beta > 5 + \tfrac{7}{2}\sqrt{2}. \tag{8.3}$$

There exists $Q = Q(\beta) > 1$ such that the following holds. For $q \in [1,Q)$ and $\gamma > 0$, there exists $\epsilon = \epsilon(\beta,\Gamma,\gamma,q) > 0$ and $\eta = \eta(\beta,q) > 1$ such that: if

$$P\Bigl(\bigl[\ln(1+(\Lambda/\Delta))\bigr]^\beta\Bigr) < \epsilon, \tag{8.4}$$

there exist identically distributed, positive random variables $D_x \in L^\eta(P)$, $x \in \mathbb{Z}$, such that

$$\mathbb{P}_{\lambda,\delta}\bigl((x,s) \leftrightarrow (y,t)\bigr) \le \exp\bigl[-\gamma d_q(x,s;y,t)\bigr] \quad \text{if } d_q(x,s;y,t) \ge D_x, \tag{8.5}$$

for $(x,s),(y,t) \in \mathbb{Z}\times\mathbb{R}$.

The lower bound (8.3) for β is enough to imply that $P(D_x^\eta) < \infty$ for some $\eta > 1$. The larger β, the larger η may be taken.

For the remainder of this section we assume that the conditions of the above theorem are valid, and we shall work with the conclusion (8.5), with

$q = 1$, $\gamma > 1$, and the D_x given accordingly. We let $L \geq 8$ and $K = \lceil \ln L \rceil$, and consider the event

$$A_L = \bigcap_{x=K}^{L-K} \big\{ D_x < \min\{x, L - x\} \big\},$$

noting that

$$P(A_L) \geq 1 - 2 \sum_{x=K}^{\infty} P(D \geq x),$$

where D has the distribution of the D_x. Since $P(D) < \infty$,

$$P(A_L) \to 1 \qquad \text{as } L \to \infty. \tag{8.6}$$

An estimate for the rate of convergence may be obtained (here and later) by the fact that $P(D^\eta) < \infty$ for some $\eta > 1$.

We comment next on the adaptation of our earlier results to the disordered setting. Theorem 7.1 holds within the random environment, without change. The conclusion of Lemma 7.24 is valid with $K = \lceil \ln L \rceil$ whenever the event A_L occurs. Lemma 7.25 holds unconditionally. The conclusion of Lemma 6.8 holds on A_L with the lower bound $C_1 L^{-\alpha}$ replaced by CX_L and the upper bound $C_2 L^\alpha$ replaced by $(CX_L)^{-1}$, with C a constant and

$$X_L = \prod_{x \in \Theta} \frac{\delta_x}{\delta_x + \lambda_{x,x-1} + \lambda_{x,x+1}},$$

where, in the notation of the proof of Lemma 6.8, $\Theta = (S_L^+ \setminus \Delta) \cup (S_L^- \setminus \Gamma)$. Now,

$$\ln X_L = -2 \sum_{x=0}^{K-1} Z_x - 2 \sum_{x=L-K+1}^{L} Z_x \tag{8.7}$$

where

$$Z_x = \ln \left(1 + \frac{\lambda_{x,x-1} + \lambda_{x,x+1}}{\delta_x} \right).$$

The two summations in (8.7) are independent of one another, and each is the sum of a 1-dependent sequence of random variables. Also,

$$Z_x \leq \ln \left(1 + \frac{\lambda_{x,x-1}}{\delta_x} \right) + \ln \left(1 + \frac{\lambda_{x,x+1}}{\delta_x} \right),$$

so that, by (8.4) and the Minkowski inequality,

$$\sqrt{P(Z_x^2)} \leq 2 \sqrt{P\Big(\big[\ln(1 + (\Lambda/\Delta)) \big]^2 \Big)} < \infty.$$

By the central limit theorem for 1-dependent sequences (see, for example, Theorem 19.2.1 of [15]),

$$P(B_L^\rho) \to 1 \qquad \text{as } L \to \infty, \tag{8.8}$$

where $B_L^\rho = \{X_L \geq L^{-\rho}\}$ and $\rho \in (0, \infty)$ satisfies

$$\rho > 4P(Z_0). \tag{8.9}$$

Some changes are necessary to the proof of Lemma 6.9, reflecting the fact that the decay in (8.5) is sub-exponential in time. The circuit illustrated in Figure 4 is generated by translation, discretisation, and reflection of the Cartesian line $y = 2x$. In the disordered setting, we work instead with the curve $y = e^x$, and we assume $\beta > 5e^{m+\frac{1}{2}L}$. We define two further events that depend on the environment. Assume for simplicity that m is even, write $k = \frac{1}{2}m$, and let

$$C_{L,m} = \bigcap_{x=0}^{L} \big\{ D_x < \tfrac{1}{2} \min\{k + x, L + k - x\} \big\},$$
$$D_{L,m} = \bigcap_{x=-k}^{L+k} \big\{ D_x < \min\{m + x, L + m - x\} \big\}.$$

In the current setting, (7.32) becomes

$$t_1 \leq C_1 e^{-\frac{1}{4}\gamma m} \quad \text{on the event} \quad C_{L,m},$$

for some constant C_1 depending on γ. Similarly, (7.33) is replaced by

$$t_2^2 \leq C_2 e^{-\frac{1}{2}\gamma m} \quad \text{on the event} \quad D_{L,m}.$$

An amended version of Lemma 6.9 thus holds, so long as the event $C_{L,m} \cap D_{L,m}$ occurs.

We estimate $P(C_{L,m} \cap D_{L,m})$ as follows. First, since $P(D) < \infty$,

$$P(C_{L,m}) \geq 1 - 2 \sum_{x=0}^{\lfloor \frac{1}{2}L \rfloor} P(D_x \geq \tfrac{1}{2}(k + x)) \to 1 \qquad \text{as } m \to \infty. \tag{8.10}$$

Similarly,

$$P(D_{L,m}) \geq 1 - 2 \sum_{x=-k}^{\lfloor \frac{1}{2}L \rfloor} P(D_x \geq m + x) \to 1 \qquad \text{as } m \to \infty. \tag{8.11}$$

Suppose that $A_L \cap B_L^\rho \cap C_{L,m} \cap D_{L,m}$ occurs for some ρ satisfying (8.9). The principal estimate (2.3) follows with CL^α replaced by CL^ρ as above. On the above event, the proof of Theorem 2.8 may be followed to obtain the logarithmic decay of entanglement. Note from (8.6) and (8.8) that $P(A_L \cap B_L^\rho) \to 1$ as $L \to \infty$, and by (8.10)–(8.11) that $P(C_{L,m} \cap D_{L,m}) \to 1$ as $m \to \infty$.

Proof of Theorem 8.2. This is essentially Theorem 1.1 of [17] with $d = 1$, subject to two differences: the right side of (8.5) is expressed differently in [17], and the condition on β is different. The present statement is obtained as follows from the proof of [17], using the notation of that proof. With β satisfying (8.3) and $\alpha = 1 + \sqrt{2}$, we pick $p > 2\alpha$ and $\nu = q^{-1}$ satisfying (3.3) of [17]. Let K_x denote the minimal k_1 in the second paragraph of the proof of Theorem 3.3 of [17]. As there,

$$P(K_x > r) \le \frac{c}{L_r^{p-\alpha}}, \qquad r \ge 1,$$

where c is a constant, and $(L_r : r \ge 1)$ is a sequence of positive reals given by $L_r = L^{\alpha^r}$ for some large L. Let $D_x = bL_{K_x}$. Inequality (8.5) holds by the argument of [17]. Furthermore, for $\eta > 1$, $P(D_x^\eta)$ has the same order as

$$b^\eta \sum_{a=0}^{\infty} a^{\eta-1} P(L_{K_x} > a) \le b^\eta \sum_{r=0}^{\infty} L^{\eta\alpha^{r+1}} \cdot \frac{1}{L^{(p-\alpha)\alpha^r}}, \tag{8.12}$$

which is finite whenever $\eta - 1$ is small and positive. □

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