BROWNIAN FROGS WITH REMOVAL: PANDEMICS IN A DIFFUSING POPULATION

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ABSTRACT. A stochastic model of susceptible/infected/removed (SIR) type, inspired by COVID-19, is introduced for the spread of infection through a spatially-distributed population. Individuals are initially distributed at random in space, and they move according to independent random processes. The disease may pass from an infected individual to an uninfected individual when they are sufficiently close. Infected individuals are permanently removed at some given rate α. Two models are studied here, termed the 'delayed diffusion' and the 'diffusion' models. In the first, individuals are stationary until they are infected, at which time they begin to move; in the second, all individuals start to move at the initial time 0. Using a perturbative argument, conditions are established under which the disease infects a.s. only finitely many individuals. It is proved for the delayed diffusion model that there exists a critical value $\alpha_c \in (0, \infty)$ for the existence of a pandemic.

1. INTRODUCTION

Numerous mathematical models have been introduced to describe the spread of a disease about a population. Such models may be deterministic or stochastic, or a mixture of each; they may incorporate a range of factors including susceptibility, infectivity, recovery, and removal; the population members (termed 'particles') may be distributed about some given space; and so on. Inspired in part by the COVID-19 pandemic of 2020, we propose two models in which the particles move randomly about the space that they inhabit; infection may be passed between particles that are sufficiently close; after the elapse of a random time since infection, a particle is removed from the process. These models differ from that of Beckman, Dinan, Durrett, Huo, and Junge [3] through the introduction of the permanent 'removal' of particles, and this new feature brings a significant new difficulty to the analysis. (The degree of immunity of an individual previously infected by COVID-19 is not known at the time of writing.)

Date: 4 September 2020.

2010 *Mathematics Subject Classification.* 60K35, 60G15.

Key words and phrases. Percolation, infectious disease, SIR model, frog model, epidemic, pandemic, COVID-19, diffusion.

We shall concentrate on the case in which the particles inhabit the d-dimensional reals $\mathbb{R}^d$ where $d \geq 2$. Here is a concrete example of the processes studied here.

(a) Particles are initially distributed in $\mathbb{R}^d$ in the manner of a Poisson process with rate λ conditioned to contain a point at the origin 0.
(b) Particles move randomly within $\mathbb{R}^d$ according to independent Brownian motions with variance-parameter σ^2.
(c) At time 0 the particle at the origin (the initial 'infective') suffers from an infectious disease, which may be passed to others when sufficiently close.
(d) When two particles, labelled P and P', are within a given distance δ, and P is already infected, then particle P' becomes infected.
(e) Each particle is infected for a total period of time having the exponential distribution with parameter $\alpha \in [0, \infty)$, and is then permanently removed.

The fundamental question is to determine for which vectors $(\lambda, \delta, \sigma, \alpha)$ it is the case that (with strictly positive probability) infinitely many particles become infected. For simplicity, we shall assume henceforth that

$$\delta = \sigma = 1. \tag{1.1}$$

We shall generally assume $\alpha > 0$. In the special case $\alpha = 0$, (studied in [3]) a particle once infected remains infected forever, and the subsequent analysis is greatly facilitated by a property of monotonicity that is absent in the more challenging case $\alpha > 0$ considered in the current work.

Two protocols for movement feature in this article.

A. *Delayed diffusion model.* The initial infective starts to move at time 0, and all other particles remain stationary until they are infected, at which times they begin to move.
B. *Diffusion model.* All particles begin to move at time 0.

The related literature is somewhat ramified, and a spread of related problems have been studied by various teams. We mention a selection of papers but do not attempt a full review, and we concentrate on work associated with the lattices $\mathbb{Z}^d$ rather than with trees or complete graphs.

The delayed diffusion model may be viewed as a continuous-time version of the 'frog' random walk process studied in Alves et al. [1, 2], Ramirez and Sidoravicius [23], Fontes et al. [5], and Hoffman, Johnson, and Junge [13, 14]. See Popov [22] for an early review. Kesten and Sidoravicius [15, 16] considered the frog model as a model for infection, both with and without recuperation (that is, when infected frogs recover and become available for reinfection). The paper of Beckman et al. [3] is devoted to the delayed diffusion model without removal (that is, with $\alpha = 0$). Peres et al. [21] studied three geometric properties of a Poissonian/Brownian cloud

of particles, in work inspired in part by the dynamic Boolean percolation model of van den Berg et al. [4]. Related work has appeared in Gracar and Stauffer [7].

A number of authors have considered the frog model with recuperation under the title 'activated random walks'. The reader is referred to the review by Rolla [24], and for recent work to Stauffer and Taggi [27] and Rolla et al. [25].

The main new difficulty in the models considered here is that particles are removed after a random period of infectivity. This introduces a non-monotonicity into the model in that: the longer that particles remain infective, the more they may create islands of 'removed' particles which can serve as barriers to the further spread of infection. A related situation (but without the movement of particles) was considered by Kuulasma [18] in a discrete setting, and the methods derived there are useful in our Section 3.6. See also Alves et al. [1, p. 4].

Let I denote the set of particles that are ever infected, and

$$\theta(\lambda, \alpha) := \mathbb{P}_{\lambda,\alpha}(|I| = \infty). \tag{1.2}$$

We say the process

$$\begin{aligned} \text{becomes extinct} \quad & \text{if } \theta(\lambda, \alpha) = 0, \\ \text{survives} \quad & \text{if } \theta(\lambda, \alpha) > 0. \end{aligned}$$

Let λ_c denote the critical value of λ for the disk (or 'Boolean') percolation model with radius 1 on $\mathbb{R}^d$ (see, for example, [20]). It is immediate for both models above that $\theta(\lambda, \alpha) > 0$ if $\lambda > \lambda_c$ and $\alpha \geq 0$, since in that case the disease spreads instantaneously to the percolation cluster C containing the initial infective, and in addition we have $\mathbb{P}_{\lambda,\alpha}(|C| = \infty) > 0$.

We write θ_d (respectively, θ_{dd}) for the function θ of (1.2) in the case of the diffusion model (respectively, delayed diffusion model). The following two theorems are proved in Sections 3 and 4 as special cases of results for more general epidemic models than those given above.

Theorem 1.1. *For the above* delayed diffusion model *on* $\mathbb{R}^d$ *with* $d \geq 2$, *there exists* $\underline{\lambda} \in (0, \lambda_c]$ *and a non-decreasing function* $\alpha_c : (0, \underline{\lambda}) \to (0, \infty)$ *such that, for* $0 < \lambda < \underline{\lambda}$,

$$\theta_{dd}(\lambda, \alpha) \begin{cases} = 0 & \text{if } \alpha > \alpha_c(\lambda), \\ > 0 & \text{if } \alpha < \alpha_c(\lambda). \end{cases} \tag{1.3}$$

Theorem 1.2. *For the above* diffusion model *on* $\mathbb{R}^d$ *with* $d \geq 2$, *there exists* $\underline{\lambda} \in (0, \lambda_c]$ *and a non-decreasing function* $\alpha_c : (0, \underline{\lambda}) \to (0, \infty)$ *such that* $\theta_d(\lambda, \alpha) = 0$ *when* $\alpha > \alpha_c(\lambda)$ *and* $0 < \lambda < \underline{\lambda}$.

For the diffusion model, we have no proof that $\theta_{\mathrm{d}}(\lambda,\alpha) > 0$ for $0 < \lambda < \underline{\lambda}$ and sufficiently small α. The above theorems are proved using a perturbative argument, and thus fall short of the assertion that $\underline{\lambda} = \lambda_{\mathrm{c}}$.

The methods of proof may be made quantitative, leading to bounds for the numerical values of the critical points α_{c}. Such bounds are far from precise, and therefore we do not explore them here. The intensity λ of the Poisson process may be assumed non-constant so long as it is bounded uniformly between two strictly positive constants. The existence of the subcritical phase may be proved for more general diffusions than Brownian motion.

We write $\mathbb{Z}_0 = \{0, 1, 2, \dots\}$ and 1_A for the indicator function of an event or set A. Let $S(r)$ denote the closed r-ball of $\mathbb{R}^d$ with centre at the origin, and $S = S(1)$. The d-dimensional Lebesgue measure of a set A is written $|A|_d$, and the Euclidean norm $\|\cdot\|_d$. The *radius* of $M \subseteq \mathbb{R}^d$ is given by

$$\mathrm{rad}(M) := \sup\{\|m\|_d : m \in M\}.$$

We abbreviate $\mathbb{P}_{\lambda,\alpha}$ (respectively, $\mathbb{E}_{\lambda,\alpha}$) to the generic notation $\mathbb{P}$ (respectively, $\mathbb{E}$).

The contents of this paper are as follows. The two models are defined in Section 2 with a degree of generality that includes general diffusions and a more general process of infection. The delayed diffusion model is studied in Section 3, and the diffusion model in Section 4. Theorem 1.1 (respectively, Theorem 1.2) is contained within Theorem 3.1 (respectively, Theorem 4.1).

2. General models

2.1. **The general set-up.** Let $d \geq 2$. A *diffusion process* in $\mathbb{R}^d$ is a solution ζ to the stochastic differential equation

$$d\zeta(t) = a(\zeta(t))\,dt + B(\zeta(t))\,dW_t, \tag{2.1}$$

where W is a standard Brownian motion in $\mathbb{R}^d$. (We may write either W_t or $W(t)$.) For definiteness, we shall assume that: $\zeta(0) = 0$; ζ has continuous sample paths; the instantaneous drift vector a and variance matrix B are continuous. We do not allow a, B to be time-dependent. We call the process 'Brownian' if ζ is a standard Brownian motion, which is to say that a is the zero vector and B is the identity matrix.

Let ζ be such a diffusion, and let $(\zeta_i : i \geq 0)$ be independent copies of ζ. Let $\alpha \in (0,\infty)$, $\rho \in [0,\infty)$, and let $\mu : \mathbb{R}^d \to [0,\infty)$ be integrable with

$$\mathrm{Int}(\mu) := \int_{\mathbb{R}^d} \mu(x)\,dx \in (0,\infty). \tag{2.2}$$

We call μ *radially decreasing* if

$$\mu(rx) \leq \mu(x) \qquad x \in \mathbb{R}^d,\ r \in [1,\infty). \tag{2.3}$$

Let $\Pi = (X_0 = 0, X_1, X_2, \dots)$ be a Poisson process on $\mathbb{R}^d$ (conditioned to possess a point at the origin 0) with constant intensity $\lambda \in (0, \infty)$. At time 0, particles labelled $\mathcal{P} = \{P_0, P_1, P_2, \dots\}$ are placed at the respective points $X_0 = 0, X_1, X_2, \dots$. We may refer to a particle P_i by either its index i or its initial position X_i.

For $i \geq 0$, at any given time t particle P_i is in one of three states S (susceptible), I (infected), and R (removed). Thus the state space is $\Omega = \{\mathrm{S}, \mathrm{I}, \mathrm{R}\}^{\mathbb{Z}_0}$, and we write $\omega(t) = (\omega_i(t) : i \geq 0) \in \Omega$ for the state of the process at time t. Let S_t (respectively, I_t, R_t) be the set of particles in state S (respectively, I, R) at time t. We take

$$\omega_i(0) = \begin{cases} \mathrm{I} & \text{if } i = 0, \\ \mathrm{S} & \text{otherwise,} \end{cases}$$

so that $I_0 = \{P_0\}$ and $S_0 = \mathcal{P} \setminus \{P_0\}$. The only particle-transitions that may occur are $\mathrm{S} \to \mathrm{I}$ and $\mathrm{I} \to \mathrm{R}$. The transitions $\mathrm{S} \to \mathrm{I}$ occur at rates that depend on the locations of the currently infected particles.

2.2. **Delayed diffusion model.** Each particle P_j is stationary if and only if it is in state S. If it become infected (at some time B_j, see (2.5)), henceforth it follows the diffusion $X_j + \zeta_j$. We write

$$\pi_j(t) = \begin{cases} X_j & \text{if } t \leq B_j, \\ X_j + \zeta_j(t - B_j) & \text{if } t > B_j, \end{cases}$$

for the position of P_j at time t.

A particle changes its state according to the following rates.

($\mathrm{S} \to \mathrm{I}$) Let $t > 0$, and let P_j be a particle that is in state S at all times $s < t$. Each $P_i \in I_t$ (with $i \neq j$) infects P_j at rate $\rho\mu(X_j - \pi_i(t))$. The aggregate rate at which P_j becomes infected is

$$\sum_{i \in I_t,\, i \neq j} \rho\mu(X_j - \pi_i(t)). \tag{2.4}$$

($\mathrm{I} \to \mathrm{R}$) An infected particle is removed at rate α.

Transitions of other types are not permitted. We take the sample path $\omega = (\omega(t) : t \geq 0)$ to be pointwise right-continuous. The *infection time* B_j of particle P_j is given by

$$B_j = \inf\{t \geq 0 : P_j \in I_t\}. \tag{2.5}$$

The infection rates $\rho\mu(X_j - \pi_i(t))$ of (2.4) are finite, and hence infections take place at a.s. distinct times. We may thus speak of P_j as being 'directly infected' by P_i. We speak of a point $z \in \Pi$ as being *directly infected* by a point $y \in \Pi$ when the associated particles have that property. If P_j is infected directly by P_i, we call P_j a *child* of P_i, and P_i the *parent* of P_j.

Following its infection, particle P_i remains infected for a further random time T_i, called the *lifetime* of P_i, and is then removed. The times T_i are random variables with the exponential distribution with parameter $\alpha > 0$, and are independent of one another and of the X_j and ζ_j.

In the above version of the delayed diffusion model, ρ is assumed finite. Consider the case where $\rho = \infty$ and $\mu = 1_M$ where $M \subseteq \mathbb{R}^d$ is compact. In this situation, a susceptible particle P_j becomes infected at the earliest instant that it belongs to $\pi_i(t) + M$ for some $P_i \in I_t$, $i \neq j$. This happens when either (i) $X_j \in X_i + M$ at the infection time B_i of P_i, or (ii) an infected particle P_i infects P_j (or initiates a chain of instantaneous infections leading to P_j), while the former is diffusing post-infection around $\mathbb{R}^d$.

The role of the Boolean model of continuum percolation becomes clear when $\rho = \infty$, and we illustrate this, subject to the simplifying assumption that M is symmetric in the sense that $x \in M$ if and only if $-x \in M$. Let $\Pi = (X_i : i \geq 0)$ be a Poisson process in $\mathbb{R}^d$ with constant intensity λ, and declare two points X_i, X_j to be *adjacent* if and only if $X_j - X_i \in M$. This adjacency relation generates a graph G with vertex-set Π. In the delayed diffusion process on the set Π, entire clusters of the percolation process are infected simultaneously.

In either case $\rho < \infty$ or $\rho = \infty$, we write $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha)$ for the probability that infinitely many particles are infected. For concreteness, we note our special interest in the case in which:

(i) ζ is a standard Brownian motion,
(ii) $\mu = 1_S$ with S the closed unit ball of $\mathbb{R}^d$.

2.3. **Diffusion model.** The diffusion model differs from the delayed diffusion model of Section 2.2 in that all particles begin to move at time $t = 0$. The location of P_j at time t is $X_j + \zeta_j(t)$, and the transition rates are given as follows.

(S $\to$ I) Let $t > 0$, and let P_j be susceptible at all times $s < t$. Each $P_i \in I_t$ (with $i \neq j$) infects P_j at rate $\rho\mu(X_j + \zeta_j(t) - \zeta_i(t))$. The aggregate rate at which P_j becomes infected is

$$\sum_{i \in I_t,\, i \neq j} \rho\mu\big(X_j + \zeta_j(t) - \zeta_i(t)\big). \tag{2.6}$$

(I $\to$ R) An infected particle is removed at rate α.

As in Section 2.2, we may allow $\rho = \infty$ and $\mu = 1_M$ with M compact. In either case $\rho < \infty$ or $\rho = \infty$ we write $\theta_{\mathrm{d}}(\lambda, \rho, \alpha)$ for the probability that infinitely many particles are infected.

3. The delayed diffusion model

3.1. **Main result.** We consider the general delayed diffusion model of Section 2.2, and we adopt the notation of that section. Recall the critical point λ_c of the Boolean continuum percolation on $\mathbb{R}^d$ in which a closed unit ball is placed at each point of a rate-λ Poisson process.

Theorem 3.1. *Consider the Brownian delayed diffusion model on $\mathbb{R}^d$ where $d \geq 2$.*

(a) *Let $\rho \in (0,\infty)$. There exists a function $\alpha_c : (0,\infty)^2 \to (0,\infty)$ such that*

$$\theta_{\mathrm{dd}}(\lambda,\rho,\alpha) \begin{cases} = 0 & \text{if } \alpha > \alpha_c(\lambda,\rho), \\ > 0 & \text{if } \alpha < \alpha_c(\lambda,\rho). \end{cases} \tag{3.1}$$

The function $\alpha_c = \alpha_c(\lambda,\rho)$ is non-decreasing in ρ.

(b) *Let $\rho = \infty$ and $\mu = 1_S$ where S is the closed unit ball in $\mathbb{R}^d$. There exists a non-decreasing function $\alpha_c : (0,\infty) \to (0,\infty]$ such that, for $0 < \lambda < \lambda_c$,*

$$\theta_{\mathrm{dd}}(\lambda,\infty,\alpha) \begin{cases} = 0 & \text{if } \alpha > \alpha_c(\lambda), \\ > 0 & \text{if } \alpha < \alpha_c(\lambda). \end{cases} \tag{3.2}$$

Furthermore, there exists $\underline{\lambda} \in (0,\lambda_c]$ such that

$$\alpha_c(\lambda) \begin{cases} < \infty & \text{if } 0 < \lambda < \underline{\lambda}, \\ = \infty & \text{if } \lambda > \lambda_c. \end{cases}$$

In both cases (a) *and* (b), *the function $\theta_{\mathrm{dd}}(\lambda,\rho,\alpha)$ is non-decreasing in α.*

This theorem extends Theorem 1.1. Its proof is found in Sections 3.5–3.6, and it uses results derived earlier in Section 3.

3.2. **A condition for subcriticality when $\rho < \infty$.** Consider the general delayed diffusion model of Section 2.2, and assume first that $\rho \in (0,\infty)$. Let $I_0 = \{0\}$. We call $y \in \Pi$ a *first generation infected point up to time t* if y is directly infected by P_0 at or before time t. Let $I_{1,t}$ be the set of all first generation infected points up to time t. For $n \geq 2$, we call $z \in \Pi$ an *nth generation infected point up to time t* if, at or before time t, z is directly infected by some $y \in I_{n-1,t}$, and we define $I_{n,t}$ accordingly. Write $I_n = \lim_{t\to\infty} I_{n,t}$, the set of all nth generation infected points, and let $I = \bigcup_n I_n$ be the set of points that are ever infected.

Proposition 3.2. *Let $\rho \in (0,\infty)$ and*

$$L_t(x) = \mathbb{E}\left(1 - \exp\left(-\int_0^t \rho\mu(x-\zeta(s))\,ds\right)\right). \tag{3.3}$$

We have that $\mathbb{E}|I_{1,t}| \le R_t$ *and* $\mathbb{E}|I_1| \le R$, *where*

$$R_t = \lambda \int_{\mathbb{R}^d} \left[\int_0^t L_s(x) \alpha e^{-\alpha s}\, ds + L_t(x) e^{-\alpha t} \right] dx, \tag{3.4}$$

$$R = \lim_{t\to\infty} R_t = \lambda \int_{\mathbb{R}^d} \int_0^\infty L_s(x) \alpha e^{-\alpha s}\, ds\, dx. \tag{3.5}$$

The constant R in (3.5) is an upper bound for the so-called *reproductive rate* of the process.

Proposition 3.3. *Let* $\rho \in (0,\infty)$.

(a) *We have that* $\mathbb{E}|I_n| \le R^n$ *for* $n \ge 0$, *where* R *is given in* (3.5).
(b) *If* $R < 1$, *then* $\mathbb{E}|I| \le 1/(1-R)$, *and hence* $\theta_{\mathrm{dd}}(\lambda,\rho,\alpha) = 0$.
(c) *We have that* $R \le \lambda\rho\,\mathrm{Int}(\mu)/\alpha$.

Note that parts (b) and (c) imply that

$$\theta_{\mathrm{dd}}(\lambda,\rho,\alpha) = 0 \quad \text{if} \quad \alpha > \lambda\rho\,\mathrm{Int}(\mu). \tag{3.6}$$

Proof of Proposition 3.2. Let $\mathcal{F}_0(t)$ be the σ-field generated by $(\zeta_0(s) : 0 \le s \le t)$. Conditional on $\mathcal{F}_0(t)$, for $i \ge 1$, let $A_i = (A_i^k : k \ge 0)$ be a Poisson process on $[0,\infty)$ with rate function

$$r_{X_i}(s) := \rho\mu(X_i - \zeta_0(s)).$$

Assume the A_i are independent conditional on $\mathcal{F}_0(t)$, and write $N_i = |\{k : A_i^k \le t\}|$. We say that P_0 'tries to infect' P_i at the times $\{A_i^k : k \ge 1\}$. Let $U_t = \{X_i : i \ge 1,\ N_i \ge 1\}$ be the set of points in Π that P_0 tries to infect up to time t. Note that $I_{1,t}$ is dominated stochastically by U_t. The domination is strict since there may exist $X_i \in U_t$ such that P_i is infected before time t by some previously infected $P_j \ne P_0$.

Consider a particle, labelled P_j say, with initial position $x \in \mathbb{R}^d$. Conditional on $\mathcal{F}_0(t)$, P_0 tries to infect P_j up to time t with probability not exceeding

$$1 - \exp\left(-\int_0^t r_x(s)\, ds \right).$$

Therefore,

$$\mathbb{P}\big(X_j \in I_{1,t} \,\big|\, X_j = x,\ \mathcal{F}_0(t)\big) \le \mathbb{E}\left(1 - \exp\left(-\int_0^t r_x(s)\, ds \right) \,\middle|\, \mathcal{F}_0(t) \right). \tag{3.7}$$

By the colouring theorem for Poisson processes (see, for example, [10, Thm 6.13.14]), conditional on $\mathcal{F}_0(t)$, U_t is a Poisson process with inhomogeneous intensity function given by

$$\Lambda_{t,\zeta_0}(x) = \lambda\mathbb{E}\left(1 - \exp\left(-\int_0^t r_x(s)\, ds \right) \,\middle|\, \mathcal{F}_0(t) \right).$$

By Fubini's theorem,

$$\mathbb{E}|I_{1,t}| \le \mathbb{E}\big(\mathbb{E}(|U_t| \,\big|\, T_0)\big) = \int_{\mathbb{R}^d} \left[\lambda \int_0^t L_s(x)\alpha e^{-\alpha s}\, ds + L_t(x)\mathbb{P}(T_0 > t)\right] dx, \tag{3.8}$$

and (3.4) follows. Equation (3.5) follows as $t \to \infty$ by the monotone convergence theorem. □

Proof of Proposition 3.3. (a) This holds by a variation of the proof of Proposition 3.2, which we outline as follows. Let $n \ge 1$. We build the cluster of infected points according to generation number, starting with $I_0 = \{0\}$. By following the trajectory ζ_0 until time T_0, and observing the infections by P_0, we discover I_1. Let $\mathcal{F}_1$ be the σ-field generated by the trajectory $(\zeta_0(t) : t \in [0, T_0])$ of P_0 until its removal, together with the set of particles that are directly infected by P_0 and the times and locations of these infections.

Let $n \ge 1$, and let $\mathcal{F}_n$ be the σ-field generated by this discovery process until the nth generation I_n has been discovered. Thus $\mathcal{F}_n$ is the σ-field generated by the sets $I_0, I_1, \dots, I_n$ together with the trajectories of particles in $I_0 \cup \cdots \cup I_{n-1}$ prior to their removals, and the infection times and locations of particles in I_n. We condition on $\mathcal{F}_n$, and write $I_n = \{y_1, y_2, \dots\}$ where the ordering of the y_i is arbitrary. We shall bound the mean numbers of children of the y_i considered in order.

Let B_{y_1} be the time of infection of y_1, and T_{y_1} its lifetime. By the marking theorem for Poisson processes (see [17, Sect. 5.2]), the positions of uninfected particles at time B_{y_1} may be regarded as a subset V_1 of a rate-λ Poisson process. By the calculation of the previous proof, the mean number of children of y_1 (given $\mathcal{F}_n$) is no greater than the value R given in (3.5).

This is now iterated from the starting points $y_2, y_3, \dots$. It follows that

$$|I_{n+1}| \le \sum_i |V_i|,$$

where V_i is given as V_1 but with parent y_i. Therefore,

$$\mathbb{E}\big(|I_{n+1}| \,\big|\, \mathcal{F}_n\big) \le R|I_n|,$$

whence $\mathbb{E}|I_{n+1}| \le R\,\mathbb{E}|I_n|$.

(b) By part (a) and the assumption $R < 1$,

$$\mathbb{E}|I| = \sum_{n=0}^{\infty} \mathbb{E}|I_n| \le \frac{1}{1-R} < \infty.$$

Therefore, $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha) = \mathbb{P}(|I| = \infty) = 0$.

(c) Since $1 - e^{-z} \le z$ for $z \ge 0$, by (3.3) and Fubini's theorem,

$$\int_{\mathbb{R}^d} L_t(x)\,dx \le \rho t \,\mathrm{Int}(\mu).$$

By (3.5),

$$R \le \lambda\rho \,\mathrm{Int}(\mu) \int_0^\infty s\alpha e^{-\alpha s}\,ds = \frac{\lambda\rho}{\alpha}\,\mathrm{Int}(\mu),$$

as claimed. □

3.3. **Infection with compact support.** Suppose $\mu = 1_M$ with M compact. The dependence of $R = R(\rho)$ (in (3.5)) on the infection rate $\rho \in (0,\infty)$ is highlighted in the formula

$$R(\rho) = \lambda \int_{\mathbb{R}^d} \int_0^\infty L_s(x)\alpha e^{-\alpha s}\,ds\,dx, \tag{3.9}$$

where

$$L_t(x) = \mathbb{E}\big(1 - \exp\left(-\rho Q_t(x)\right)\big), \tag{3.10}$$

and

$$Q_t(x) = \big|\{s \in [0,t] : x \in \zeta(s) + M\}\big|_1.$$

Note that $Q_t(x)$ is the amount of time up to t at which x lies in the 'sausage'

$$\Sigma_t := \bigcup_{s\in[0,t]} \big[\zeta(s) + M\big], \qquad t \ge 0. \tag{3.11}$$

Consider the limit $\rho \to \infty$. By (3.9) and dominated convergence,

$$R(\rho) \uparrow \overline{R} := \lambda \int_{\mathbb{R}^d} \int_0^\infty \overline{L}_s(x)\alpha e^{-\alpha s}\,ds\,dx, \tag{3.12}$$

where

$$\overline{L}_t(x) = \mathbb{P}(Q_t(x) > 0) = \mathbb{P}(x \in \Sigma_t).$$

Therefore,

$$\overline{R} = \lambda \int_0^\infty \mathbb{E}|\Sigma_s|_d\,\alpha e^{-\alpha s}\,ds, \tag{3.13}$$

where the integral is the mean volume of the sausage Σ up to time T_0. This formula is easily obtained from first principles applied to the $\rho = \infty$ delayed diffusion process (see Section 3.4).

Example 3.4 (Bounded motion)**.** *If, in addition to the assumptions above, each particle is confined within some given distance $\Delta < \infty$ of its initial location, then $\Sigma_t \subseteq S(\Delta + \mathrm{rad}(M))$. Therefore, by (3.12)–(3.13),*

$$R(\rho) \le \overline{R} \le \lambda \big| S(\Delta + \mathrm{rad}(M)) \big|_d. \tag{3.14}$$

If the right side of (3.14) is strictly less than 1, then $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha) = 0$ for $\rho \in (0, \infty)$ by Proposition 3.3. This is an improvement over (3.6) for large ρ.

3.4. **A condition for subcriticality when $\rho = \infty$.** Let $d \ge 2$, $\rho = \infty$, and $\mu = 1_M$ with M compact. The argument of Sections 3.2–3.3 is easily adapted subject to a condition on the volume of the sausage Σ of (3.11), namely

$$\mathrm{C}_{\gamma,\sigma}\text{: there exist } \gamma, \sigma \in [0, \infty) \text{ such that, for } t \ge 0, \ \mathbb{E}|\Sigma_t|_d \le \gamma e^{\sigma t}. \tag{3.15}$$

Let

$$R(\infty) = \lambda \int_0^\infty \mathbb{E}|\Sigma_s|_d \, \alpha e^{-\alpha s} \, ds, \tag{3.16}$$

in agreement with (3.12)–(3.13). Note that $R(\infty)$ equals the mean number of points of the Poisson process $\Pi \setminus \{0\}$ lying in the sausage Σ_T, where T is independent of Σ and is exponentially distributed with parameter α.

Theorem 3.5.

(a) *If $R(\infty) < 1$ then $\theta_{\mathrm{dd}}(\lambda, \infty, \alpha) = 0$.*
(b) *Assume condition $\mathrm{C}_{\gamma,\sigma}$ of (3.15) holds, and $\lambda < \underline{\lambda} := 1/\gamma$. If $\alpha > \overline{\alpha} := \sigma/(1 - \lambda\gamma)$, then $R(\infty) < 1$ for $\alpha > \overline{\alpha}$.*

Proof. (a) This holds by the argument of Proposition 3.3 adapted to the case $\rho = \infty$.
(b) Subject to condition (3.15) with $\lambda\gamma < 1$,

$$R(\infty) \le \lambda \int_0^\infty \alpha\gamma e^{-(\alpha-\sigma)s} \, ds = \frac{\lambda\alpha\gamma}{\alpha - \sigma}, \qquad \alpha > \sigma, \tag{3.17}$$

and the second claim follows. □

Example 3.6 (Brownian motion with $d = 2$)**.** *Suppose $d = 2$, ζ is a standard Brownian motion, and $M = S$. By (3.16) and the results of Spitzer* [26, p. 117]*,*

$$\begin{aligned} R(\infty) &= \lambda|S|_2 + \lambda \int_0^\infty \alpha e^{-\alpha s} \int_{\mathbb{R}^2 \setminus S} \mathbb{P}(x \in \Sigma_s) \, dx \, ds \\ &= \lambda\pi + \lambda \int_{\mathbb{R}^2 \setminus S} \frac{K_0(\|x\|_2 \sqrt{2\alpha})}{K_0(\sqrt{2\alpha})} \, dx = \lambda Z_\alpha, \end{aligned}$$

where

$$Z_\alpha = \pi + \frac{2\pi}{\sqrt{\alpha}} \frac{K_1(\sqrt{2\alpha})}{K_0(\sqrt{2\alpha})} = \pi + \frac{2\pi}{\sqrt{\alpha}} + \mathrm{o}(\alpha^{-\frac{1}{2}}) \qquad \text{as } \alpha \to \infty. \tag{3.18}$$

Here, K_1 (respectively, K_0) is the modified Bessel function of the second kind of order 1 (respectively, order 0) given by

$$K_0(x) = \int_0^\infty e^{-x\cosh s}\,ds, \qquad K_1(x) = \int_0^\infty e^{-x\cosh s}\cosh s\,ds.$$

Therefore, if $\lambda < \underline{\lambda} := 1/\pi$, there exists $\overline{\alpha} \in (0,\infty)$ such that $R(\infty) < 1$ when $\alpha > \overline{\alpha}$.

Example 3.7 (Brownian motion with $d \geq 5$)**.** *Suppose $d \geq 5$, ζ is a standard Browian motion, and $M = S$. Getoor* [6, Thm 2] *has shown an explicit constant C such that*

$$\mathbb{E}|\Sigma_t|_d - tc_d \uparrow C \qquad \text{as } t \to \infty,$$

where c_d is the Newtonian capacity of the closed unit ball S of $\mathbb{R}^d$. By (3.16)*,*

$$R(\infty) \leq \lambda\left(\frac{c_d}{\alpha} + C\right).$$

Therefore, if $\lambda < \underline{\lambda} := 1/C$, there exists $\overline{\alpha} \in (0,\infty)$ such that $R(\infty) < 1$ when $\alpha > \overline{\alpha}$. Related estimates are in principle valid for $d = 3, 4$, though the behaviour of $\mathbb{E}|\Sigma_t| - tc_d$ is more complicated (see [6]*).*

Example 3.8 (Brownian motion with constant drift)**.** *Let $d \geq 2$, $M = S$, with ζ a Brownian motion with constant drift. It is standard (with a simple proof using subadditivity) that the limit $\gamma := \mathbb{E}|\Sigma_t|_d/t$ exists and in addition is strictly positive when the drift is non-zero. Thus, for $\epsilon > 0$, there exists C_ϵ such that*

$$\mathbb{E}|\Sigma_t|_d \leq C_\epsilon + (1+\epsilon)\gamma t, \qquad t \geq 0.$$

As in Example 3.7, if $\lambda < \underline{\lambda} := 1/C_\epsilon$, there exists $\overline{\alpha} \in (0,\infty)$ such that $R(\infty) < 1$ when $\alpha > \overline{\alpha}$. See also [11, 12].

Example 3.9 (Ornstein–Uhlenbeck process)**.** *Let $M = S$ and consider the Ornstein–Uhlenbeck process in $\mathbb{R}^d$ satisfying*

$$d\zeta(t) = A\zeta(t)\,dt + dW_t$$

where W is standard Brownian motion in $\mathbb{R}^d$, A is a $d \times d$ real matrix, and $\zeta(0) = 0$. It is an exercise that $\mathrm{C}_{\gamma,\sigma}$ holds for suitable γ, σ.

3.5. **Proof of Theorem 3.1: a preliminary proposition.** Consider the delayed diffusion model with $d \geq 2$. Suppose that either $\rho \in (0,\infty)$ with μ as in (2.2), or $\rho = \infty$ and

$$\mu(x) = 1_S(x), \qquad x \in \mathbb{R}^2, \tag{3.19}$$

where S is the closed unit ball with centre at the origin.

The forthcoming Proposition 3.10 is motivated in part by work of Kuulasmaa [18]. Recall the initial placements $\Pi = (X_0 = 0, X_1, X_2, \dots)$ of particles P_i, with law denoted P (and corresponding expectation E); we condition on Π.

Fix $i \geq 0$, and consider the following model for infection. The particle P_i is the *unique* initially infected particle, and it diffuses according to ζ_i and has lifetime T_i. All other particles P_j, $j \neq i$, are kept stationary for all time at their respective locations X_j. As P_i moves around $\mathbb{R}^d$, it infects other particles in the usual way; newly infected particles are permitted neither to move nor to infect others. Let J_i be the (random) set of particles infected by P_i in this process. Given Π, the set J_i depends only on the pair (ζ_i, T_i) associated with P_i.

Let $\tau_{i,j}$ be the time of the first infection by P_i of P_j, *assuming that P_i is never removed*. Write $i \to j$ if $\tau_{i,j} < T_i$, which is to say that this infection takes place before P_i is removed. Thus, $J_i = \{j : i \to j\}$.

Suppose first that $\rho < \infty$. Given (Π, ζ_i, T_i), the vector $\tau_i = (\tau_{i,j} : j \neq i)$ contains conditionally independent random variables with respective distribution functions

$$F_{i,j}(t) = 1 - \exp\left(-\int_0^t \rho\mu(X_j - \zeta_i(s))\,ds\right), \qquad t \geq 0, \tag{3.20}$$

and

$$\mathbb{P}(i \to j \mid \Pi, \zeta_i, T_i) = F_{i,j}(T_i). \tag{3.21}$$

When $\rho = \infty$, we have that

$$\tau_{i,j} = \inf\{t > 0 : X_j \in X_i + \zeta_i(t) + S\}, \tag{3.22}$$

the first hitting time of $X_j - X_i$ by the radius-1 Wiener sausage of ζ_i. As above, we write $i \to j$ if $\tau_{i,j} < T_i$, with J_i and τ_i given accordingly.

One may thus construct sets J_i for all $i \geq 0$; given Π, the set J_i depends only on (ζ_i, T_i), and therefore the J_i are conditionally independent given Π. The sets $\{J_i : i \geq 0\}$ generate a directed graph $\vec{G} = \vec{G}_\Pi$ with vertex-set $\mathbb{Z}_0$ and directed edge-set $\vec{E} = \{[i, j\rangle : i \to j\}$. Write $\vec{I}$ for the set of vertices k of $\vec{G}$ such that there exists a directed path of $\vec{G}$ from 0 to k. To the edges of $\vec{G}$ we attach random labels, with edge $[i, j\rangle$ receiving the label $\tau_{i,j}$.

From the vector $(\tau_i, T_i : i \geq 0)$, we can construct a copy of the general delayed diffusion process by allowing an infection by P_i of P_j whenever P_j has not been infected earlier by another particle. Let I denote the set of ultimately infected particles in this coupled process.

Proposition 3.10. *For $\rho \in (0, \infty]$, we have $I = \vec{I}$.*

By rescaling in space/time, we obtain the following. The full parameter-set of the process is $\{\lambda, \rho, \alpha, \mu, \sigma\}$, where σ is the standard-deviation parameter of the Brownian motion, and we shall sometimes write $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha, \mu, \sigma)$ accordingly.

Proposition 3.11. *Let $\rho \in (0, \infty]$.*

(a) *For given $\lambda \in (0,\infty)$, the function $\theta_{\mathrm{dd}}(\lambda,\rho,\alpha)$ is non-decreasing in ρ and non-increasing in α.*
(b) *We have that*

$$\theta_{\mathrm{dd}}(\lambda,\rho,\alpha,\mu,1) = \theta_{\mathrm{dd}}(\lambda/r^d,\rho/r^2,\alpha/r^2,\mu_r,1), \qquad r \geq 1, \tag{3.23}$$

where $\mu_r(x) := \mu(x/r)$.
(c) *If μ is radially decreasing (see (2.3)), then*

$$\alpha_{\mathrm{c}}(\lambda,\rho) \geq r^2\alpha_{\mathrm{c}}(\lambda/r^d,\rho/r^2), \qquad r \geq 1.$$

(d) *If $\rho = \infty$ and μ is radially decreasing, then $\theta_{\mathrm{dd}}(\lambda,\infty,\alpha)$ and $\alpha_{\mathrm{c}}(\lambda,\infty)$ are non-decreasing in λ.*

Proof of Proposition 3.10. This is a deterministic claim. Assume Π is given. If $i \in I$, there exists a chain of direct infection from 0 to i, and this chain generates a directed path of $\vec{G}$ from 0 to i. Suppose, conversely, that $k \in \vec{I}$. Let $\mathcal{P}_k$ be the set of directed paths of $\vec{G}$ from 0 to k. Let $\pi \in \mathcal{P}_k$ be a shortest such path (where the length of an edge $[i,j\rangle$ is taken to be the label $\tau_{i,j}$ of that edge). We may assume that the $\tau_{i,j}$, for $i \to j$, are distinct; no difficulty emerges on the complementary null set. Then the path π is a geodesic, in that every sub-path is the shortest directed path joining its endvertices. Therefore, when infection is initially introduced at P_0, it will be transmitted directly along π to P_k. □

Proof of Proposition 3.11. (a) By Proposition 3.10(a), if the parameters are changed in such a way that each J_i is stochastically increased (respectively, decreased), then the set I is also stochastically increased (respectively, decreased). The claims follow by (3.20)–(3.21) when $\rho < \infty$, and by (3.22) when $\rho = \infty$.

(b) Let $r \geq 1$, and consider the effect of dilating space by the ratio r. After stretching space by a factor r, the resulting stretched Poisson process $r\Pi$ has intensity λ/r^d, the resulting Brownian motion $r\zeta_i(t)$ is distributed as $\zeta_i(r^2t)$, and μ is replaced by μ_r. Therefore,

$$\theta_{\mathrm{dd}}(\lambda,\rho,\alpha,\mu,1) = \theta_{\mathrm{dd}}(\lambda/r^d,\rho,\alpha,\mu_r,r). \tag{3.24}$$

Next, we use the construction of the process in terms of the J_i given above Proposition 3.10. If $\rho < \infty$ then, by (3.21) and the change of variables $u = r^2 s$,

$$\begin{aligned}\mathbb{P}(i \to j \mid \Pi, \zeta_i, T_i) &= 1 - \exp\left(-\int_0^{T_i} \rho\mu(X_j - \zeta_i(s))\, ds\right) \\ &\stackrel{\mathrm{d}}{=} 1 - \exp\left(-\int_0^{T_i} \rho\mu_r(rX_j - \zeta_i(r^2 s))\, ds\right) \\ &\stackrel{\mathrm{d}}{=} 1 - \exp\left(-\int_0^{r^2 T_i} \rho\mu_r(rX_j - \zeta_i(u))\, \frac{du}{r^2}\right),\end{aligned}$$

where $\stackrel{\mathrm{d}}{=}$ means equality in distribution. Since $r^2 T_i$ is exponentially distributed with parameter α/r^2, the right side of (3.24) equals $\theta_{\mathrm{dd}}(\lambda/r^d, \rho/r^2, \alpha/r^2, \mu_r, 1)$, as claimed. The same conclusion is valid for $\rho = \infty$, by (3.22).

(c) Since $\mu_r \geq \mu$ by assumption, it follows by (3.23) that

$$\theta_{\mathrm{dd}}(\lambda, \rho, \alpha, \mu, 1) \geq \theta_{\mathrm{dd}}(\lambda/r^d, \rho/r^2, \alpha/r^2, \mu, 1), \qquad r \geq 1.$$

By the monotonicity of θ_{dd} in α, if $\alpha > \alpha_{\mathrm{c}}(\lambda, \rho)$ then $\alpha/r^2 \geq \alpha_{\mathrm{c}}(\lambda/r^d, \rho/r^2)$ as claimed.

(d) This holds as in part (b).

□

Remark 3.12. *In the forthcoming proof of Section 3.6.3 we shall use the following consequence of Proposition 3.10. By part* (a)*,*

$$\theta_{\mathrm{dd}}(\lambda, \rho, \alpha) = \mathrm{E}\big(\mathbb{Q}_\Pi(|\vec{I}| = \infty)\big). \tag{3.25}$$

In proving survival, it therefore suffices to prove the right side of (3.6) *is strictly positive.*

3.6. **Proof of Theorem 3.1.**

3.6.1. *Existence of* α_{c}. Consider the Brownian delayed diffusion model with $d \geq 2$, $\rho \in (0, \infty]$. When $\rho = \infty$, we assume in addition that

$$\mu(x) = 1_S(x), \qquad x \in \mathbb{R}^2, \tag{3.26}$$

where S is the closed unit ball with centre at the origin. Note that μ is radially decreasing.

By Proposition 3.11, $\theta_{\mathrm{dd}}(\lambda, \rho, \alpha)$ is non-decreasing in ρ, and non-increasing in α, and is moreover non-decreasing in λ if $\rho = \infty$ and μ is radially decreasing (as is the case with (3.26)). With

$$\alpha_{\mathrm{c}}(\lambda, \rho) := \inf\big\{\alpha : \theta_{\mathrm{dd}}(\lambda, \rho, \alpha) = 0\big\},$$

we have that

$$\theta_{\mathrm{dd}}(\lambda,\rho,\alpha)\begin{cases}>0 & \text{if } \alpha<\alpha_{\mathrm{c}}(\lambda,\rho),\\ =0 & \text{if } \alpha>\alpha_{\mathrm{c}}(\lambda,\rho),\end{cases}$$

and, furthermore, α_{c} is non-decreasing in ρ.

In case (a) of the theorem, by Proposition 3.3, $\alpha_{\mathrm{c}}(\lambda,\rho)<\infty$ for all λ, ρ. In case (b), by Theorem 3.5 and Example 3.8, there exists $\underline{\lambda}\in(0,\lambda_{\mathrm{c}}]$ such that $\alpha_{\mathrm{c}}(\lambda,\infty)<\infty$ when $\lambda\in(0,\underline{\lambda})$. As remarked after (1.2), $\alpha_{\mathrm{c}}(\lambda,\infty)=0$ when $\lambda>\lambda_{\mathrm{c}}$.

It remains to show that $\alpha_{\mathrm{c}}(\lambda,\rho)>0$ for all $\lambda\in(0,\infty)$, $\rho\in(0,\infty]$, and the rest of this proof is devoted to that. This will be achieved by comparison with a directed site percolation model on $\mathbb{Z}_0^2$ viewed as a directed graph with edges directed away from the origin. When $d=2$, the key fact is the *recurrence* of Brownian motion, which permits a static block argument. This fails when $d\geq 3$, in which case we employ a dynamic block argument and the *transience* of Brownian motion.

3.6.2. *The case $d=2$.* Assume first that $d=2$, for which we use a static block argument. Let $\epsilon>0$. We choose $a>0$ such that

$$\mathbb{P}(\Pi\cap aS\neq\varnothing)>1-\epsilon. \tag{3.27}$$

For $\mathbf{x}\in\mathbb{Z}^2$, let $S_{\mathbf{x}}=3a\mathbf{x}+aS$ be the ball with radius a and centre at $3a\mathbf{x}$. We declare $\mathbf{x}$ *occupied* if $\Pi\cap S_{\mathbf{x}}\neq\varnothing$, and *vacant* otherwise. Note that the occupied/vacant states of different $\mathbf{x}$ are independent. If a given $\mathbf{x}\neq 0$ is occupied, we let $Q_{\mathbf{x}}\in\Pi\cap S_{\mathbf{x}}$ be the earliest such point in the lexicographic ordering, and we set $Q_0=0$. If $\mathbf{x}$ is occupied, we denote by $\zeta_{\mathbf{x}}$ the diffusion associated with the particle at $Q_{\mathbf{x}}$, and $T_{\mathbf{x}}$ for the lifetime of this particle.

Let ζ be a standard Brownian motion on $\mathbb{R}^2$ with $\zeta(0)=0$, and let

$$W_t(\zeta):=\bigcup_{s\in[0,t]}\big[\zeta(s)+S\big],\qquad t\in[0,\infty), \tag{3.28}$$

be the corresponding Wiener sausage.

Suppose for now that $\rho=\infty$; later we explain how to handle the case $\rho<\infty$. First we explain what it means to say that the origin 0 is *open*. Let

$$F(\zeta,z)=\inf\{t:z\in W_t(\zeta)\},\qquad z\in\mathbb{R}^2,$$

be the first hitting time of z by $W(\zeta)$.

For $\mathbf{y}\in\mathbb{Z}^2$, we define the event

$$K(\zeta_0,\mathbf{y})=\bigcap_{x\in S_{\mathbf{y}}}\{F(\zeta_0,z)<T_0\},$$

and

$$K(\zeta_0)=\bigcap_{\mathbf{y}\in N}K(\zeta_0,\mathbf{y}),$$

where $N = \{(0,1),(1,0)\}$ is the neighbour set of 0 in the directed graph on $\mathbb{Z}_0^2$. By the recurrence of ζ_0, we may choose α such that

$$p_\alpha(0) := \mathbb{P}(K(\zeta_0)) \quad \text{satisfies} \quad p_\alpha(0) > 1 - \epsilon. \tag{3.29}$$

We call 0 *open* if 0 is occupied, and in addition the event $K(\zeta_0)$ occurs. If 0 is not open, it is called *closed*.

We now explain what is meant by declaring $\mathbf{x} \in \mathbb{Z}^2 \setminus \{0\}$ to be open. Assume $\mathbf{x}$ is occupied and pick $Q_\mathbf{x}$ as above. For $\mathbf{y} \in \mathbf{x} + N$, we define the event

$$K(\zeta_\mathbf{x}, \mathbf{y}) = \bigcap_{z \in S_\mathbf{y}} \{F(Q_\mathbf{x} + \zeta_\mathbf{x}, z) < T_\mathbf{x}\}, \tag{3.30}$$

and

$$K(\zeta_\mathbf{x}) = \bigcap_{\mathbf{y} \in N} K(\zeta_\mathbf{x}, \mathbf{y}).$$

By the recurrence of ζ, we may choose α such that

$$p_\alpha(\mathbf{x}) := \mathbb{P}\big(K(\zeta_\mathbf{x}) \,\big|\, \mathbf{x} \text{ is occupied}\big) \quad \text{satisfies} \quad p_\alpha(\mathbf{x}) > 1 - \epsilon. \tag{3.31}$$

We declare $\mathbf{x} \in \mathbb{Z}^2$ *open* if $\mathbf{x}$ is occupied, and in addition the event $K(\zeta_\mathbf{x})$ occurs. A vertex of $\mathbb{Z}^2$ which is not open is called *closed*. Conditional on the set of occupied vertices, the open/closed states are independent.

The open/closed state of a vertex $\mathbf{x} \in \mathbb{Z}^2$ depends only on the existence of $Q_\mathbf{x}$ and on the diffusion $\zeta_\mathbf{x}$, whence the open/closed states of different $\mathbf{x} \in \mathbb{Z}^2$ are independent. By (3.27)–(3.29), the configuration of open/closed vertices forms a family of independent Bernoulli random variables with density at least $(1-\epsilon)^2$. Choose $\epsilon > 0$ such that $(1-\epsilon)^2$ exceeds the critical probability of directed site percolation on $\mathbb{Z}_0^2$ (cf. [9, Thm 3.30]). With strictly positive probability, the origin is the root of an infinite directed cluster of the latter process. Using the definition of the state 'open' for the delayed diffusion model, we conclude that the graph $\vec{G}$ contains an infinite directed path from the origin with strictly positive probability. The corresponding claim of Theorem 3.1(b) follows by Lemma 3.10(a).

Suppose now that $\rho \in (0, \infty)$. We adapt the above argument by redefining the times $F(\zeta, z)$ and the events $K(\zeta)$ as follows. Consider first the case of the origin, assumed to be occupied. Let

$$E(\zeta, z, t) = \big|\{s \in [0,t] : z \in \zeta(s) + S\}\big|_1. \tag{3.32}$$

Pick $F > 0$ such that $e^{-\rho F} < \epsilon$, and write

$$\overline{K}(\zeta_0, t) = \bigcap_{\mathbf{y} \in N,\, z \in S_\mathbf{x}} \{E(\zeta_0, z, t) > F\}.$$

In words, $\overline{K}(\zeta_0, t)$ is the event that the Wiener sausage, started at 0 and run for time t, contains every $z \in S_{(0,1)} \cup S_{(1,0)}$ for an aggregate time exceeding F. It follows that,

given that $Q_{\mathbf{y}} \in \Pi \cap S_{\mathbf{y}}$ for some $\mathbf{y} \in N$, then P_0 infects $Q_{\mathbf{y}}$ with probability at least $1 - e^{-\rho F} > 1 - \epsilon$.

By elementary properties of a recurrent Brownian motion, we may pick t and then $\alpha = \alpha(t)$ such that (cf. (3.29))

$$p_\alpha(0) := \mathbb{P}\big(\overline{K}(\zeta_0, t) \cap \{t < T_0\}\big) \quad \text{satisfies} \quad p_\alpha(0) > 1 - \epsilon. \tag{3.33}$$

Turning to general $\mathbf{x} \in \mathbb{Z}^2 \backslash \{0\}$, a similar construction is valid for an event $\overline{K}(\zeta_{\mathbf{x}}, t)$ as in (3.33), and we replicate the above comparison with directed percolation with $(1-\epsilon)^2$ replaced by $(1-\epsilon)^3$.

3.6.3. *The case $d \geq 3$.* Let $d = 3$; the case $d \geq 4$ is handled similarly. This time we use a *dynamic* block argument, combined with Remark 3.12. The idea is the following. Let ζ_0 be the diffusion of particle P_0. We track the projection of ζ_0, denoted $\overline{\zeta}_0$, on the plane $\mathbb{R}^2 \times \{0\}$. By the recurrence of $\overline{\zeta}_0$, the Wiener sausage $W(\zeta_0)$ a.s. visits every line $z \times \{0\}$ infinitely often, for $z \in \mathbb{R}^2$. At such a visit, we choose a point Q_z of Π lying in $W(\zeta_0)$ 'near to' the line $z \times \{0\}$. The construction is then iterated with Q_z as the starting particle. We build this process in each of two independent directions, and may choose the parameter values such that it dominates the cluster at 0 of a supercritical directed site percolation process.

For $A \subseteq \mathbb{R}^3$, we write $\overline{A}$ for its projection onto the first two coordinates. That is, $\overline{\mathbb{R}}^2 = \mathbb{R}^2 \times \{0\}$ is the plane of the first two coordinates, and similarly $\overline{\mathbb{Z}}^2 = \mathbb{Z}^2 \times \{0\}$, $\overline{\mathbb{Z}}^2_0 = \mathbb{Z}^2_0 \times \{0\}$, and $\overline{S} = S \cap \overline{\mathbb{R}}^2$. We abuse notation by identifying $\mathbf{x} = (x_1, x_2, 0, \dots, 0) \in \overline{\mathbb{R}}^2$ (respectively, $\overline{\mathbb{Z}}^2$ etc) with the 2-vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ (respectively, $\mathbb{Z}^2$ etc).

For $\mathbf{x} \in \overline{\mathbb{Z}}^2$, let $\overline{S}_{\mathbf{x}} = 3a\mathbf{x} + a\overline{S}$ be the two-dimensional ball with radius $a > 1$ and centre at $3a\mathbf{x}$, and let $C_{\mathbf{x}} = S_{\mathbf{x}} \times \mathbb{R}^1$ be the *cylinder* generated by $\mathbf{x}$. Let $\zeta = (\zeta^{(i)} : i = 1, 2, 3)$ be a standard Brownian motion in $\mathbb{R}^3$ with $\zeta(0) = 0$ and coordinate processes $\zeta^{(i)}$, and let $\overline{\zeta} = (\zeta^{(1)}, \zeta^{(2)}, 0)$ be its projection onto the first two coordinates. Note that $\overline{\zeta}$ is a recurrent process on $\overline{\mathbb{R}}^2$.

We declare the particle at 0 to be *open*, and let $\mathbf{y} \in N := \{(1,0), (0,1)\}$. We shall see that, with a probability to be bounded below, there exists a (random) particle at some $Q_{\mathbf{y}} \in C_{\mathbf{y}}$ such that P_0 infects this particle. If this occurs, we declare $\mathbf{y}$ to be open. On the event that $\mathbf{y}$ is open, we may iterate the construction starting at $Q_{\mathbf{y}}$, to find a number of further random vertices of $\vec{G}$. By a comparison with a supercritical directed site percolation model, we shall show (for large α) that $\vec{G}$ contains an infinite directed cluster with root 0. The claim then follows by Proposition 3.10 and Remark 3.12.

Suppose for now that $\rho = \infty$. Let $\epsilon > 0$. With ζ a standard Brownian motion on $\mathbb{R}^3$ with $\zeta(0) = 0$, let $W_t(\zeta)$ be the corresponding Wiener sausage (3.28). We explain

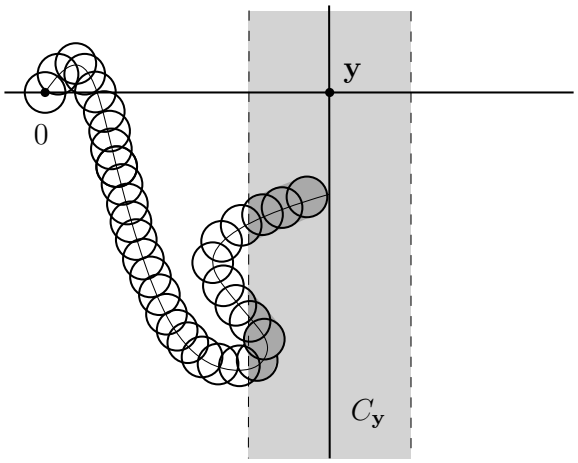

FIGURE 3.1. The Wiener sausage $W(\zeta_0)$ stopped when it hits the line $\mathbf{y} \times \mathbb{R}$. The dark shaded areas constitute the region $L(\zeta_0, \mathbf{y})$.

next the state open/closed for a vertex $\mathbf{y} \in N$. Let

$$F(\zeta_0, \mathbf{y}) = \inf\big\{t : (\mathbf{y} \times \{0\}) \cap W_t(\zeta_0) \neq \varnothing\big\}. \tag{3.34}$$

Since $\overline{\zeta}_0$ is recurrent, we have $F(\zeta_0, \mathbf{y}) < \infty$ a.s. Let T_0 be the lifetime of P_0, and define the event

$$K(\zeta_0, \mathbf{y}) = \big\{F(\zeta_0, \mathbf{y}) < T_0\big\}. \tag{3.35}$$

We explain next how a is chosen (see Figure 3.1). By a geometrical observation, there exists an absolute constant $c > 0$ such that the following holds. Let $a > 1$. For $\mathbf{y} \in N$, the intersection

$$L(\zeta_0, \mathbf{y}) := W_{F(\zeta_0, \mathbf{y})}(\zeta_0) \cap C_{\mathbf{y}}$$

has volume at least ca. We now pick $a > 1$ sufficiently large that

$$\mathbb{P}\big(N_{\mathbf{y}} \mid K(\zeta_0, \mathbf{y})\big) > 1 - \epsilon \qquad \text{where} \qquad N_{\mathbf{y}} := \{\Pi \cap L(\zeta_0, \mathbf{y}) \neq \varnothing\}.$$

If $\Pi \cap L(\zeta_0, \mathbf{y}) \neq \varnothing$, we pick the earliest point in the intersection (in lexicographic order) and denote it $Q_{\mathbf{y}}$, and we say that $Q_{\mathbf{y}}$ has been *occupied from* 0. We call $\mathbf{y}$ *open* if $K(\zeta_0, \mathbf{y}) \cap N_{\mathbf{y}}$ occurs, and *closed* otherwise.

By the recurrence of $\overline{\zeta}$, we may choose $\alpha > 0$ such that

$$p_\alpha(\mathbf{y}) := \mathbb{P}(\mathbf{y} \text{ is open}) \quad \text{satisfies} \quad p_\alpha(0) > 1 - \epsilon. \tag{3.36}$$

In order to define the open/closed states of other $\mathbf{x} \in \overline{\mathbb{Z}}^2$, it is necessary to generalize the above slightly, and we do this next. Instead of considering a Brownian

motion ζ starting at $\zeta(0) = 0$, we move the starting point to some $q \in \overline{\mathbb{R}}^2$. Thus ζ becomes $q + \zeta$, and (3.34)–(3.35) become

$$F(\zeta, q, \mathbf{y}) = \inf\big\{t : (\mathbf{y} \times \{0\}) \cap (q + W_t(\zeta)) \neq \varnothing\big\},$$
$$K(\zeta, q, \mathbf{y}, T) = \{F(\zeta, q, \mathbf{y}) < T\}.$$

By the recurrence of $\overline{\zeta}$, we may choose α such that

$$\overline{p}_\alpha(\mathbf{y}) := \inf\big\{\mathbb{P}(K(\zeta_0, q, \mathbf{y}, T_0) : q \in \overline{S}\big\} \quad \text{satisfies} \quad \overline{p}_\alpha(\mathbf{y}) > 1 - \epsilon. \tag{3.37}$$

The extra notation introduced above will be used at the next stage.

We construct a non-decreasing sequence pair (V_n, W_n) of disjoint subsets of $\overline{\mathbb{Z}}_0^2$ in the following way. The set V_n is the set of vertices known to be open at stage n of the construction, and W_n is the set known to be closed.

The vertices of $\overline{\mathbb{Z}}_0^2$ are ordered in L^1 order: for $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, we declare

$$\mathbf{x} < \mathbf{y} \qquad \text{if} \qquad \text{either } x_1 + x_2 < y_1 + y_2, \quad \text{or } x_1 + x_2 = y_1 + y_2 \text{ and } x_1 < y_1.$$

We refer to a point $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}_0^2$ as belonging to *generation* n if $x_1 + x_2 = n$.

First, let

$$V_0 = \{0\}, \qquad W_0 = \varnothing.$$

We choose the least $\mathbf{y} \in N$, and set:

$$\text{if } \mathbf{y} \text{ is open:} \quad V_1 = V_0 \cup \{\mathbf{y}\},\ W_1 = W_0,$$
$$\text{otherwise:} \quad V_1 = V_0,\ W_1 = W_0 \cup \{\mathbf{y}\}.$$

In the first case, we say that '$\mathbf{y}$ is occupied from 0'.

For $A \subset \overline{\mathbb{Z}}_0^2$, let ΔA be the set of vertices $b \in \overline{\mathbb{Z}}_0^2 \setminus A$ such that b has some neighbour $a \in A$ with $a < b$. Suppose (V_k, W_k) have been defined for $k = 1, 2, \dots, n$, and define (V_{n+1}, W_{n+1}) as follows. Select the least $\mathbf{z} \in \Delta V_n \setminus W_n$. If such $\mathbf{z}$ exists, find the least $\mathbf{x} \in V_n$ such that $\mathbf{z} = \mathbf{x} + \mathbf{y}$ for some $\mathbf{y} \in N$. Thus $\mathbf{x}$ is known to be open, and there exists a vertex of $\vec{G}$ at the point $Q_\mathbf{x} \in C_\mathbf{x}$.

As above,

$$L(\zeta_\mathbf{x}, Q_\mathbf{x}, \mathbf{z}) := W_{F(\zeta_\mathbf{x}, Q_\mathbf{x}, \mathbf{y})}(\zeta_\mathbf{x}) \cap C_\mathbf{z},$$
$$N_\mathbf{z} := \{\Pi \cap L(\zeta_\mathbf{x}, Q_\mathbf{x}, \mathbf{y}) \neq \varnothing\}.$$

If $K(\zeta_\mathbf{x}, Q_\mathbf{x}, \mathbf{z}, T_\mathbf{x}) \cap N_\mathbf{z}$ occurs we call $\mathbf{z}$ *open*, and we say that $\mathbf{z}$ is occupied from $\mathbf{x}$; otherwise we say that $\mathbf{z}$ is *closed*.

$$\text{If } \mathbf{z} \text{ is open:} \quad V_{n+1} = V_n \cup \{\mathbf{z}\},\ W_{n+1} = W_n,$$
$$\text{otherwise:} \quad V_{n+1} = V_n,\ W_{n+1} = W_n \cup \{\mathbf{z}\}.$$

By (3.36)–(3.37), the vertex $\mathbf{z}$ under current scrutiny is open with conditional probability at least $(1-\epsilon)^2$.

This process is iterated until the earliest stage at which no such $\mathbf{z}$ exists. If this occurs for some $n < \infty$, we declare $V_m = V_n$ for $m \geq n$, and in any case we set $V_\infty = \lim_{m\to\infty} V_m$.

The resulting set V_∞ is the cluster at the origin of a type of dependent directed site percolation process which is built by generation-number. Having discovered the open vertices $\mathbf{z}$ in generation n together with the associated points $Q_{\mathbf{z}}$, the law of the next generation is (conditionally) independent of the past and is 1-dependent.

By [19, Thm 0.0] (see also [8, Thm 7.65] and the references therein), there exists $\eta = \eta(\epsilon)$, satisfying $\eta(\epsilon) \to 0$ as $\epsilon \to 0$, such that V_∞ dominates stochastically the cluster at the origin of a 'normal' directed site percolation process on $\mathbb{Z}_0^2$ with density $1-\eta(\epsilon)$. Therefore, for sufficiently small $\epsilon > 0$, $\mathbb{P}(|V_\infty| = \infty) > 0$. By a consideration of the geometry of the above construction, and the definition of the local states open/occupied, by (3.25) this entails $\theta_{\mathrm{dd}}(\lambda, \infty, \alpha) > 0$.

When $\rho \in (0, \infty)$, we extend the earlier argument (around (3.35) and later). Rather than presenting all the required details, we consider the special case of (3.35); the general case is similar. Let $\mathbf{y} \in N$ and $X_t := W_t(\zeta_0) \cap C_{\mathbf{y}}$. We develop the previous reference to the first hitting time $F(\zeta_0, \mathbf{y})$ with a consideration of the limit set $X_\infty = \lim_{t\to\infty} X_t$. Since $\overline{\zeta}_0$ is recurrent and ζ_0 is transient, there exists a deterministic $\eta > 0$ such that:

(a) a.s., X_∞ contains infinitely many disjoint closed connected regions, each with volume exceeding $\frac{1}{2}ca$, and
(b) every point of $\mathbb{R}^3$ in the union of these regions belongs to X_∞ for a total time exceeding η.

Each such region contains a point of Π with probability at least $1 - e^{-\frac{1}{2}\lambda ca}$. Each such point is infected by P_0 with probability at least $1 - e^{-\rho\eta}$. Pick N such that, in N independent trials each with probability of success $1 - e^{-\frac{1}{2}\lambda ca} - e^{-\rho\eta}$, there exists at least one success with probability exceeding $1-\epsilon$. Finally, pick the deterministic time τ such that there is probability at least $1-\epsilon$ that X_τ contains at least N disjoint closed connected regions each with volume exceeding $\frac{1}{2}ca$.

Finally, we pick α such that

$$\mathbb{P}(T_0 > \tau) \geq 1 - \epsilon.$$

With these choices, the probability that $W_\tau(\zeta_0) \cap C_{\mathbf{y}}$ contains some particle that is infected from 0 is at least $(1-\epsilon)^3$. The required argument proceeds henceforth as before.

4. The diffusion model

4.1. **A condition for subcriticality.** We consider the diffusion model in the general form of Sections 2.1 and 2.3, and we adopt the notation of those sections. Recall the critical point λ_c of the Boolean continuum percolation on $\mathbb{R}^d$ in which a closed unit ball is centred at each point of a rate-λ Poisson process on $\mathbb{R}^d$. We prove the existence of a subcritical phase.

Condition (3.15) is now replaced as follows. Let ζ' be an independent copy of ζ, and define the sausage

$$\Sigma'_t := \bigcup_{s\in[0,t]} \big[\zeta(s) - \zeta'(s) + S\big], \qquad s \geq 0. \tag{4.1}$$

We shall assume

$$\mathrm{C}'_{\gamma,\sigma}\text{: there exist } \gamma,\sigma \in [0,\infty) \text{ such that, for } t \geq 0,\ \mathbb{E}|\Sigma'_t|_d \leq \gamma e^{\sigma t}, \tag{4.2}$$

and we make a note about this condition in Remark 4.3.

Theorem 4.1. *Consider the general Brownian diffusion model on $\mathbb{R}^d$ where $d \geq 2$.*

(a) *Let $\rho \in (0,\infty)$. There exists a non-decreasing function $\alpha_c : (0,\infty)^2 \to (0,\infty)$ such that $\theta_d(\lambda,\rho,\alpha) = 0$ if $\alpha > \alpha_c(\lambda,\rho)$.*

(b) *Let $\rho = \infty$ and $\mu = 1_S$. Assume in addition that condition $\mathrm{C}'_{\gamma,\sigma}$ of (4.2) holds. Let $\alpha_c(\lambda) = \sigma/(1-\lambda\gamma)$ and $\underline{\lambda} = 1/\gamma$. Then $\theta_d(\lambda,\infty,\alpha) = 0$ if $\alpha > \alpha_c(\lambda)$ and $0 < \lambda < \underline{\lambda}$.*

This theorem extends Theorem 1.2. Its proof follows that given in Section 3.2 for the delayed diffusion model, and thus we present only an outline.

Proof. (a) Let $\lambda \in (0,\infty)$, and suppose first that $\rho < \infty$. Proposition 3.2 holds with the same proof but with $L_t(x)$ replaced by

$$\widetilde{L}_t(x) = \mathbb{E}\left(1 - \exp\left(-\int_0^t \rho\mu(x + \zeta(s) - \zeta'(s))\,ds\right)\right), \tag{4.3}$$

where ζ' is an independent copy of ζ.

With this new $\widetilde{L}_t(x)$, Proposition 3.3 is unchanged in the current context. As there, the bound $R = R(\rho)$ now satisfies

$$R(\rho) = \lambda \int_{\mathbb{R}^d}\int_0^\infty \widetilde{L}_s(x)\alpha e^{-\alpha s}\,ds\,dx \leq \frac{\lambda\rho}{\alpha}\,\mathrm{Int}(\mu). \tag{4.4}$$

We may take $\alpha_c = \lambda\rho\,\mathrm{Int}(\mu)$, and the claim follows by Proposition 3.3(b) adapted to the diffusion model.

(b) Let $\rho = \infty$. As at (3.16),

$$R(\infty) := \lambda \int_0^t \mathbb{E}|\Sigma'_s|_d \, \alpha e^{-\alpha s} \, ds. \tag{4.5}$$

As in Theorem 3.5(b) adapted to the diffusion model, we have by $\mathrm{C}'_{\gamma,\sigma}$ that $R(\infty) < 1$ if $\lambda < \underline{\lambda} := 1/\gamma$ and $\alpha > \alpha_{\mathrm{c}}(\lambda) := \sigma/(1 - \lambda\gamma)$. As in Theorem 3.5(a), $\theta_{\mathrm{d}}(\lambda, \rho, \alpha) = 0$ for $\lambda \in (0, \underline{\lambda})$ and $\alpha > \alpha_{\mathrm{c}}(\lambda)$. □

Example 4.2 (Bounded motion)**.** *Let $\rho = \infty$ and $\mu = 1_M$ as above, and suppose in addition that each particle is confined within some given distance $\Delta < \infty$ of its initial location. By (4.5),*

$$R(\infty) \le \lambda \big| S\big(2(\Delta + \mathrm{rad}(M))\big) \big|_d.$$

If the right side is strictly less than 1, then $\theta_{\mathrm{d}}(\lambda, \infty, \alpha) = 0$ by Proposition 3.3 adapted to the current context.

Remark 4.3 (Condition $\mathrm{C}'_{\gamma,\sigma}$)**.** *Let $M_t = \sup\{\|\zeta(s)\|_d : s \in [0,t]\}$, the maximum displacement of ζ up to time t, and let M'_t be given similarly in terms of ζ'. By Minkowski's inequality,*

$$\mathbb{E}|\Sigma'_t|_d \le \mathbb{E}\big([M_t + M'_t + 1]^d\big) \le \big(2\|M_t\| + 1\big)^d,$$

were $\|\cdot\|$ denotes the L^d norm. Therefore, $\mathrm{C}'_{\gamma,\sigma}$ holds for some γ, σ if $\|M_t\| \le \gamma' e^{\sigma' t}$ for suitable γ', σ'.

4.2. **The Brownian diffusion model.** Suppose that $\rho \in (0, \infty]$, $\mu = 1_S$, and ζ is a standard Brownian motion (one may allow it to have constant non-zero drift, but for simplicity we set the drift to 0). Since $(\zeta - \zeta')/\sqrt{2}$ is a standard Brownian motion, it is easily seen that $\mathbb{E}|\Sigma'_s|_d = \mathbb{E}|W_{2s}|_d$ where W is the usual radius-1 Wiener sausage. Therefore,

$$R(\infty) = \lambda \int_0^\infty \mathbb{E}|W_{2s}|_d \, \alpha e^{-\alpha s} \, ds = \lambda \int_0^\infty \mathbb{E}|W_s|_d \, (\alpha/2) e^{-\alpha s/2} \, ds.$$

Hence, $\alpha_{\mathrm{c}}(\lambda) = 2\overline{\alpha}_{\mathrm{dd}}(\lambda)$ where $\overline{\alpha}_{\mathrm{dd}}(\lambda)$ is the corresponding quantity $\overline{\alpha}$ of Example 3.8 for the delayed diffusion model.

This section closes with a remark on the missing 'supercritical' parts of Theorems 1.2 and 4.1. An iterative construction similar to that of Section 3.6 may be proved for the diffusion model. However, Proposition 3.10 is not easily extended or adapted.

Acknowledgements

ZL's research was supported by National Science Foundation grant 1608896 and Simons Foundation grant 638143. The work of GRG was done partly under the influence of Covid-19.

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