THE RANDOM-CLUSTER MODEL

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Abstract. It is well known that percolation, Ising, and Potts models are special cases of the random-cluster model of Fortuin and Kasteleyn. This paper is an account of the basic properties of this model. Of primary interest is the study of the associated phase transitions. A version of the random-cluster model was discovered by Peter Whittle in his study of the mathematics of polymerization.

1. Introduction

The Ising model [30] is a standard mathematical model for ferromagnetism; it exhibits a phase transition and an interesting range of critical phenomena. Whereas the Ising model permits only two possible spins at each site, the Ashkin–Teller and Potts models allow a general number of spin values ([5, 43]). In the late 1960s, Piet Kasteleyn observed that certain observables of electrical networks, percolation processes, and Ising models have features in common, namely versions of the series and parallel laws. In an investigation pursued jointly with Kees Fortuin, he came upon a one-parameter class of measures which includes in its ranks the percolation, Ising, and Potts models, together with (in a certain limit) electrical networks. This class is simple to describe and has rich structure; it is the class of *random-cluster models*.

Unlike its more classical counterparts in statistical physics, the random-cluster model is a process which lives on the *edges* of a graph rather than on its *vertices*. It is a random graph whose structure provides information concerning the nature of the phase transition in physical systems. Not only does the model incorporate a unifying description of physical models, but also it provides a natural setting for certain techniques of value ([3, 7, 9, 23, 36, 45]). The present paper has as purpose to present a compact account of some of the basic properties of random-cluster measures.

We start with a finite graph $G = (V, E)$, and let p and q be real parameters satisfying $0 \leq p \leq 1$ and $q > 0$. The set of realizations of a random-cluster process on G is the space $\Omega_E = \{0,1\}^E$ of 'edge-configurations'. A typical realization is a vector $\omega = (\omega(e) : e \in E)$ of 0's and 1's. Instead of working with such a vector ω, it is often convenient to work with the set $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ of 'open' edges.

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On Ω_E, we define the *random-cluster measure* $\phi_{p,q}$ by

$$\phi_{p,q}(\omega) = \frac{1}{Z_{p,q}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \quad \omega \in \Omega_E, \tag{1.1}$$

where

$$Z_{p,q} = \sum_{\omega \in \Omega_E} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)} \tag{1.2}$$

is the normalizing factor (or 'partition function') and $k(\omega)$ is the number of components of the graph $(V, \eta(\omega))$. In the special case when $q = 1$, the edge-variables $\omega(e)$ are independent. For more general values of q, the measure $\phi_{p,q}$ is obtained from product measure by the inclusion of a 'Radon–Nikodym derivative' that weights the probability of any given configuration according to the number of its components.

The main physical importance of random-cluster measures lies in their relationship with Potts models. We do not explore this here, choosing instead to concentrate on random-cluster measures *per se*. For details of the physical motivation, see [3, 16, 26]. Random-cluster measures have been considered independently by Peter Whittle [46] in his work on polymerization. Peter's 'first-shell model' is intimately related to the measure $\phi_{p,q}$ on the complete graph with n vertices, where $p = \lambda/n$ for some fixed positive λ. The consequent process, in the limit as $n \to \infty$, exhibits a phase transition which models the sol–gel transition of polymers. (See Section 6 and [12] for another treatment of this process.)

This paper is organized as follows. The next section contains statements of two fundamental properties of random-cluster measures, namely the FKG inequality and the comparison inequalities. In Section 3, we turn to the general notion of random-cluster measures on the infinite hypercubic lattice $\mathbb{Z}^d$; incorporated here is an account of the thermodynamic limit and of the uniqueness of random-cluster measures. Section 4 contains results about the phase transition in a general number of dimensions, and Section 5 contains corresponding material in two dimensions. Then there are sections about the mean-field theory, and about the history of random-cluster processes; the latter account is based on information kindly furnished to the present author by Piet Kasteleyn. Proofs are omitted throughout; further details may be found in [3, 25, 26].

This is not the first general account of the area. Much of the basic methodology was published first in the remarkable series of papers of Fortuin and Kasteleyn [18, 19, 20, 21, 22, 31]. Aizenman et al. [3] have provided a useful modern account of some of this material.

2. Useful inequalities

One of the most valuable properties of random-cluster measures is the FKG inequality, which is satisfied (in general) if and only if $q \geq 1$. This inequality has many applications. There appears to have been no serious study of the case $0 < q < 1$, presumably because the FKG inequality does not hold in this regime. Before stating the FKG inequality, we require some further notation.

There is a partial order on Ω_E given by $\omega \le w'$ if and only if $\omega(e) \le \omega'(e)$ for all $e \in E$. A function $f : \Omega_E \to \mathbb{R}$ is called *increasing* if $f(\omega) \le f(\omega')$ whenever $\omega \le \omega'$; f is *decreasing* if $-f$ is increasing. An event F $(\subseteq \Omega_E)$ is called *increasing* (respectively *decreasing*) if its indicator function I_F is increasing (respectively decreasing). Finally, we write $\mathbf{E}_{p,q}$ for expectation with respect to $\phi_{p,q}$.

Theorem 2.1 (FKG inequality). *Suppose that $q \ge 1$. If f and g are increasing functions on Ω_E, then*

$$\mathbf{E}_{p,q}(fg) \ge \mathbf{E}_{p,q}(f)\mathbf{E}_{p,q}(g)\,. \tag{2.1}$$

Replacing f and g by $-f$ and $-g$, we deduce that (2.1) holds for decreasing f and g, also. Specializing to indicator functions, we obtain that

$$\phi_{p,q}(A \cap B) \ge \phi_{p,q}(A)\phi_{p,q}(B) \quad \text{for increasing events } A, B, \tag{2.2}$$

whenever $q \ge 1$. It is not difficult to see that the FKG inequality does not generally hold when $0 < q < 1$.

A second valuable property of random-cluster measures is the pair of 'comparison inequalities', as follows. Given two mass functions μ_1 and μ_2 on Ω_E, we say that μ_2 *dominates* μ_1, and write $\mu_1 \le \mu_2$, if

$$\sum_{\omega \in \Omega_E} f(\omega)\mu_1(\omega) \le \sum_{\omega \in \Omega_E} f(\omega)\mu_2(\omega)$$

for all increasing functions $f : \Omega_E \to \mathbb{R}$. One may establish certain domination inequalities involving the measures $\phi_{p,q}$ for different values of the parameters p and q.

Theorem 2.2 (Comparison inequalities). *It is the case that*

$$\phi_{p',q'} \le \phi_{p,q} \qquad \textit{if} \quad q' \ge q,\ q' \ge 1,\ p' \le p, \tag{2.3}$$

$$\phi_{p',q'} \ge \phi_{p,q} \qquad \textit{if} \quad q' \ge q,\ q' \ge 1,\ \frac{p'}{q'(1-p')} \ge \frac{p}{q(1-p)}. \tag{2.4}$$

3. Infinite graphs and the thermodynamic limit

In studying random-cluster measures on lattices, we restrict ourselves to the case of the hypercubic lattice in d dimensions, where $d \ge 2$; similar observations are valid in greater generality. Let $d \ge 2$, and let $\mathbb{Z}^d$ be the set of all d-vectors of integers; if $x \in \mathbb{Z}^d$, we normally write $x = (x_1, x_2, \dots, x_d)$. For $x, y \in \mathbb{Z}^d$, let

$$\|x - y\| = \sum_{i=1}^{d} |x_i - y_i|\,.$$

We place an edge $\langle x, y\rangle$ between x and y if and only if $\|x-y\| = 1$; the set of such edges is denoted by $\mathbb{E}^d$, and we write $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ for the ensuing lattice. For any subset S of $\mathbb{Z}^d$, we write ∂S for its boundary, i.e.,

$$\partial S = \{s \in S : \langle s, t\rangle \in \mathbb{E}^d \text{ for some } t \notin S\}.$$

Let $\Omega = \{0,1\}^{\mathbb{E}^d}$ be the set of 'edge-configurations' of $\mathbb{L}^d$, and let $\mathcal{F}$ be the σ-field of subsets of Ω generated by the finite-dimensional cylinders. Following the general definition of a Gibbs state introduced by Dobrushin [15] and Lanford and Ruelle [38], we now define a random-cluster measure as follows. A probability measure ϕ on $(\Omega, \mathcal{F})$ is called a *random-cluster measure* if it has the property that, conditional on the states of edges lying outside any given finite set E ($\subseteq \mathbb{E}^d$), the distribution of the states of edges within E satisfies (1.1) with the appropriate boundary condition specifying which endpoints of edges in E are joined by edges outside E. In order to achieve a more precise definition, we introduce further notation. Any subset E of $\mathbb{E}^d$ generates a graph $G(E) = (V(E), E)$ having edge-set E and vertex-set $V(E)$ the set of all endvertices of members of E. For $\omega \in \Omega$, we denote by ω_E the cylinder event $\{\nu \in \Omega : \nu(e) = \omega(e) \text{ for all } e \in E\}$. Let $\omega \in \Omega$, and define the equivalence relation $\xrightarrow{\omega}$ on $V(E)$ given by $u \xrightarrow{\omega} v$ if and only if u and v are in the same component of the graph $(\mathbb{Z}^d, \eta(\omega) \setminus E)$. A probability measure ϕ on $(\Omega, \mathcal{F})$ is called a random-cluster measure (with parameters p and q) if, for all finite subsets E of $\mathbb{E}^d$, we have that

$$\phi(\omega_E \mid \omega_{\overline{E}}) = \phi_{E,\omega}(\xi) \quad \phi\text{-a.s.,} \tag{3.1}$$

where $\phi_{E,\omega}$ is the random-cluster measure given in (1.1) on the graph obtained from $G(E)$ by identifying any pair u, v of vertices satisfying $u \xrightarrow{\omega} v$, and where $\xi = (\omega(e) : e \in E)$, (and $\overline{E}$ is the complement of E). We write $\mathcal{R}_{p,q}$ for the class of random-cluster measures on $(\Omega, \mathcal{F})$ with parameters p and q.

We turn now to the thermodynamic limit. Let Λ be a finite box of $\mathbb{L}^d$, which is to say that

$$\Lambda = \prod_{i=1}^{d} [x_i, y_i]$$

for some $x, y \in \mathbb{Z}^d$; we interpret $[x_i, y_i]$ as the set $\{x_i, x_i+1, \ldots, y_i\}$. The set Λ generates a subgraph of $\mathbb{L}^d$ having vertex set Λ and edge set $\mathbb{E}_\Lambda$ containing all $\langle x, y\rangle$ with $x, y \in \Lambda$. We are interested in the limit (as $\Lambda \uparrow \mathbb{Z}^d$) of the random-cluster measure on the finite box Λ. Two such limits are relevant to the question of phase transition, depending on the 'boundary conditions'. Let Ω_Λ^1 be the subset of $\Omega = \{0,1\}^{\mathbb{E}^d}$ containing all $\omega \in \Omega$ for which $\omega(e) = 1$ for $e \notin \mathbb{E}_\Lambda$; similarly define Ω_Λ^0 as the subset of Ω containing all ω with $\omega(e) = 0$ for $e \notin \mathbb{E}_\Lambda$. One speaks of configurations in Ω_Λ^1 as having 'wired' boundary conditions, and configurations in Ω_Λ^0 as having 'free' boundary conditions. We now define two random-cluster measures. Let $0 \le p \le 1$ and $q > 0$. For $b = 0, 1$, define

$$\phi_{\Lambda,p,q}^b(\omega) = \frac{1}{Z_\Lambda^b} \left\{ \prod_{e \in \mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)}, \quad \omega \in \Omega_\Lambda^b, \tag{3.2}$$

where $k(\omega, \Lambda)$ is the number of components of ω which intersect Λ, and

$$Z_\Lambda^b = \sum_{\omega \in \Omega_\Lambda^b} \left\{ \prod_{e \in \mathbb{E}_\Lambda} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)} \tag{3.3}$$

is the appropriate normalizing constant.

Theorem 3.1. Thermodynamic limit. *Suppose $q \geq 1$. The weak limits*

$$\phi_{p,q}^b = \lim_{\Lambda \uparrow \mathbb{Z}^d} \phi_{\Lambda,p,q}^b, \quad \textit{for } b = 0, 1, \tag{3.4}$$

exist and satisfy $\phi_{p,q}^0 \leq \phi_{p,q}^1$. Furthermore $\phi_{p,q}^0, \phi_{p,q}^1 \in \mathcal{R}_{p,q}$.

The limits in (3.4) are to be interpreted along any increasing sequence of finite boxes, and the weak convergence is in the sense that $\phi_{\Lambda,p,q}^b(A) \to \phi_{p,q}^b(A)$ for all finite-dimensional cylinders A. The assumption that $q \geq 1$ is necessary for the proof, which relies on the validity of the FKG inequality.

One may discuss other boundary conditions, 'mixed' conditions which are more complicated than either wired or free; it is easy to see by the FKG inequality that $\phi_{p,q}^0$ and $\phi_{p,q}^1$ are the most 'extreme' measures obtainable in the infinite-volume limit. In particular we have (by applying the FKG inequality to (3.1) and passing to the limit as $E \uparrow \mathbb{E}^d$) that

$$\phi_{p,q}^0 \leq \phi \leq \phi_{p,q}^1 \quad \text{for all } \phi \in \mathcal{R}_{p,q}; \tag{3.5}$$

therefore there is a unique random-cluster measure if and only if $\phi_{p,q}^0 = \phi_{p,q}^1$.

An indicator of phase transition in the Potts model is its 'magnetization'. The corresponding macroscopic quantity for the random-cluster process is the *percolation probability* defined as

$$\theta(p,q) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \theta_\Lambda(p,q), \tag{3.6}$$

where $\theta_\Lambda(p,q) = \phi_\Lambda^1(0 \leftrightarrow \partial\Lambda)$ is the ϕ_Λ^1–probability of an open path from the origin to a vertex of $\partial\Lambda$. The limit exists in (3.6) if $q \geq 1$ (see [3, p. 22]). We have that $\theta(p,q) = \phi^1(0 \leftrightarrow \infty)$, the ϕ^1–probability that the origin is in an infinite cluster; in the case $q = 1$, this coincides with the 'percolation probability' of the percolation model (see [24]). Using the comparison inequality (2.3), $\theta(p,q)$ is a non-decreasing function of p, and we may therefore define the critical value

$$p_c(q) = \sup\{p : \theta(p,q) = 0\}, \quad \text{for } q \geq 1. \tag{3.7}$$

In defining the critical point, one might have used the measure ϕ^0 in place of ϕ^1, and the corresponding macroscopic quantity $\theta^0(p,q) = \phi^0(0 \leftrightarrow \infty)$; it is a consequence of the forthcoming Theorem 4.2 that the value of $p_c(q)$ would be unchanged by doing this.

4. General results in d dimensions

It would be unreasonable to expect exact calculations of quantities such as $p_c(q)$ for general dimensions d, although some such results may be aspired to when $d = 2$ (see the next section). Instead one may seek to understand the nature of the phase transition in more general terms. There is a bulk of information available for certain special values of q, but the overall picture is exceedingly patchy. The case $q = 1$ is, of course, special; in this case of percolation, edges are present or absent *independently* of each other, and this aids the analysis substantially. It is less evident why integer values of q (and particularly the case $q = 2$) present more tractable problems than non-integral values, but this is indeed the case. The case of the Ising model ($q = 2$) has been much studied (see, for example, [1, 17]), and so has the Potts model. Also, the contour method of Pirogov and Sinai [41, 42] may be applied elegantly to random-cluster models, yielding a wealth of results valid for all sufficiently large real q ([36]).

The class $\mathcal{R}_{p,q}$ of random-cluster measures has a subclass $\mathcal{T}_{p,q}$ of translation-invariant measures. Measures in this subclass conform to the principle of the uniqueness of the infinite cluster, a property of some importance in their study.

Theorem 4.1. *Let $0 \le p \le 1$, $q > 0$, and let $N = N(\omega)$ be the number of infinite clusters of the graph $(\mathbb{Z}^d, \eta(\omega))$.*

(a) We have that $\phi\big(N \in \{0,1\}\big) = 1$ for $\phi \in \mathcal{T}_{p,q}$.

(b) Suppose that $q \ge 1$. The measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are translation-invariant and ergodic, and therefore, for $b = 0, 1$,

$$\textit{either} \quad \phi^b_{p,q}(N = 0) = 1 \quad \textit{or} \quad \phi^b_{p,q}(N = 1) = 1.$$

Next we discuss results concerning the uniqueness of translation-invariant measures, and to this end we introduce the notion of free energy. Let $0 < p < 1$, and define the (finite-box) partition function Z^b_Λ by (3.3), for boundary conditions $b = 0, 1$. Rather than working with Z^b_Λ, we work instead with

$$Y^b_\Lambda = (1-p)^{-|\mathbb{E}_\Lambda|} Z^b_\Lambda = \sum_{\omega \in \Omega^b_\Lambda} q^{k(\omega,\Lambda)} \exp\big\{\pi |\eta(\omega) \cap \mathbb{E}_\Lambda|\big\} \tag{4.1}$$

where $\pi = \log\{p/(1-p)\}$. The *free energy* $f(p,q)$ of the corresponding measure is defined as the following limit, which is independent of the boundary condition:

$$f(p,q) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \left\{ \frac{1}{|\mathbb{E}_\Lambda|} \log Y^b_\Lambda \right\}, \qquad b = 0, 1.$$

It is easily seen that $f(p,q)$ is a convex function of $\pi = \log\{p/(1-p)\}$ for $\pi \in \mathbb{R}$, and therefore f is differentiable with respect to p except on some countable set Π of p-values. As a consequence of this, one obtains a partial conclusion concerning the uniqueness of random-cluster measures.

Theorem 4.2. *Fix $q \geq 1$ and let $0 < p < 1$. The following three statements are equivalent.*

(a) The free energy $f(p, q)$ is differentiable at p.

(b) The edge-density $h(p) = \phi^1_{p,q}\big(\omega(e) = 1\big)$ is continuous at p.

(c) There exists a unique random-cluster measure with parameters p and q.

Note that $h(p)$ is monotonic non-decreasing. See [25] for proofs of the above theorems.

There is incomplete information about the countable set Π of points of non-differentiability of the free energy. It is thought to be the case that Π is the empty set for sufficiently small q (≥ 1), and otherwise is a singleton set containing the critical point $p_c(q)$ only. Proofs of this have been given in special cases ([29, 37, 36, 39]), particularly for $d = 2$ and $q \geq 4$, and for $d \geq 2$ and sufficiently large q. A useful general conclusion is that of [3, p. 37], which states that

$$\phi^0 = \phi^1, \text{ and hence } |\mathcal{R}_{p,q}| = 1, \text{ whenever } \theta(p, q) = 0. \tag{4.2}$$

During the 1980s was published a striking series of papers in which the Pirogov–Sinai theory of contours [41, 42] was applied to Potts models. In the culminating paper [36], it was shown that Pirogov–Sinai theory may be applied succinctly to the random-cluster process in a general number d of dimensions ($d \geq 2$). As a consequence, one obtains many conclusions of value *so long as q is sufficiently large*. That is to say, there exists $q_0(d)$ such that for following holds, for $q \geq q_0(d)$.

(i) *Discontinuity of the percolation probability*: $\theta^1(p, q)$ is discontinuous at $p = p_c(q)$.

(ii) *Exponential decay and the mass gap*: Let $\tau_{p,q}(x, y)$ be the $\phi^1_{p,q}$–probability of a path joining the vertices x and y, and denote by e_n the vertex $(n, 0, 0, \ldots, 0)$. The *correlation length* $\xi(p, q)$, defined by

$$\xi(p, q)^{-1} = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log \tau_{p,q}(0, e_n) \right\}, \tag{4.3}$$

exists and satisfies $0 < \xi(p) < \infty$ for $0 < p < p_c(q)$. Furthermore there is a 'mass gap' in the sense that $\lim_{p\uparrow p_c(q)} \xi(p, q)^{-1}$ is *strictly positive*.

It is expected that the phase transition is second-order for small q, and first-order for large q. More specifically, it is thought that there exists a number $Q(d)$ such that

$$\theta^1(p, q) \text{ is } \begin{cases} \text{continuous at } p_c(q) & \text{if } q < Q(d) \\ \text{discontinuous at } p_c(q) & \text{if } q > Q(d). \end{cases}$$

Furthermore, it is conjectured that

$$Q(d) = \begin{cases} 4 & \text{if } d = 2 \\ 2 & \text{if } d \geq 6. \end{cases}$$

Much is known about percolation and the Ising model (see [1, 4, 17, 24] for example), and consequently about the random-cluster process when $q = 1, 2$. For example, the correlation length $\xi(p, q)$, defined in (4.3), is strictly positive and finite when $p < p_c(q)$ and $q = 1, 2$. (However, is it finite when $q = \frac{3}{2}$, say?) There are surprising gaps of knowledge. For example, it is an open problem to prove that $\theta(p, 1)$ is continuous at $p = p_c(1)$. This has been proved by special arguments when $d = 2$ (see [24]) and for large d (see [28]), but there is no proof known which is valid for general d (see [6, 27] for the latest results).

5. The case of two dimensions

Consider the random-cluster process on the two-dimensional lattice $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$, with parameters p and q satisfying $q \geq 1$. There exists the following remarkable conjecture.

Conjecture 5.1. *The critical value $p_c(q)$ in two dimensions is given by*

$$p_c(q) = \frac{\sqrt{q}}{1 + \sqrt{q}} \quad \textit{for} \quad q \geq 1\,. \tag{5.1}$$

This conjecture has been validated when $q = 1$ by Kesten [32] in his famous proof that the critical probability of bond percolation on $\mathbb{L}^2$ is $\frac{1}{2}$. For $q = 2$, the value of $p_c(2)$ given in (5.1) agrees with the celebrated calculation by Onsager [40] of the critical temperature of the Ising model on $\mathbb{Z}^2$. The transfer-matrix approach developed by Onsager and others leads to an exact formula for the free energy $f(p, 2)$. Such a formula is only part of the complete verification of the above conjecture in this case; a fuller proof may be achieved using the results of [2, 13].

The formula for $p_c(q)$ has been established rigorously in [37, 36] for sufficiently large (real) values of q. In [37], a theory of 'contours' developed by Pirogov and Sinai (see [41, 42]) was used to prove the corresponding result for Potts models and large integral q; this proof exploits the self-duality of $\mathbb{L}^2$, and may be rewritten more neatly in terms of random-cluster measures. See also [35]. The extension to general dimensions d was proved in [36], as discussed in the previous section.

Further results for Potts models have been obtained by Hintermann, Kunz, and Wu [29]. By studying the zeros of the free energy, they provided arguments for identifying the value of $p_c(q)$ for integers q satisfying $q \geq 4$, and in addition obtaining the exponential decay of correlation functions in the high-temperature phase (i.e., $p < p_c(q)$). It seems possible that the arguments of [29] may be generalized in a rigorous manner to real values of q satisfying $q \geq 4$. The first step is to map the random-cluster process on to an ice-type model; this has been done already by Baxter (see [7, 8]).

It is the self-duality of $\mathbb{L}^2$ that leads to the proposed exact formula (5.1) for $p_c(q)$; similar formulae may be proposed for an inhomogeneous square lattice (where the value of p may differ for horizontal and vertical edges) and for triangular and hexagonal lattices (making use of the star-triangle transformation). Recall that the dual G^{d} of a planar graph G is obtained by placing a vertex in each face of G, and by joining two such vertices by an edge whenever the two corresponding faces of G have a boundary edge in common. (If G is finite, its dual graph possesses a vertex in the *infinite* face of G as well as vertices in its finite faces.) It is easy to see that the dual of $\mathbb{L}^2$ is isomorphic to $\mathbb{L}^2$. What is the effect of graphical duality on the random-cluster measure?

Using graphical duality (see [14, 25, 45]), one finds that the random-cluster measure $\phi^0_{p,q}$ is dual in a certain way to the measure $\phi^1_{p',q}$, where

$$\frac{p'}{1-p'} = \frac{q(1-p)}{p}. \tag{5.2}$$

As a consequence, *the complement of a random-cluster process on $\mathbb{L}^2$ is itself a random-cluster process, but with different parameters and boundary conditions.*

The formula for $p_c(q)$ is now obtainable via a crude and non-rigorous argument. We may accept the following picture. If $p < p_c(q)$, then all components of the process are finite, and they are islands which float in an infinite open ocean of the dual lattice. Similarly, if $p > p_c(q)$, then there is an infinite component of the process which constrains the components of the dual process to be finite. If such a picture is valid, then

$$p < p_c(q) \quad \text{if and only if} \quad p' > p_c(q), \tag{5.3}$$

where p' is given by (5.2). It would follow that $p_c(q)$ is the fixed point of the mapping given in (5.2), so that

$$\frac{p_c(q)}{1-p_c(q)} = \frac{q(1-p_c(q))}{p_c(q)} ,$$

implying that $p_c(q) = \sqrt{q}/(1+\sqrt{q})$ for $q \geq 1$.

The above crude argument may be improved in places, but the conjecture remains unproved. Using Theorem 4.2 and an argument of Zhang (see [24, p. 195]), one may obtain that $p_c(q) \geq \sqrt{q}/(1+\sqrt{q})$, and it remains to prove the reversed inequality. Inequalities of such type have been explored further by Welsh [45], using 'sponge' arguments. The method of graphical duality yields in conjunction with (4.2) that $|\mathcal{R}_{p,q}| = 1$ if $p \neq \sqrt{q}/(1+\sqrt{q})$ and $q \geq 1$.

Since $\theta(p, q)$ is a decreasing limit (3.6) of continuous functions of p, it is an upper semi-continuous (and hence right-continuous) function of p. It is a problem of substantial importance to determine whether $\theta(p, q)$ is continuous at $p = p_c(q)$, or whether there is a jump discontinuity; this amounts to deciding whether or not $\theta(p_c(q), q) = 0$. This is known to be valid when $q = 1$ [24, 32], $q = 2$, and known to be invalid for large q ([35, 37, 36]). It may be conjectured ([7, 47]) that this holds if and only if $q \leq 4$. In the language of statistical physics, one believes that the phase transition is second-order if $q \leq 4$, and is first-order if $q > 4$.

6. Mean-field theory

The mean-field Potts model may be formulated as a Potts model on the complete graph K_n, being the graph with n labelled vertices every pair of which is joined by an edge. The study of such a process dates back at least to 1954 ([34]), and has been continued since ([33, 47]). This model is exactly soluble, in the sense that quantities of interest may be calculated exactly. It is therefore not surprising that the corresponding random-cluster processes (for real q) have 'exact solutions' also ([12]). Before discussing this, it is appropriate to note that a close relative of such a random-cluster process was described and studied by Peter Whittle [46]; some further details of Peter's 'first-shell' model for polymerization may be found in the appendix.

Consider the random-cluster measure $\phi_{n,\lambda,q}$ on the complete graph K_n, having parameters $p = \lambda/n$ and q. In the case $q = 1$, this measure is product measure, and therefore the ensuing graph is an Erdős–Rényi random graph ([11]). The overall picture for general values of q is rather richer than for the case $q = 1$. It turns out that the phase transition is of first-order if and only if $q > 2$, and the behaviour of the system depends on how λ compares with a 'critical value' $\lambda_c(q)$ taking the value

$$\lambda_c(q) = \begin{cases} q & \text{if } 0 < q \leq 2 \\ 2\left(\dfrac{q-1}{q-2}\right)\log(q-1) & \text{if } q > 2. \end{cases} \tag{6.1}$$

From the detailed picture described in [12] we extract the following information. The given properties occur with $\phi_{n,\lambda,q}$–probability tending to 1 as $n \to \infty$.

A. *The case* $\lambda < \lambda_c(q)$. The largest component of the graph is of order $\log n$.

B. *The case* $\lambda > \lambda_c(q)$. There is a 'giant component' having order $\theta(\lambda, q)n$ where θ is defined to be the largest root of the equation

$$e^{\lambda\theta} = \frac{1 + (q-1)\theta}{1-\theta}, \quad \lambda \geq \lambda_c(q)\,. \tag{6.2}$$

C. *The case* $\lambda = \lambda_c(q)$, $0 < q \leq 2$. The largest component has order $n^{2/3}$.

D. *The case* $\lambda = \lambda_c(q)$, $q > 2$. The largest component is either of order $\log n$ or of order $\theta(\lambda, q)n$, where θ is given as in part B above.

The dichotomy between first- and second-order phase transition is seen by studying the function $\theta(\lambda, q)$ defined in (6.2). When $0 < q \leq 2$, then $\theta(\lambda, q)$ descends continuously to 0 as $\lambda \downarrow \lambda_c(q)$. On the other hand, this limit is strictly positive when $q > 2$.

7. Historical observations

The basic theory of the random-cluster process was enunciated in the series of papers of Kees Fortuin and Piet Kasteleyn around 1970, and in the 1971 doctoral thesis ([18]) of Fortuin. Contemporaneously, and in collaboration with Ginibre, these authors established the FKG inequality for functions and measures on finite distributive lattices. The early work on random-cluster processes contains the main elements of much of the theory surveyed in the present paper, particularly that described in Section 2. The impact of this approach was perhaps attenuated by the combinatorial style of the first papers. Although the method led to certain famous successes, public understanding of it has been greatly aided by the more recent review and methodology of Aizenman et al. [3] and Edwards and Sokal [16].

An independent discovery of the random-cluster process was made by Peter Whittle in the 1970s, while he was working on his 'first-shell model' for polymerization. Such a model may be formulated as follows. Given n labelled vertices, we place edges between them at random. Let $\mathbf{s} = (s_{ij} : i, j = 1, 2, \ldots, n)$ be a vector of non-negative integers; the vector $\mathbf{s}$ corresponds to a directed multigraph on the vertex set, in which there are exactly s_{ij} directed edges from vertex i to vertex j. Now define a probability measure on the set of all such $\mathbf{s}$ by

$$P(\mathbf{s}) \propto \left\{ \prod_{i,j} \frac{(h/2)^{s_{ij}}}{s_{ij}!} \right\} \left\{ \prod_{r} H_r^{n_r} \right\} \nu^k , \tag{7.1}$$

where h, H_r, ν are positive constants, n_r is the number of vertices incident to exactly r edges (with either orientation), and k is the total number of components of the corresponding graph. In the special case when $H_j = 1$ (for all j), the measure P differs from a random-cluster measure only in the fact that the first brace is 'Poisson' rather than 'Bernoulli'. It is easily seen that, when $H_j \equiv 1$, this model is equivalent to a random-cluster model with $p = 1 - e^{-h}$; since $p = \lambda/n$, we should take $h = \lambda/n$. Therefore Peter Whittle's polymer model (with $H_j \equiv 1$) is essentially the random-cluster process on a complete graph (see [12]). Peter's work exploits a duality with a "compartmental model", and as a consequence he is able to study the nature of the phase transition for different values of ν; his compartmental model is essentially the Potts model, and the 'duality' is basically that discovered in a more general context by Fortuin and Kasteleyn.

It is interesting to reflect on the manner in which the random-cluster model was discovered, and the following remarks are based on information kindly provided to the present author by Piet Kasteleyn. When in the late 1960's Kees Fortuin went to Leiden for Ph.D. study, Kasteleyn had for some time been interested in a similarity between a number of elementary facts concerning three different models defined on finite graphs.

A. *Electrical networks.* Given two resistors of sizes r_1 and r_2, their combined resistance r in series or parallel satisfies

$$r = r_1 + r_2 \quad \text{(series)}, \qquad r^{-1} = r_1^{-1} + r_2^{-1} \quad \text{(parallel)}\,. \tag{7.1}$$

B. *Ising model.* Two edges of a graph with respective interactions J_1 and J_2, placed in either series or parallel, may be replaced by a single edge whose interaction J satisfies

$$f(J) = \frac{f(J_1) + f(J_2)}{1 + f(J_1)f(J_2)} \quad \text{(series)}, \qquad f(J) = f(J_1)f(J_2) \quad \text{(parallel)}, \tag{7.2}$$

where $f(x) = e^{-2\beta x}$ and β is the reciprocal of temperature.

C. *Percolation model.* Two edges having edge-probabilities p_1 and p_2 may be replaced by a single edge whose edge-probability p satisfies

$$p = p_1 p_2 \quad \text{(series)}, \qquad (1-p) = (1-p_1)(1-p_2) \quad \text{(parallel)}. \tag{7.3}$$

For the q-state Potts model, the rules in (7.2) become

$$f(J) = \begin{cases} \dfrac{f(J_1) + f(J_2) + (q-2)f(J_1)f(J_2)}{1 + (q-1)f(J_1)f(J_2)} & \text{(series)} \\ f(J_1)f(J_2) & \text{(parallel)}. \end{cases} \tag{7.4}$$

Setting $q = 1$ in (7.4), and making a suitable change of variables $p = 1 - f(J)$, we recover the percolation rule (7.3). The electrical network rules may be obtained from (7.4) also, by setting $r = q^{\frac{1}{2}}/(1 - f(J))$ and passing to the formal limit as $q \downarrow 0$.

Turning to typical quantities of interest, such as current flow, two-point correlations, and pair-connectivity, Fortuin and Kasteleyn found that each can be expressed as the ratio of polynomials of the edge-variables (suitably transformed), and that such polynomials satisfy a recursion relation based on the deletion and contraction of edges. By iterating accordingly, they arrived at the partition function (1.2) of the random-cluster model.

The random-cluster process generalizes Potts models in at least two ways. First, it provides an interpolation of Potts models to non-integral values of q. Secondly, there are questions to be asked about random-cluster models for which there are no corresponding questions for Potts models. That is to say, whereas for any observable f of a Potts model there exists a corresponding observable F of the corresponding random-cluster model (see [16]), the converse is false: there exist functions F of the random-cluster model with no corresponding f independent of p.

In addition to providing the above information, Professor Kasteleyn has highlighted two areas for future research. First, what can be said about the random-cluster model when $0 < q < 1$ (i.e., outside the FKG domain), and particularly in the electrical network limit as $q \downarrow 0$? It may even be rewarding to consider the case $q < 0$, for which ϕ can no longer be a probability measure (see the work on rank-generating functions contained in [10, 44]).

Secondly, corresponding to an 'antiferromagnetic' Potts model is a random-cluster model with $p < 0$. Once again ϕ would be a signed rather than a probability measure. In the limit as $p \to -\infty$, one is led to the theory of q-colourings of the graph (when q is integral).

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References

1. Aizenman, M. (1982), Geometric analysis of ϕ^4 fields and Ising models, *Communications in Mathematical Physics* **86**, 1–48.
2. Aizenman, M., Barsky, D. J., and Fernández, R. (1987), The phase transition in a general class of Ising-type models is sharp, *Communications in Mathematical Physics* **47**, 343–374.
3. Aizenman, M., Chayes, J. T., Chayes, L., and Newman, C. M. (1988), Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models, *Journal of Statistical Physics* **50**, 1–40.
4. Aizenman, M. and Fernández, R. (1986), On the critical behavior of the magnetization in high-dimensional Ising models, *Journal of Statistical Physics* **44**, 393–454.
5. Ashkin, J. and Teller, E. (1943), Statistics of two-dimensional lattices with four components, *The Physical Review* **64**, 178–184.
6. Barsky, D. J., Newman, C. M., and Grimmett, G. R. (1991), Percolation in half spaces: equality of critical probabilities and continuity of the percolation probability, *Probability Theory and Related Fields* **90**, 111–148.
7. Baxter, R. J. (1982), *Exactly Solved Models in Statistical Mechanics*, Academic Press, London.
8. Baxter, R. J., Kelland, S. B., and Wu, F. Y. (1976), Equivalence of the Potts model or Whitney polynomial with an ice-type model, *Journal of Physics A: Mathematical and General* **9**, 397–406.
9. Bezuidenhout, C. E., Grimmett, G. R., and Kesten, H. (1992), Strict inequality for critical values of Potts models and random-cluster processes, *Communications in Mathematical Physics* (to appear).
10. Biggs, N. L. (1984), *Algebraic Graph Theory*, Cambridge University Press, Cambridge.
11. Bollobás, B. (1985), *Random Graphs*, Academic Press, London.
12. Bollobás, B., Grimmett, G. R., and Janson, S., The random-cluster process on the complete graph (to appear).
13. Burton, R. M. and Keane, M. (1989), Density and uniqueness in percolation, *Communications in Mathematical Physics* **121**, 501–505.
14. Chayes, J. T., Chayes, L., and Schonmann, R. H. (1987), Exponential decay of connectivities in the two dimensional Ising model, *Journal of Statistical Physics* **49**, 433–445.
15. Dobrushin, R. L. (1968), Gibbsian random fields for lattice systems and pairwise interactions, *Functional Analysis and its Applications (in translation)* **2**, 292–301.
16. Edwards, R. G. and Sokal, A. D. (1988), Generalization of the Fortuin–Kasteleyn–Swendsen–Wang representation and Monte Carlo algorithm, *The Physical Review* D **38**, 2009–2012.
17. Fernández, R., Fröhlich, J., and Sokal, A. D. (1992), *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory*, Springer–Verlag, Berlin.
18. Fortuin, C. M. (1971), *On the random-cluster model*, Doctoral thesis, University of Leiden.
19. Fortuin, C. M. (1972a), On the random cluster model. II. The percolation model, *Physica* **58**, 393–418.
20. Fortuin, C. M. (1972b), On the random cluster model. III. The simple random-cluster process, *Physica* **59**, 545–570.
21. Fortuin, C. M. and Kasteleyn, P. W. (1972), On the random cluster model. I. Introduction and relation to other models, *Physica* **57**, 536–564.
22. Fortuin, C. M., Kasteleyn, P. W., and Ginibre, J. (1971), Correlation inequalities on some partially ordered sets, *Communications in Mathematical Physics* **22**, 89–103.

23. Gandolfi, A., Keane, M., and Newman, C. M. (1992), Uniqueness of the infinite component in a random graph with applications to percolation and spin glasses, *Probability Theory and Related Fields* **92**, 511–527.
24. Grimmett, G. R. (1989), *Percolation*, Springer–Verlag, Berlin.
25. Grimmett, G. R. (1993), Uniqueness of random-cluster measures, and related results (to appear).
26. Grimmett, G. R. (1994), Percolative problems, *Probability and Phase Transition* (G. R. Grimmett, ed.), Kluwer, Dordrecht (to appear).
27. Grimmett, G. R. and Marstrand, J. M. (1990), The supercritical phase of percolation is well behaved, *Proceedings of the Royal Society (London), Series A* **430**, 439–457.
28. Hara, T. and Slade, G. (1990), Mean-field critical behaviour for percolation in high dimensions, *Communications in Mathematical Physics* **128**, 333-391.
29. Hintermann, D., Kunz, H., and Wu, F. Y. (1978), Exact results for the Potts model in two dimensions, *Journal of Statistical Physics* **19**, 623–632.
30. Ising, E. (1925), Beitrag zur theorie des ferromagnetismus, *Zeitschrift für Physik* **31**, 253–258.
31. Kasteleyn, P. W. and Fortuin, C. M. (1969), Phase transitions in lattice systems with random local properties, *Journal of the Physical Society of Japan* **26,** 11–14, Supplement,.
32. Kesten, H. (1980), The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$, *Communications in Mathematical Physics* **74**, 41–59.
33. Kesten, H. and Schonmann, R. H. (1990), Behavior in large dimensions of the Potts and Heisenberg models, *Reviews in Mathematical Physics* **1**, 147–182.
34. Kihara, T., Midzuno, Y., and Shizume, J. (1954), Statistics of two-dimensional lattices with many components, *Journal of the Physical Society of Japan* **9**, 681–687.
35. Kotecký, R. and Shlosman, S. (1982), First order phase transitions in large entropy lattice systems, *Communications in Mathematical Physics* **83**, 493–515.
36. Laanait, L., Messager, A., Miracle-Sole, S., Ruiz, J., and Shlosman, S. (1991), Interfaces in the Potts model I: Pirogov–Sinai theory of the Fortuin–Kasteleyn representation, *Communications in Mathematical Physics* **140**, 81–91.
37. Laanait, L., Messager, A., and, Ruiz, J. (1986), Phase coexistence and surface tensions for the Potts model, *Communications in Mathematical Physics* **105**, 527–545.
38. Lanford, O. E. and Ruelle, D. (1969), Observables at infinity and states with short range correlations in statistical mechanics, *Communications in Mathematical Physics* **13**, 194–215.
39. Martirosian, D. H. (1986), Translation invariant Gibbs states in the q-state Potts model, *Communications in Mathematical Physics* **105**, 281–290.
40. Onsager, L. (1944), Crystal Statistics, I. A two-dimensional model with an order-disorder transition, *The Physical Review* **65**, 117–149.
41. Pirogov, S. A. and Sinai, Ya. G. (1975), Phase diagrams of classical lattice systems, *Theoretical and Mathematical Physics* **25**, 1185–1192.
42. Pirogov, S. A. and Sinai, Ya. G. (1976), Phase diagrams of classical lattice systems, continuation, *Theoretical and Mathematical Physics* **26**, 39–49.
43. Potts, R. B. (1952), *Proceedings of the Cambridge Philosophical Society* **48**, 106–109.
44. Tutte, W. T. (1984), *Graph Theory*, Addison-Wesley, Menlo Park, California.
45. Welsh, D. J. A. (1992), Percolation in the random-cluster process, *Journal of Physics A: Mathematical and General* (to appear).
46. Whittle, P. (1986), *Systems in Stochastic Equilibrium*, John Wiley, Chichester.
47. Wu, F. Y. (1982), The Potts model, *Reviews in Modern Physics* **54**, 235–268.

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