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On the connectedness of a random graph

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Abstract

Let $\mathbf{p} = (p(i): i \geqslant 0)$ be a sequence of numbers satisfying $0 \leqslant p(i) < 1$ for $i = 0, 1, 2, \ldots$, and let G be a random graph with vertex set $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ and with edge set defined as follows: for each pair i, j of vertices, where $i \leqslant j$, there is an edge joining i and j with probability $p(j-i)$, independently of the presence or absence of all other edges. We explore the connectedness of G, showing that G is almost surely connected if and only if $\Sigma_i p(i) = \infty$ and the (positive) greatest common divisor of the set $\{i \geqslant 1: p(i) > 0\}$ equals 1; if one of these two conditions fails to hold then G is almost surely disconnected. Corresponding results hold in higher dimensions, for random graphs defined on the vertex sets $\mathbb{Z}^d$ where $d \geqslant 2$.

1. *Introduction*

We shall consider a random graph on the vertex set $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ in which, for pairs $i, j \in \mathbb{Z}$, the events $\{i \sim j\}$, that vertices i and j are joined by an edge, are such that

(i) the collection of events $\{i \sim j\}$ for $-\infty < i \leqslant j < \infty$ is a family of independent events;

(ii) the probability of $\{i \sim j\}$ depends only on the distance $|i-j|$ between i and j.

Let $\mathbf{p} = (p(i): i \geqslant 0)$ be a sequence of numbers satisfying $p(0) = 0$ and $0 \leqslant p(i) < 1$ for $i \geqslant 1$. Let G be a random graph with vertex set $\mathbb{Z}$ and edge set defined as follows: for each pair $i, j \in \mathbb{Z}$ where $i \leqslant j$, there is an edge between i and j with probability $p(j-i)$, independently of the presence or absence of all other edges. In this paper, we ask ourselves whether or not G is a connected graph, and present a necessary and sufficient condition, in terms of the probability sequence $\mathbf{p}$, for G to be almost surely (a.s.) connected; it turns out that if $\Sigma_i p(i) < \infty$ then G is a.s. disconnected, whilst if $\Sigma_i p(i) = \infty$ and the (positive) greatest common divisor of the set $\{i \geqslant 1: p(i) > 0\}$ equals 1, then G is a.s. connected. It is not surprising that we make some use of the Borel–Cantelli lemmas; indeed our main results (Theorems 1 and 2) may be seen as extensions of these lemmas. Corresponding results hold for random graphs in higher dimensions, with vertex sets $\mathbb{Z}^d$ where $d \geqslant 2$.

There are many types of random graph. Possibly the most studied has finite vertex set $\{1, 2, \ldots, n\}$, where each pair of vertices is joined by an edge with probability p, and $p = p(n)$ is a parameter which depends on n alone. Many interesting questions about such graphs deal with asymptotic properties as $n \to \infty$ (see Grimmett[**3**] for a review). Our graphs differ in two important respects: they are infinite, and the probability that two given vertices are adjacent depends on the distance separating them (we note here that we use the word 'adjacent' in its graph-theoretic sense only). We are not the first to study such graphs and their connectedness. Holmes[**4**] has made some progress with a related problem, and we believe that Erdös has posed the question of ascertaining whether or not G is a.s. connected when $p(i) = (i+1)^{-1}$ for all $i \geqslant 1$.

Our graph-theoretic terminology is fairly standard. All the graphs in this paper are undirected without multiple edges; moreover, all our graphs a.s. contain no loops. Two vertices (i,j say) are called *adjacent* (written $i \sim j$) if they are joined by an edge; a vertex is *isolated* if it is adjacent to no other vertex. If i and j are distinct vertices of the graph G, a *path* from i to j is an alternating sequence $i = v_1, e_1, v_2, e_2, \ldots, e_n, v_{n+1} = j$ of distinct vertices and edges of G such that e_i joins v_i to v_{i+1} for $1 \leqslant i \leqslant n$. A *component* of $G = (V, E)$ is a subgraph $G' = (V', E')$ of G such that, for each distinct pair $i, j \in V'$, there exists a path in G' from i to j, and both V' and E' are maximal with respect to this property. We say that G is *connected* if its only component is G itself. If U is a subset of vertices, then $G(U)$ denotes the graph 'G restricted to U', being the subgraph of G with vertex set U and maximal edge set. See Bollobás[**1**] for more facts about graphs.

2. *The results*

Unless otherwise stated, $\mathbf{p} = (p(i): i \geqslant 0)$ is a given sequence of numbers satisfying $p(0) = 0$ and $0 \leqslant p(i) < 1$ for $i \geqslant 1$, and G is a random graph defined as in the Introduction from the sequence $\mathbf{p}$; we shall sometimes refer to $\mathbf{p}$ as the *probability sequence* of G. The following type of result is standard.

LEMMA 1. *Either G is a.s. connected or G is a.s. disconnected.*

Before stating our principal result it is convenient to introduce a classification of probability sequences.

Definition 1. For any probability sequence $\mathbf{p}$, we define $\pi(\mathbf{p})$ to be the (positive) greatest common divisor of the set $\{i: p(i) > 0\}$. If $\pi(\mathbf{p}) = 1$ then $\mathbf{p}$ is called *aperiodic*.

THEOREM 1. *G is a.s. connected if and only if* $\mathbf{p}$ *satisfies the two following conditions*:

(i) $\sum_i p(i) = \infty$,

(ii) $\mathbf{p}$ *is aperiodic.*

Furthermore, if (i) *does not hold then G a.s. contains infinitely many isolated vertices, whilst if* (i) *holds but* (ii) *does not hold then G a.s. has exactly* $\pi(\mathbf{p})$ *components with vertex sets* $Z_i = \{i + n\pi(\mathbf{p}): -\infty < n < \infty\}$ *as i ranges over* $\{0, 1, \ldots, \pi(\mathbf{p}) - 1\}$.

Condition (i) is intuitively attractive, since the degree $\rho(j)$ of each vertex j in G satisfies

$$P(\rho(j) = \infty) = \begin{cases} 0 & \text{if} \quad \sum_i p(i) < \infty \\ 1 & \text{if} \quad \sum_i p(i) = \infty. \end{cases}$$

Condition (ii) is natural also. It is quite easy to see that there is positive probability that vertices i and j are joined by a path in G if and only if $|i-j|$ is a multiple of $\pi(\mathbf{p})$; thus, if $\pi(\mathbf{p}) > 1$ then (for example) 0 and 1 are a.s. in different components of G.

The following theorem is of crucial importance in the proof of Theorem 1, and we state it separately.

THEOREM 2. *Let C be the vertex set of the component of G containing* 0. *If $\Sigma_i p(i) = \infty$ then*

$$\sum_{i \in C} p(|i|) = \infty \quad a.s.$$

For simplicity of exposition we have not allowed any of the probabilities $p(i)$ to equal 1. However, it is fairly easy to see that in the case when $p(i) = 1$ for some $i \geqslant 1$, then G is a.s. connected if and only if $\pi(\mathbf{p}) = 1$.

A result similar to Theorem 1 holds in higher dimensions. Let d be a positive integer and let $(p(\mathbf{i}): \mathbf{i} \in \mathbb{Z}^d)$ be a family of numbers satisfying

$$0 \leqslant p(\mathbf{i}) < 1, p(\mathbf{i}) = p(-\mathbf{i}) \quad \text{for all} \quad \mathbf{i} \in \mathbb{Z}^d,$$

and $p(\mathbf{0}) = 0$, where $\mathbf{0} = (0, 0, \ldots, 0)$. Let $G(d)$ be a random graph with vertex set $\mathbb{Z}^d$ and edge set defined as follows: for each unordered pair $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ there is an edge between $\mathbf{i}$ and $\mathbf{j}$ with probability $p(\mathbf{j} - \mathbf{i})$, independently of the presence or absence of all other edges. Let $J = \{\mathbf{i}: p(\mathbf{i}) > 0\}$ and write

$$K = \left\{ \sum_{k=1}^{m} \lambda_k \mathbf{n}_k : \mathbf{n}_k \in J, \lambda_k \in \mathbb{Z} \quad \text{for} \quad 1 \leqslant k \leqslant m, \quad \text{and} \quad m \geqslant 1 \right\}$$

for the abelian group of linear combinations of vectors in J. Note that the collection $\mathbb{Z}^d/K$ of cosets of K in $\mathbb{Z}^d$ is finite if and only if J contains d linearly independent vectors.

THEOREM 3. *$G(d)$ is a.s. connected if and only if the following two conditions hold*:

(i) $\sum_{\mathbf{i} \in \mathbb{Z}^d} p(\mathbf{i}) = \infty$,

(ii) $K = \mathbb{Z}^d$.

Furthermore, if (i) *fails to hold then $G(d)$ a.s. contains infinitely many isolated vertices, whilst if* (i) *holds but* (ii) *does not hold then $G(d)$ a.s. contains $|\mathbb{Z}^d/K|$ components whose vertex sets are the cosets of K in $\mathbb{Z}^d$.*

Holmes [4] was interested in a problem which is related to ours. His results imply that the graph G and probability sequence $\mathbf{p}$ satisfy

(*a*) if $\Sigma_i p(i) < \infty$ then G a.s. contains infinitely many isolated vertices,

(*b*) if $p(i) = \alpha/i$ and $\alpha > 1$ then G is a.s. connected.

Our proof of (*a*) differs from his. It is not too difficult to prove that (*b*) holds; the main difficulty in proving Theorem 1 is to prove that, subject to aperiodicity, the condition of (*b*) may be replaced by the condition that $\Sigma_i p(i) = \infty$.

The rest of this paper contains the proofs of the results above. Theorem 2 is proved in the next section, and Lemma 1 and Theorems 1 and 3 are proved in the final section.

3. *Proof of Theorem* 2

3·1. *The binomial distribution*

The proof of Theorem 2 proceeds by a series of lemmas, two of which are general results about the binomial distribution. The first of these is an estimate of the tail of the distribution; such estimates are standard but their form is of some importance to us.

LEMMA 2. *Suppose that Y is binomially distributed with parameters n and p.*

(*a*) *If $0 < \alpha \leqslant \frac{1}{2}$ and $0 < p < \frac{1}{100}$ then*

$$P(Y \geqslant p^{\alpha} n) \leqslant 6 \exp(-\tfrac{1}{2} p^{\alpha} n).$$

(*b*) *If* $p = \frac{1}{2}$ *then*

$$P(Y \geqslant \tfrac{3}{4}n) \leqslant 4\exp(-\tfrac{1}{32}n).$$

Proof. We prove (*a*) only; the proof of (*b*) is similar (see Marstrand [5], p. 4). Writing $a(j) = \binom{n}{j} p^j (1-p)^{n-j}$, we have that

$$\frac{a(j+1)}{a(j)} = \frac{n-j}{j+1}\frac{p}{1-p} < \frac{np}{j(1-p)},$$

and so, for all $j > \frac{1}{2}p^{\alpha}n$, we have that

$$\frac{a(j+1)}{a(j)} < \frac{2p^{1-\alpha}}{1-p} < \frac{1}{3}.$$

We write $[x]$ for the integer part of x, and set $\beta = [\frac{1}{2}p^{\alpha}n]$, $\gamma = [p^{\alpha}n]$, to obtain

$$P(Y \geqslant p^{\alpha}n) < \sum_{j=\gamma}^{n} a(j) < \sum_{j=0}^{\infty} a(\gamma)\, 3^{-j} < 2a(\gamma).$$

Moreover

$$a(\gamma) < a(\beta)\,(\tfrac{1}{3})^{\gamma-\beta} < 3\exp(-\tfrac{1}{2}p^{\alpha}n)$$

as required, since $a(\beta) < 1$ and $\gamma - \beta > \frac{1}{2}p^{\alpha}n - 1$. ▌

LEMMA 3. *Let* $X_1, X_2, \ldots$ *be a sequence of random variables, each taking the values* 0 *or* 1, *and suppose that*

$$P(X_{i+1} = 1 \mid X_j = x_j \quad \textit{for} \quad 1 \leqslant j \leqslant i) \leqslant p \quad \textit{for all} \quad \mathbf{x} = (x_1, \ldots, x_i) \quad \textit{and} \quad i \geqslant 0,$$

where p *is a constant satisfying* $0 < p \leqslant 1$. *Then, for all* n,

$$P(X_1 + \ldots + X_n \geqslant m) \leqslant P(Y \geqslant m) \quad \textit{for} \quad 0 \leqslant m \leqslant n$$

where Y *is binomially distributed with parameters* n *and* p.

Proof. Let $A_1, A_2, \ldots$ be independent Bernoulli variables with

$$P(A_i = 1) = 1 - P(A_i = 0) = p \quad \text{for all} \quad i.$$

Let $B_1, B_2, \ldots, C_1, C_2, \ldots$ be random variables, each taking the values 0 or 1, whose distributions are defined recursively as follows:

(3·1) the distribution of B_i depends only on $\{A_j, B_j : j < i\}$,

(3·2) $P(B_1 = 1) = p^{-1}P(X_1 = 1)$,

(3·3) for $i \geqslant 1$ we have that

$$\begin{aligned} &P(B_{i+1} = 1 \mid A_j = a_j, B_j = b_j \quad \text{for} \quad 1 \leqslant j \leqslant i) \\ &\qquad = p^{-1}P(X_{i+1} = 1 \mid X_j = a_j b_j \quad \text{for} \quad 1 \leqslant j \leqslant i), \end{aligned}$$

(3·4) $C_i = A_i B_i$ for $i \geqslant 1$.

We claim that $(C_1, \ldots, C_n)$ and $(X_1, \ldots, X_n)$ have the same joint distribution for all n, and prove this by induction. The claim holds for $n = 1$ by (3·2) and (3·4). Suppose it holds for $n = i$. Then

$$\begin{aligned} &P(C_{i+1} = 1, C_j = c_j \quad \text{for} \quad 1 \leqslant j \leqslant i) \\ &\quad = P(C_{i+1} = 1 \mid C_j = c_j \quad \text{for} \quad 1 \leqslant j \leqslant i)\, P(C_j = c_j \quad \text{for} \quad 1 \leqslant j \leqslant i) \\ &\quad = P(X_{i+1} = 1 \mid X_j = c_j \quad \text{for} \quad 1 \leqslant j \leqslant i)\, P(X_j = c_j \quad \text{for} \quad 1 \leqslant j \leqslant i) \\ &\quad = P(X_{i+1} = 1, X_j = c_j \quad \text{for} \quad 1 \leqslant j \leqslant i) \end{aligned}$$

by (3·3), (3·4), and the induction hypothesis, and the claim has been justified. The lemma now follows since $C_i \leqslant A_i$ for all i and therefore, for all n,

$$\sum_{i=1}^{n} C_i \leqslant \sum_{i=1}^{n} A_i,$$

where the latter sum has the required binomial distribution. ❙

3·2. *Trees and structures*

The idea of the proof of Theorem 2 is to show that, with probability close to 1, for all large M there exists a positive integer $N = N(M)$ such that 'nearly all' the vertices $j \in I = \{-N, -N+1, \ldots, N\}$ are such that

$$\sum_{i \in C(j, I)} p(|i-j|) \geqslant M, \tag{3·5}$$

where $C(j, I)$ is the vertex set of the component of $G(I)$ containing j. By considering the vertex 0, it will follow that the probability that

$$\sum_{i \in C} p(|i|) \geqslant M$$

is near 1, and the result will then follow by letting $M \to \infty$. An intuitive reason why (3·5) often holds is the following. Suppose that M is fixed and that $j \in I$ and $A \subseteq I$ are such that

$$\sum_{i \in A} p(|i-j|) < M;$$

then, by a previous choice of N large enough,

$$\sum_{i \in I \backslash A} p(|i-j|)$$

may be made as large as we like, implying that it is very likely that j has a neighbour in $I \backslash A$.

The following lemma is fundamental. Suppose $j \in \mathbb{Z}$, $E \subseteq \mathbb{Z}$ and $\nu(j, E)$ is the number of vertices in E which are adjacent to j.

Lemma 4. *If $j \in \mathbb{Z}$, $E \subseteq \mathbb{Z}$ and M is a positive integer such that*

$$\sum_{i \in E} p(|i-j|) > 8M$$

then $P(\nu(j, E) < M) < 4\exp(-\tfrac{1}{8}M)$.

Proof. Without loss of generality we may assume that $j = 0$. We can partition E into a disjoint union

$$E = E_1 \cup E_2 \cup \ldots \cup E_{4M}$$

such that

$$\sum_{\in E_k} p(|i|) > 1 \quad \text{for} \quad 1 \leqslant k \leqslant 4M.$$

For each k,

$$\begin{aligned} P(\nu(0, E_k) = 0) &= \prod_{i \in E_k} (1 - p(|i|)) \\ &\leqslant \exp\Big(-\sum_{i \in E_k} p(|i|)\Big) \\ &< e^{-1} < \tfrac{1}{2}. \end{aligned}$$

Thus, by Lemma 3,

$$P(\nu(0,E) < M) \leqslant P(\nu(0,E_k) = 0 \quad \text{for at least } 3M \text{ values of } k) \leqslant P(Y \geqslant 3M)$$

where Y is binomially distributed with parameters $4M$ and $\frac{1}{2}$. The result now follows from Lemma 2 (*b*). ∎

Henceforth we assume that

$$\sum_i p(i) = \infty \tag{3·6}$$

and let G be the associated random graph. Let m be a positive integer (later we shall take the limit as $m \to \infty$), $M = m^2$, let N be a positive integer such that

$$\sum_{i=1}^{N} p(i) > 9M, \tag{3·7}$$

and write $I = \{-N, -N+1, \ldots, N\}$. Until further notice we confine our attention to the graph $G(I)$, that is G restricted to I. For each $j \in I$ we shall construct a spanning tree of the component of $G(I)$ containing j, and shall call this spanning tree the *structure* at j. Each structure will be made up of smaller trees, and we begin by describing how to construct a new 'generation' of a tree, given the previous history. Let $J = \{j(1), j(2), \ldots, j(q)\} \subseteq A \subseteq I$. Think of A as the set of vertices already dealt with, and J as the current generation. We define disjoint sets $A(s) \subseteq I$, $s = 0, 1, \ldots, q$, inductively as follows. $A(0) = \phi$. Having defined $A(0), A(1), \ldots, A(s-1)$, we write

$$A^*(s-1) = A \cup \left(\bigcup_{u=0}^{s-1} A(u)\right)$$

and consider $\nu(j(s), I \backslash A^*(s-1)) = \nu$, say. We distinguish between three cases, and shall speak of the vertex $j(s)$ as being 'treated' under the case which applies.

Case 1. $\nu \geqslant M$. We then define $A(s)$ to be a set of M vertices in $I \backslash A^*(s-1)$, each of which is adjacent to $j(s)$; such a set exists by the definition of ν, and we take $A(s)$ to be the set of such vertices which are least in the lexicographic ordering of I.

Case 2. $\nu < M$ *and* $\sum_{i \in A^*(s-1)} p(|i - j(s)|) \geqslant M$. We then define $A(s) = \phi$.

Case 3. $\nu < M$ *and* $\sum_{i \in A^*(s-1)} p(|i - j(s)|) < M$. Again we define $A(s) = \phi$, but we note the separate case.

Thus the 'generation' J gives birth to the 'new generation' J^* given by

$$J^* = \bigcup_{u=1}^{q} A(u), \tag{3·8}$$

and we assume that J^* is given in increasing order. We call $A(u)$ the *family* of $j(u)$.

LEMMA 5. *Let* $J = \{j(1), \ldots, j(q)\} \subseteq A \subseteq I$ *and let* J^* *be given by the above procedure. Let* $\kappa(3)$ *be the number of values of* s (*with* $1 \leqslant s \leqslant q$) *such that* $j(s)$ *is treated under Case* 3. *Then, if* $m > 10$,

$$P(\kappa(3) \geqslant m^{-1}q) \leqslant 6 \exp(-\tfrac{1}{2}m^{-1}q).$$

Proof. Let X_s be the indicator function of the event that $j(s)$ is treated under Case 3.

If the second inequality in the definition of Case 3 holds at the stage of treating $j(s+1)$, then

$$\sum_{i \in I \backslash A^*(s)} p(|i-j(s+1)|) > 8M,$$

by (3·7), and Lemma 4 implies that

$$P(X_{s+1} = 1 \mid X_t = x_t \quad \text{for} \quad 1 \leqslant t \leqslant s) < 4\exp(-\tfrac{1}{8}M) < M^{-1}$$

for all $(x_1, \ldots, x_s)$ and $0 \leqslant s < q$. Now apply Lemma 3 to find that

$$P(X_1 + \ldots + X_q = \kappa(3) \geqslant m^{-1}q) \leqslant P(Y \geqslant m^{-1}q)$$

where Y is binomially distributed with parameters q and M^{-1}. The result follows by an application of Lemma 2 with $\alpha = \frac{1}{2}$, $p = M^{-1}$ and $n = q$. ▎

Next we put our generations together to form a tree. Let $j_0 \in A_0 \subseteq I$. We define a sequence of disjoint sets ('generations') $J_0, J_1, \ldots, J_W \subseteq I$ inductively until the process terminates. First, $J_0 = \{j_0\}$. If $\nu(j_0, I\backslash A_0) \geqslant M = m^2$, then we denote by J_1 the first m^2 vertices of $I \backslash A_0$ (in the lexicographic ordering) which are adjacent to j_0; if $\nu(j_0, I\backslash A_0) < m^2$ then the process terminates with $W = 0$. The subsequent stages in the construction are slightly different. Suppose that we have constructed $J_t = \{j_t(1), j_t(2), \ldots, j_t(q_t)\}$ where $q_t = m^{t+1}$, for $1 \leqslant t \leqslant v$. We then construct a new generation with $A = A_0 \cup (\bigcup_{t=0}^{v} J_t)$, $q = q_v$, $J = J_v$, forming sets $A(s) = A_{v+1}(s)$, say, and $A^*(s) = A^*_{v+1}(s)$, say, for $0 \leqslant s \leqslant q_v$. If there exist at least $m^{-1}q_v = m^v$ values of s (such that $1 \leqslant s \leqslant q_v$) under Case 1, that is with

$$\nu(j_v(s), I\backslash A^*_{v+1}(s-1)) \geqslant M = m^2, \tag{3·9}$$

then, denoting the first m^v of these by Q, we define

$$J_{v+1} = \bigcup_{s \in Q} A_{v+1}(s),$$

this being a subset of the new generation J^* defined in the manner of (3·8). Note that $|Q| = m^v$ and, for each $s \in Q$, $|A_{v+1}(s)| = m^2$. Hence

$$|J_{v+1}| = m^{v+2} = q_{v+1},$$

and we have defined $J_{v+1} = \{j_{v+1}(1), \ldots, j_{v+1}(q_{v+1})\}$, assumed arranged in increasing order.

If, however, (3·9) holds for fewer than m^v values of s, then the process terminates with $W = v$ and $J_W = J_v$. The resulting set

$$T(j_0, A_0) = \bigcup_{v=0}^{W} J_v \tag{3·10}$$

is called a *tree*. The vertex j_0 is called the *root* of $T(j_0, A_0)$.

LEMMA 6. *Consider* $T(j_0, A_0)$, *given by* (3·10), *and let* w *be a positive integer. Let* $\kappa(2)$ *be the number of vertices in* J_W *which, when applying the new generation procedure in the attempt to form the next generation, are treated under Case* 2. *Then, if* $m > 10$,

$$P(W = w, \kappa(2) \leqslant m^{w+1} - 2m^w) \leqslant 6\exp(-\tfrac{1}{2}m^w).$$

Proof. If the tree terminates after the construction of J_w then there are fewer than

m^w vertices of the last generation J_w under Case 1, leaving at least $m^{w+1}-m^w$ vertices under Cases 2 or 3. Thus

$$\begin{aligned} P(W = w, \kappa(2) \leqslant m^{w+1}-2m^w) &\leqslant P(W = w, \kappa(3) \geqslant m^w) \\ &\leqslant P(\kappa(3) \geqslant m^w \mid W \geqslant w), \end{aligned}$$

where $\kappa(3)$ is the number of vertices of J_w which are treated under Case 3. Now apply Lemma 5 to find that

$$P(\kappa(3) \geqslant m^w \mid W \geqslant w) \leqslant 6\exp(-\tfrac{1}{2}m^w). \,|$$

We need a lemma to deal with the special case $w = 0$.

LEMMA 7. *Consider* $T(j_0, A_0)$, *given by* (3·10). *If*

$$\sum_{i\in A_0} p(|i-j_0|) < M, \tag{3·11}$$

then the probability that the tree terminates immediately at j_0 *satisfies*

$$P(W = 0) < 4\exp(-\tfrac{1}{8}m^2).$$

Proof. If (3·11) holds then, by (3·7),

$$\sum_{i\in I\setminus A_0} p(|i-j_0|) > 8M$$

and Lemma 4 gives the result. |

We are now ready to put such trees together to form *structures*. Let $j \in I$ and define an infinite sequence $T_0, T_1, \ldots$ of (possibly empty) trees as follows. $T_0 = T(j_0, \{j_0\})$. Suppose that $T_0, T_1, \ldots, T_y \subseteq I$ have been constructed. We set $T_{y+1} = \phi$ if either $T_y = \phi$ or there is no pair k, l of adjacent vertices of G such that

$$k \in S_y = T_0 \cup T_1 \cup \ldots \cup T_y, \, l \in I\setminus S_y.$$

If $T_y \neq \phi$ and there exists such a pair k, l, then let j be the least vertex in $I\setminus S_y$ which is adjacent to some $k \in S_y$ and define $T_{y+1} = T(j, \{j\} \cup S_y)$. We write $Z = \min\{y: T_{y+1} = \phi\}$ and denote by $S(j)$ the rooted spanning tree of $C(j, I)$, having root j, vertex set

$$\bigcup_{r=0}^{\infty} T_r = \bigcup_{r=0}^{Z} T_r,$$

and edge set containing exactly those edges (and no others) of $G(I)$ which were utilized in the construction of the sets $T_0, T_1, \ldots, T_Z$. We call $S(j)$ the *structure* of $G(I)$ at j. We shall regard $S(j)$ as either a spanning tree or a set of vertices, depending on the context, and will make use of the representation

$$S(j) = \bigcup_{r=0}^{Z} T_r \tag{3·12}$$

when thinking of $S(j)$ as a set. Care is needed here, since there will generally be pairs $x, y \in I$ for which $S(x)$ and $S(y)$ are the same sets of vertices but are different structures.

We make two remarks concerning independence. First, at every stage in the construction of $S(j)$ we examine *new* pairs of vertices and ask whether or not they are connected; at no stage do we consider a pair which we have considered before. Secondly, for all $x, y, j \in I$,

$$P(x \sim y \mid S(j), x, y \notin S(j)) = P(x \sim y) = p(|x-y|),$$

and, conditional on $S(j)$, the events $\{x \sim y\}$ for $x, y \notin S(j)$ are independent. In fact, a type of Markov property holds in that, however far one has gone with the construction of some structure (in the manner of this section), the rules governing future construction remain unchanged (given the current state).

3·3. *Density points*

We recall that each vertex $k \in S(j)$ either belongs to a unique family A of a unique generation of a unique tree T of $S(j)$, or is the root of a unique tree of $S(j)$. We denote by $S_k(j)$ the set of vertices of $S(j)$ which were constructed prior to the construction of the family A (or prior to the construction of k itself, if k is the root of a tree).

Definition 2. Let $j \in I$ and consider $S(j)$. We call a vertex $k \in S(j)$ a *density point* of $S(j)$ if

$$\sum_{i \in S_k(j)} p(|i-k|) \geqslant M. \tag{3·13}$$

We define the property of being a density point in terms of the substructure $S_k(j)$ in order to preserve the Markov property which was formulated loosely at the end of the last section.

We aim to show that, with probability close to 1, 'nearly all' vertices in I are density points of some structure. We shall require a classification of trees also.

Definition 3. Fix $j \in I$ and consider $S(j) = \bigcup_{r=0}^{Z} T_r$ given by (3·12). For $0 \leqslant r \leqslant Z$ we have from (3·10) that

$$T_r = \bigcup_{r=0}^{W} J_v;$$

we call W the *height* of T_r. If $W = 0$ then we call T_r *thin* if the single vertex of T_r is not a density point of $S(j)$. If $W > 0$ then we call T_r *thin* if it contains not more than $m^{W+1} - 2m^W$ density points of $S(j)$. A tree which is not thin is called *thick*. All empty trees T_r, where $r > Z$, are defined to be *thick*.

LEMMA 8. *Fix $j \in I$ and consider $S(j)$, given by* (3·12). *Let w and r be non-negative integers. If $m > 10$ then*

$$P(T_r \textit{ is thin and has height } w) \leqslant 6\exp\left(-\tfrac{1}{8}m^{(w+1)/2}\right).$$

Proof. We may suppose that $T_r \neq \phi$, since if $T_r = \phi$ then it is trivially thick. We write W for the height of T_r and k for the root of T_r. If $w = 0$ then

$$\begin{aligned} P(T_r \text{ is thin, } W = 0 \mid T_r \neq \phi, S_k(j)) &\leqslant P(W = 0 \mid T_r \neq \phi, S_k(j), \\ &\qquad k \text{ is not a density point of } S(j)) \\ &< 4\exp\left(-\tfrac{1}{8}m^2\right) \\ &< 6\exp\left(-\tfrac{1}{8}m^{\frac{1}{2}}\right) \end{aligned} \tag{3·14}$$

by Lemma 7. Next suppose that $w > 0$.

$$\begin{aligned} P(T_r \text{ is thin, } W = w \mid T_r \neq \phi, S_k(j)) &\leqslant P(W = w, \kappa(2) \leqslant m^{w+1} - 2m^w \mid \\ &\qquad T_r \neq \phi, S_k(j)) \\ &\leqslant 6\exp\left(-\tfrac{1}{2}m^w\right) \\ &< 6\exp\left(-\tfrac{1}{8}m^{(w+1)/2}\right) \end{aligned} \tag{3·15}$$

by Lemma 6, where $\kappa(2)$ is the number of vertices in the final generation J_w of T_r which are treated under Case 2. The result follows immediately. ∎

Next we estimate the number of thin trees in the structure $S(j)$, and use this estimate to show that, with probability close to 1, 'nearly all' vertices in $S(j)$ are density points of $S(j)$.

Lemma 9. *Fix $j \in I$ and consider $S(j)$, given by* (3·12). *Let w and q be non-negative integers, and let $\theta(w,q)$ be the number of trees T_r such that $0 \leqslant r < q$, and such that T_r is thin and has height w. Then, if $m > 10^4$*

$$P(\theta(w,q) > q\delta(w)^{\frac{1}{4}}) \leqslant \min\,(q\delta(w), \delta(w,q))$$

where $\delta(w) = 6\exp\,(-\tfrac{1}{8}m^{(w+1)/2})$ and $\delta(w,q) = 6\exp\,(-\tfrac{1}{2}q\delta(w)^{\frac{1}{4}})$.

Proof. First note that

$$\begin{aligned} P(\theta(w,q) > 0) &= P\left(\bigcup_{r=0}^{q-1} \{T_r \text{ is thin with height } w\}\right) \\ &\leqslant \sum_{r=0}^{q-1} P(T_r \text{ is thin with height } w) \\ &\leqslant q\delta(w) \end{aligned}$$

by Lemma 8. To see the other part, let X_i be the indicator function of the event that T_i is thin with height w. From equations (3·14) and (3·15), we have that

$$P(X_i = 1 \mid X_k = x_k \text{ for } 0 \leqslant k \leqslant i-1) < \delta(w) \text{ for all } (x_0, \ldots, x_{i-1}) \text{ and } i \geqslant 0.$$

We apply Lemma 3 to deduce that

$$P(\theta(w,q) > q\delta(w)^{\frac{1}{4}}) \leqslant P(Y > q\delta(w)^{\frac{1}{4}})$$

where Y is binomially distributed with parameters q and $\delta(w)$; the result now follows by an application of Lemma 2(*a*) with $\alpha = \frac{1}{4}$, $n = q$, $p = \delta(w)$, noting that $\delta(w) < 10^{-2}$ for all $w \geqslant 0$ if $m > 10^4$. ∎

Definition 4. For $0 < \eta < 1$, we call the structure $S(j)$ *η-dense* if it contains at least $\eta|S(j)|$ density points.

Lemma 10. *Let $\epsilon > 0$. For all sufficiently large $m = m(\epsilon)$ and for all $j \in I$*

$$P(S(j) \textit{ is not } (1-\epsilon)\textit{-dense}) < \epsilon.$$

Proof. In the notation of Lemma 9, for all large m we have that

$$\begin{aligned} &P(\exists\, q, w \geqslant 0 \text{ such that } \theta(w,q) > q\delta(w)^{\frac{1}{4}}) \\ &\qquad \leqslant \sum_{w=0}^{\infty} \Big(\sum_{q<Q} q\delta(w) + \sum_{q\geqslant Q} \delta(w,q)\Big) \text{ where } Q = \delta(w)^{-\frac{1}{3}} \\ &\qquad \leqslant \sum_{w=0}^{\infty} \left(Q^2\delta(w) + \frac{\delta(w,Q)}{(1-\exp\,(-\frac{1}{2}\delta(w)^{\frac{1}{4}}))}\right) \\ &\qquad \leqslant \sum_{w=0}^{\infty} (\delta(w)^{\frac{1}{3}} + 24\delta(w)^{-\frac{1}{4}}\exp\,(-\tfrac{1}{2}\delta(w)^{-\frac{1}{12}})) \quad \text{since } 1-e^{-x} \geqslant \tfrac{1}{2}x \text{ for small } x \\ &\qquad \leqslant 400 \sum_{w=0}^{\infty} \delta(w)^{\frac{1}{12}} \quad \text{since } e^{-y} \leqslant y^{-4} \text{ for large } y \\ &\qquad < \epsilon. \end{aligned}$$

Thus, the event F that $\theta(w,q) \leqslant q\delta(w)^{\frac{1}{4}}$ for all $w, q \geqslant 0$ has probability satisfying $P(F) > 1-\epsilon$. Let $Y = \min\{r\colon T_r = \phi\}$; on F we have that

$$\theta(w, Y) \leqslant Y\delta(w)^{\frac{1}{4}} \quad \text{for all} \quad w \geqslant 0. \tag{3·16}$$

We wish to estimate the number of non-density points in $S(j)$, on F. Let U be the set of vertices in $S(j)$ which are not density points of $S(j)$; we may partition U into the disjoint union $U = U_1 \cup U_2$ where U_1 is the set of vertices of $S(j)$ which are not density points of $S(j)$ and which lie in some thin tree of $S(j)$, and U_2 is the remainder of U. On F, we have from (3·16) that

$$\begin{aligned} |U_1| &\leqslant \sum_{\substack{\text{thin trees}\\ T}} |T| \\ &\leqslant Y\left(\delta(0)^{\frac{1}{4}} + 2\sum_{w=1}^{\infty} m^{w+1}\delta(w)^{\frac{1}{4}}\right) \\ &\leqslant Y\left(4\sum_{w=0}^{\infty} m^{w+1}\exp\left(-\tfrac{1}{32}m^{(w+1)/2}\right)\right) \\ &\leqslant \tfrac{1}{2}\epsilon|S(j)| \end{aligned} \tag{3·17}$$

for all large m, since trees with height w have $1+m^2+m^3+\ldots+m^{w+1} < 2m^{w+1}$ vertices, and $Y \leqslant |S(j)|$. Finally, let T be a non-empty thick tree of $S(j)$ and let W be its height. By definition, T contains fewer than $2m^W$ vertices of $S(j)$ which are not density points of $S(j)$. For all large m and all possible values of W

$$2m^W \leqslant \tfrac{1}{2}\epsilon m^{W+1} < \tfrac{1}{2}\epsilon|T|$$

and thus

$$|U_2| < \tfrac{1}{2}\epsilon \sum_{\substack{\text{thick trees}\\ T}} |T| \leqslant \tfrac{1}{2}\epsilon|S(j)|. \tag{3·18}$$

We conclude from (3·17) and (3·18) that, on F, the number of non-density points satisfies

$$|U| < \epsilon|S(j)|,$$

giving that $S(j)$ is $(1-\epsilon)$-dense, and the lemma is proved. ▌

So far we have considered the structure $S(j)$ associated with a single progenitor $j \in I$, and shown that, with large probability, $S(j)$ is $(1-\epsilon)$-dense. Next we show that, with large probability, the same holds simultaneously for 'nearly all' $j \in I$. Remember that $|I| = 2N+1$.

LEMMA 11. *Let $\epsilon > 0$. For all sufficiently large $m = m(\epsilon)$, the number Y of vertices $j \in I$ such that $S(j)$ is not $(1-\epsilon)$-dense satisfies*

$$P(Y > (2N+1)\epsilon^{\frac{1}{2}}) < \epsilon^{\frac{1}{2}}.$$

Proof. For $j \in I$, let X_j be the indicator function of the event that $S(j)$ is not $(1-\epsilon)$-dense. Then

$$\begin{aligned} P(Y > (2N+1)\epsilon^{\frac{1}{2}}) &\leqslant ((2N+1)\epsilon^{\frac{1}{2}})^{-1}E(Y) \\ &= ((2N+1)\epsilon^{\frac{1}{2}})^{-1}\sum_{j=-N}^{N} E(X_j) \\ &< \epsilon^{\frac{1}{2}} \end{aligned}$$

by Lemma 10. ▌

Definition 5. The vertex $j \in \mathbb{Z}$ is called an *M-density point* of G if

$$\sum_{i \in C(j)} p(|i-j|) \geqslant M$$

where $C(j)$ is the component of G which contains j.

LEMMA 12. *If* $\epsilon > 0$ *then for all sufficiently large* $M = M(\epsilon)$

$$P(0 \text{ is an } M\text{-density point of } G) > 1-3\epsilon^{\frac{1}{2}}.$$

Proof. Clearly, for each $j \in I$, every density point of $S(j)$ is an M-density point of G. From Lemma 11, with probability at least $1-\epsilon^{\frac{1}{2}}$ it is the case that the set Δ of vertices $j \in I$ such that $S(j)$ is $(1-\epsilon)$-dense satisfies $|\Delta| > (2N+1)(1-\epsilon^{\frac{1}{2}})$. But for each $j \in \Delta$, $S(j)$ contains at least $(1-\epsilon)|S(j)|$ M-density points of G, and the sets $\{S(j): j \in \Delta\}$ form a partition of $\bigcup_{j \in \Delta} C(j, I) = D$, say, where $\Delta \subseteq D$. Thus D contains at least $(1-\epsilon)|D| \geqslant (1-\epsilon)|\Delta|$ M-density points of G, giving that the number X of M-density points of G in I satisfies

$$X \geqslant (1-\epsilon)|\Delta| > (1-\epsilon)(1-\epsilon^{\frac{1}{2}})(2N+1) > (1-2\epsilon^{\frac{1}{2}})(2N+1)$$

with probability at least $1-\epsilon^{\frac{1}{2}}$.

Finally, by the translation invariance of the probability measure, $P(j$ is an M-density point of $G)$ is constant for all $j \in \mathbb{Z}$, and hence

$$\begin{aligned}(2N+1)P(0 \text{ is an } M\text{-density point}) &= \sum_{j=-N}^{N} P(j \text{ is an } M\text{-density point}) \\ &= E(X) \\ &\geqslant (2N+1)(1-2\epsilon^{\frac{1}{2}})P(X > (2N+1)(1-2\epsilon^{\frac{1}{2}})) \\ &> (2N+1)(1-2\epsilon^{\frac{1}{2}})(1-\epsilon^{\frac{1}{2}}) \\ &> (2N+1)(1-3\epsilon^{\frac{1}{2}})\end{aligned}$$

and the lemma is proved. ∎

We are now ready to prove Theorem 2.

Proof of Theorem 2. In the conclusion of Lemma 12, let $M \to \infty$ and $\epsilon \to 0$ in that order to obtain that

$$\sum_{i \in C(0)} p(|i|) = \infty \quad \text{a.s.}$$

as required. ∎

4. *The remaining proofs*

We write E^c for the *complement* of an event E, and $1(E)$ for the *indicator function* of E.

Proof of Lemma 1. Lemma 1 is actually a consequence of Theorem 1, but we include another demonstration which is based upon the general theory of stationary random processes. Let $X_i = \{1(i \sim j): j > i\}$. The sequence $\mathbf{X} = \{\dots, X_{-1}, X_0, X_1, \dots\}$ is a sequence of independent identically distributed families of random variables. The event that G is connected is defined in terms of $\mathbf{X}$ and is invariant under the shift $X_s \to X_{s+1}$. Following Doob ([**2**], pp. 458–60), the event that G is connected is contained in the trivial tail σ-field of an independent sequence, and thus has probability either 0 or 1. ∎

Proof of Theorem 1. First we prove that G is a.s. connected if (i) and (ii) hold. Suppose then that $\Sigma_i p(i) = \infty$ and $\mathbf{p}$ is aperiodic, and let $E(i,j)$ denote the event that i and j are in the same component of G. We shall show that

$$P(E(0,1)) = 1. \tag{4·1}$$

To see that this is sufficient for G to be a.s. connected, note that (4·1) implies, by translation invariance, that

$$P(E(i,i+1)) = P(E(0,1)) = 1 \quad \text{for all} \quad i,$$

giving that

$$P\left(\bigcap_{i=-\infty}^{\infty} E(i,i+1)\right) = 1.$$

To prove (4·1) we proceed as follows. By the aperiodicity of $\mathbf{p}$, there exist non-zero integers $\lambda_1, \lambda_2, \ldots, \lambda_m$ and distinct positive integers $n_1, n_2, \ldots, n_m$ such that

$$\sum_{k=1}^{m} \lambda_k n_k = 1 \tag{4·2}$$

and

$$p(n_k) > 0 \quad \text{for} \quad 1 \leqslant k \leqslant m. \tag{4·3}$$

We write $l_k = \operatorname{sign}(\lambda_k)$, and use the λ's and n's to construct a sequence $S(0) = (r_0, r_1, \ldots, r_s)$ of vertices as follows:

(*a*) $r_0 = 0$ is the first vertex of $S(0)$,

(*b*) the next $|\lambda_1|$ vertices of $S(0)$ are

$$r_1 = l_1 n_1, \quad r_2 = 2l_1 n_1, \ldots, r_{|\lambda_1|} = \lambda_1 n_1,$$

(*c*) the next $|\lambda_2|$ vertices of $S(0)$ are

$$r_{|\lambda_1|+1} = \lambda_1 n_1 + l_2 n_2, \quad r_{|\lambda_1|+2} = \lambda_1 n_1 + 2l_2 n_2, \ldots, r_{|\lambda_1|+|\lambda_2|} = \lambda_1 n_1 + \lambda_2 n_2,$$

and so on,

(*d*) $s = \sum_{k=1}^{m} |\lambda_k|$ and $r_s = \sum_{k=1}^{m} \lambda_k n_k = 1$ by (4·2).

More formally, $r_0 = 0$ and for $0 \leqslant i < s$ we have that

$$r_{i+1} = r_i + l_t n_t \quad \text{if} \quad \sum_{k=1}^{t-1} |\lambda_k| \leqslant i < \sum_{k=1}^{t} |\lambda_k|,$$

where an empty summation is interpreted as 0. We may assume that the λ's and n's were chosen in such a way that no vertex appears more than once in $S(0)$, since if some vertex, v say, appears twice then the 'loop' in $S(0)$ between these two appearances of v may be removed to obtain a shorter sequence $S(0)$ joining 0 to 1; that is, *we assume that* $r_i \neq r_j$ *if* $i \neq j$. It is an immediate consequence of this and (4·3) that the probability that $S(0)$ is the vertex set, in the correct sequence, of a path in G between 0 and 1 satisfies

$$P(S(0) \text{ is a path in } G) = p(n_1)^{|\lambda_1|} \ldots p(n_m)^{|\lambda_m|} > 0, \tag{4·4}$$

and we shall use this fact later. Another useful fact to be used later is the following. Let $\mathbf{p}' = (p'(i) : i \geqslant 0)$ be the probability sequence defined by

$$p'(i) = \begin{cases} 0 & \text{if} \quad i = n_k \quad \text{for some} \quad 1 \leqslant k \leqslant m \\ p(i) & \text{otherwise.} \end{cases} \tag{4·5}$$

Then

$$\sum_i p'(i) = \infty. \tag{4·6}$$

We shall think of the edge set of G as being constructed in three independent stages. Let $E = \{(i,j): i \leqslant j\}$, the set of possible edges of G, and partition E into three disjoint sets as follows:

$$E = E_1 \cup E_2 \cup E_3$$

where

$$E_1 = \{(i,j): i < j \text{ and } j-i = n_k \text{ for some } 1 \leqslant k \leqslant m\},$$
$$E_2 = \{(i,j): i < j, i = 1 \text{ or } j = 1\}\backslash E_1,$$
$$E_3 = E\backslash(E_1 \cup E_2).$$

Thus E_1 contains all pairs (i,j) whose separating distance equals some n_k, and E_2 is the set of all remaining pairs which use vertex 1. Let G_3 (respectively G_{23}) be the subgraph of G obtained by deleting all edges of G which belong to $E_1 \cup E_2$ (respectively E_1); clearly G_3 is a subgraph of G_{23}. Let C_3 (respectively C_{23}) be the vertex set of the component of G_3 (respectively G_{23}) which contains 0, and define the events A_1 and A_2 by

$$A_1 = \{\sum_{i \in C_3} p(|i|) < \infty\}, \tag{4·7}$$

$$A_2 = \{\sum_{i \in C_3} p(|i|) = \infty\}. \tag{4·8}$$

We shall show that

$$P(E(0,1) \cap A_i) = P(A_i) \quad \text{for} \quad i = 1, 2. \tag{4·9}$$

Equation (4·1) is an immediate consequence of (4·9) since if the latter holds then

$$P(E(0,1)) = \sum_i P(E(0,1) \cap A_i) = \sum_i P(A_i) = 1,$$

and it remains to prove (4·9). Consider A_1 first. G_{23} is a random graph with probability sequence $\mathbf{p}'$ given by (4·5), and Theorem 2 and (4·6) imply that

$$\sum_{i \in C_{23}} p(|i|) = \infty \quad \text{a.s.} \tag{4·10}$$

Compare this with the definition of A_1 to see that, on A_1, $E(0,1)$ occurs except on some null event (since if 0 and 1 are not in the same component of G_{23} then $C_3 = C_{23}$, and the infinite sum of (4·10) and the finite sum of (4·7) contradict each other); thus (4·9) holds for $i = 1$. Suppose next that A_2 occurs. On A_2, we have by definition

$$\sum_{i \in C_3} P(1 \sim (i+1)) = \sum_{i \in C_3} p(|i|) = \infty$$

and so, by the second Borel–Cantelli lemma, on almost all of A_2 (that is, on all except a null subset of A_2) there exist infinitely many pairs $(i, i+1)$ satisfying $i \in C_3$, $1 \sim (i+1)$. Therefore, on almost all of A_2 there exist infinitely many pairs (i_j, i_j+1) for $j = 1, 2, \ldots$ such that $i_j \in C_3$, $1 \sim (i_j+1)$ and

$$i_{j+1} \geqslant i_j + 2\sum_{k=1}^{m} |\lambda_k|\, n_k + 2. \tag{4·11}$$

Next we consider the edges of G lying in E_1; these edges are present or absent independently of G_3 and G_{23}. Writing $S(l) = l + S(0)$ for the sequence $(l = l+r_0, l+r_1, \ldots, l+r_s = l+1)$, we have from (4·4) that

$$P(S(l) \text{ is a path in } G) = P(S(0) \text{ is a path in } G) > 0.$$

Furthermore, $S(i_j) \cap S(i_k) = \phi$ if $j \neq k$, by (4·11). Thus, on almost all of A_2, infinitely many of the paths $\{S(i_j): j \geqslant 1\}$ occur, and therefore at least one such path, $S(i_K)$ say, occurs. Consequently, on almost all of A_2, 0 and 1 are in the same component of G, since $i_K \in C_3$, $1 \sim (i_K+1)$, and the path $S(i_K)$ exists. Thus (4·9) holds for $i = 2$ and the first half of the proof is complete.

If (i) of Theorem 1 fails then, for each $i \in \mathbb{Z}$,

$$\alpha = P(i \text{ is isolated in } G) = \prod_{j=1}^{\infty} (1-p(j))^2 > 0.$$

By the ergodic theorem, the number I_n of isolated vertices in $\{1, 2, \ldots, n\}$ satisfies

$$\frac{1}{n} I_n \to I \text{ a.s. and in } L^1, \quad \text{as} \quad n \to \infty,$$

where $E(I) = \alpha$ and $0 \leqslant I \leqslant 1$. Actually $I = \alpha$ a.s.; this is a consequence of the fact that the probability measure is ergodic, but a more elementary way to prove it is to estimate the variance of $n^{-1}I_n$. A simple calculation shows that

$$\begin{aligned}\operatorname{var}\left(\frac{1}{n} I_n\right) &= \frac{1}{n^2} \sum_{i,j=1}^{n} \operatorname{cov}(1(i \text{ isolated}), 1(j \text{ isolated}))\\ &= \frac{1}{n^2}\left(\sum_{i,j=1}^{n} P(i \text{ and } j \text{ isolated})\right) - \alpha^2\\ &= \frac{\alpha}{n} + \frac{2\alpha^2}{n^2}\sum_{k=1}^{n-1} \frac{n-k}{1-p(k)} - \alpha^2.\end{aligned}$$

Now $p(k) \to 0$ as $k \to \infty$, and hence $(1-p(k))^{-1} = 1+q(k)$ where $q(k) \geqslant 0$ and $q(k) \to 0$ as $k \to \infty$. Thus

$$\begin{aligned}\operatorname{var}\left(\frac{1}{n} I_n\right) &\leqslant \frac{\alpha}{n} + \frac{2\alpha^2}{n^2}\sum_{k=1}^{n-1} (n-k) + \frac{2\alpha^2}{n}\sum_{k=1}^{n-1} q(k) - \alpha^2\\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

But $E(n^{-1}I_n) = \alpha$ for all n, and hence

$$\frac{1}{n} I_n \to \alpha \quad \text{in} \quad L^2,$$

showing that $I = \alpha$ a.s. as was claimed. It is an immediate consequence that $I_n \to \infty$ a.s. as $n \to \infty$.

Finally, suppose that (i) of Theorem 1 holds but (ii) does not hold. Clearly a.s. no pair $x \in Z_i$, $y \in Z_j$ of vertices of G is joined by an edge of G if $i \neq j$. Furthermore, the probability sequence $\mathbf{p}'' = (p''(i) = p(i\pi(\mathbf{p})): i \geqslant 0)$ is aperiodic and

$$\sum_i p''(i) = \infty,$$

implying by the first part of Theorem 1 that the graph of G restricted to Z_i is a.s. connected for each $0 \leqslant i < \pi(\mathbf{p})$. ∎

Proof of Theorem 3. The proof of Theorem 2 is easily adapted to higher dimensions, giving that if $\Sigma_{\mathbf{i}} p(\mathbf{i}) = \infty$ then

$$\sum_{\mathbf{i} \in C(d)} p(\mathbf{i}) = \infty \text{ a.s.}$$

where $C(d)$ is the vertex set of the component of $G(d)$ containing $\mathbf{0}$. It is now straightforward to deduce the result as in the previous proof. ∎

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