Journal of Theoretical Probability, Vol. 5, No. 4, 1992

Weak Convergence Using Higher-Order Cumulants

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Received July 22, 1991

Denote by $c_j(F)$ the jth cumulant (or 'semi-invariant') of the distribution function F. We say that F is 'specified by its higher-order cumulants' if it is the unique distribution function G having the following property: there exists a positive integer J such that $c_j(G) = c_j(F)$ for $j = 1, 2$ and $j \geqslant J$. Let $(F_n: n \geqslant 1)$ be a sequence of distribution functions, and suppose that there exists J such that $c_j(F_n) \to c_j(F)$ as $n \to \infty$, for $j = 1, 2$ and $j \geqslant J$. It is proved that $F_n \Rightarrow F$ so long as F is specified by its higher-order cumulants. It is an open problem to characterize the family of distributions which are specified by their higher-order cumulants.

KEY WORDS: Weak convergence; cumulants of a distribution function.

1. STATEMENT OF RESULTS

Let $m_j(F)$ and $c_j(F)$ denote the jth moment and the jth cumulant (or 'semi-invariant') of the distribution function F. It is well known (see Feller,$^{(6)}$ p. 269) that if $m_j(F_n) \to m_j(F)$ as $n \to \infty$, for all j, and in addition F is the unique distribution with moments $(m_j(F): j \geqslant 1)$, then it is necessarily the case that $F_n \Rightarrow F$ as $n \to \infty$. When applying this in practice, it is often easier to work with cumulants rather than with moments. Since knowledge of the moments of a distribution is interchangeable with knowledge of its cumulants, this may be expressed as follows. If $c_j(F_n) \to c_j(F)$ as $n \to \infty$, for all j, and in addition F is specified by its cumulants, then $F_n \Rightarrow F$. This formulation is of particular value when proving asymptotic normality, since the cumulants of the normal distribution $N(\mu, \sigma^2)$, having mean μ and variance σ^2, are

$$c_j(\mu, \sigma^2) = \begin{cases} \mu & \text{if } j = 1 \\ \sigma^2 & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

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Asymptotic normality is therefore established once it has been shown that

(a) $c_1(F_n)$ and $c_2(F_n)$ converge as $n \to \infty$, and
(b) $c_j(F_n) \to 0$ as $n \to \infty$, for $j \geqslant 3$.

This approach has been found to be convenient in various cases (including Malyshev,$^{(11)}$ Cox and Grimmett,$^{(4,5)}$ Janson$^{(8)}$).

It is a famous theorem of Marcinkiewicz$^{(12)}$ that, if there exists a positive integer J such that $c_j(F) = 0$ for $j \geqslant J$, then F is a normal distribution. That is to say, only the normal distributions have the property that all their higher-order cumulants equal 0. Janson$^{(8)}$ has made use of Marcinkiewicz's theorem to show the asymptotic normality of the sequence $(F_n: n \geqslant 1)$ of distribution functions whenever they satisfy the following:

(a) $c_1(F_n)$ and $c_2(F_n)$ converge as $n \to \infty$, and
(b) there exists an integer J such that $c_j(F_n) \to 0$ as $n \to \infty$, for $j \geqslant J$.

It is our purpose in this note to extend Janson's theorem to general limit distributions. We prove that $F_n \Rightarrow F$ as $n \to \infty$, whenever

(c) $c_j(F_n) \to c_j(F)$ for $j = 1, 2$, and
(d) there exists an integer J such that $c_j(F_n) \to c_j(F)$ as $n \to \infty$, for $j \geqslant J$,

subject to the extra assumption on F that it is specified by its higher-order cumulants in a manner to be described soon. It is an open problem to decide exactly which distributions are indeed specified by their higher-order cumulants, and we present a brief discussion of this point at the end of this section.

In this paper, we deal only with distributions G all of whose moments

$$m_j(G) = \int_{-\infty}^{\infty} x^j \, dG(x), \qquad j = 1, 2, \ldots,$$

are finite (i.e., the corresponding Lebesgue-Stieltjes integrals are absolutely convergent). The characteristic function $\phi_G(t)$ may be expressed as

$$\phi_G(t) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} m_j(G)$$

and the cumulant generating function $\kappa_G(t) = \log \phi_G(t)$ has an expansion as

$$\kappa_G(t) = \sum_{j=1}^{\infty} \frac{(it)^j}{j!} c_j(G)$$

It is easily seen that $c_1(G) = m_1(G)$, $c_2(G) = m_2(G) - m_1(G)^2$ (i.e., the variance of G), and furthermore $c_j(G)$ is a polynomial in the first j moments of G (and *vice versa*).

We say that a distribution function F is 'specified by its moments' if it is the unique distribution having moments $(m_j(F): j \geqslant 1)$. Clearly F is specified by its moments if and only if it is specified (similarly) by its cumulants.

To what extent is a distribution function specified by knowledge of all but a finite number of its cumulants? Note that $\phi_{F*G}(t) = \phi_F(t)\phi_G(t)$, and therefore

$$\kappa_{F*G}(t) = \kappa_F(t) + \kappa_G(t);$$

here, $F * G$ denotes the Lebesgue-Stieltjes convolution of F and G. If G is $N(\mu, \sigma^2)$, then $\kappa_G(t) = i\mu t - \frac{1}{2}\sigma^2 t^2$, implying that $c_j(F * G) = c_j(F)$ for $j \geqslant 3$. Therefore, cumulants of order three or more are unchanged by convolution with a normal distribution; only the first two cumulants are altered thereby. With this example in mind, we say that a distribution function F is 'specified by its higher-order cumulants' if F is the unique distribution function G satisfying the following: there exists a positive integer J such that $c_j(G) = c_j(F)$ for $j = 1, 2$ and for $j \geqslant J$.

Our principal purpose is to prove the following theorem. We write $F_n \Rightarrow F$ if the sequence $(F_n: n \geqslant 1)$ of distribution functions converges weakly to the distribution function F. See Billingsley$^{(2)}$ or Shiryayev$^{(13)}$ for accounts of weak convergence.

Theorem 1. Let $(F_n: n \geqslant 1)$ be a sequence of distribution functions, and let F be a distribution function which is specified by its higher-order cumulants. If there exists a positive integer J such that, as $n \to \infty$,

$$c_j(F_n) \to c_j(F), \qquad j = 1, 2, \qquad j \geqslant J$$

then $F_n \Rightarrow F$ as $n \to \infty$.

In the special case when F is a normal distribution, this was proved by Janson.$^{(8)}$ The following is an immediate corollary, of which the proof is exactly as in Janson.$^{(8)}$

Theorem 2. Let F be a distribution function which is specified by its higher-order cumulants. Let $\varepsilon > 0$ and $J \geqslant 3$. There exist $\delta > 0$ and $M < \infty$ such that

$$\sup_{x \in \mathbb{R}} |G(x) - F(x)| < \varepsilon$$

for any distribution function G satisfying $|c_j(G) - c_j(F)| < \delta$ for $j = 1, 2$ and $J \leqslant j \leqslant M$.

Let $\mathscr{H}$ be the family of distribution functions which are specified by their higher-order cumulants. It is a consequence of Marcinkiewicz's theorem that $\mathscr{H}$ contains the normal distributions. Lukacs(9) has proved that $\mathscr{H}$ contains the distribution of the difference of two Poisson-distributed random variables, and this conclusion has been generalized by Christensen.(3) See Lukacs(10) for a discussion of certain extensions of Marcinkiewicz's theorem.

In the negative direction, Gol'dberg(7) has exhibited a distribution which does not lie in $\mathscr{H}$. In answering a question of Linnik, he found a distribution function L with all moments finite, and with the following property: for any even polynomial p with real coefficients and satisfying $p(0) = 0$, there exists a positive ε and a distribution function H having characteristic function $\phi_H(t) = \phi_L(t)\, e^{p(\varepsilon t)}$. In particular, $c_j(H) = c_j(L)$ for all j such that the coefficient of t^j in $p(t)$ is 0. The function L has density $\lambda(x) = Af(x)$, for $x \in \mathbb{R}$, where A is a constant and f is the entire function

$$f(z) = \frac{\cosh(2\pi) - \cos(2\pi z)}{\psi(z)}, \qquad z \in \mathbb{C}$$

with

$$\psi(z) = \prod_{n=0}^{\infty} \{(1 - z/16^n)^2 + 16^{-2n}\}$$

The following property of L is striking.

Proposition 3. The distribution function L possesses finite moments of all orders, but is not specified by its moments.

In the light of this proposition, it is natural to ask whether the members of $\mathscr{H}$ may be characterized as those distributions which are specified by their moments. In the absence of an answer to this, one may ask the possibly easier question of determining whether $\mathscr{H}$ contains all distributions F with the property that the moment generating function

$$\mu_F(t) = \int_{-\infty}^{\infty} e^{tx}\, dF(x)$$

exists in a neighbourhood of the origin.

2. PROOFS

We prove Theorem 1 and Proposition 3, the proof of Theorem 2 being exactly as in Janson.(8)

Proof (*Theorem* 1). Let $(F_n: n \geqslant 1)$ be a sequence of distribution functions, and let F be a distribution function lying in $\mathscr{H}$. Suppose that there exists J $(\geqslant 3)$ such that, as $n \to \infty$,

$$c_j(F_n) \to c_j(F), \qquad j = 1, 2, \qquad j \geqslant J$$

We may suppose that $\sigma^2 = c_2(F)$ satisfies $\sigma^2 > 0$, since otherwise F is the distribution function of a constant random variable, and the result is obvious.

We let $\delta_{n,j} = c_j(F_n) - c_j(F)$, and we denote by $\boldsymbol{\delta}_n$ the vector $\boldsymbol{\delta}_n = (\delta_{n,3}, \delta_{n,4},..., \delta_{n,J-1}) \in \mathbb{R}^{J-3}$. Let Δ be the sequence $(\boldsymbol{\delta}_n: n \geqslant 1)$ of vectors.

Suppose first that the sequence Δ is bounded. We claim that the unique limit point of Δ is the origin $\mathbf{0}$ of $\mathbb{R}^{J-3}$. Once this claim is shown, the conclusion is immediate: it follows from the claim that $\boldsymbol{\delta}_n \to \mathbf{0}$ as $n \to \infty$, giving that $c_j(F_n) \to c_j(F)$ for all j, and implying therefore that $F_n \Rightarrow F$. To see the claim, argue as follows. Suppose that Δ has a limit point $\boldsymbol{\eta} = (\eta_3, \eta_4,..., \eta_{J-1}) \in \mathbb{R}^{J-3}$, and find a subsequence $(n_m: m \geqslant 1)$ along which $\boldsymbol{\delta}_{n_m} \to \boldsymbol{\eta}$. Then $c_j(F_{n_m}) \to c_j(F) + \eta_j$ as $n \to \infty$, for $3 \leqslant j < J$, implying that $c_j(F_n) \to \beta_j$ for all j, where

$$\beta_j = \begin{cases} c_j(F) + \eta_j & \text{if } \ 3 \leqslant j < J \\ c_j(F) & \text{otherwise} \end{cases}$$

It is an immediate consequence (see Janson,[(8)] Lemma 1) that there exists a distribution function G having cumulants $(\beta_j: j \geqslant 1)$. Now $F \in \mathscr{H}$, and therefore $F = G$, which is to say that $\boldsymbol{\eta} = \mathbf{0}$.

Suppose now that the sequence Δ is unbounded. By passing to a subsequence, we may assume that

$$d_n = \sup_{3 \leqslant j < J} \{|c_j(F_n) - c_j(F)|^{1/j}\}$$

satisfies $d_n \to \infty$ as $n \to \infty$.

Let X_n be a random variable with distribution function F_n, and let X have distribution function F; we assume that X is independent of each X_n. Define

$$Z_n = c_1(F) + \frac{1}{d_n}\{X_n - E(X_n)\} + \frac{\sigma_n}{\sigma}\{X - E(X)\}$$

where E denotes expectation, and $\sigma^2 = \operatorname{var}(X) = c_2(F)$, $\sigma_n^2 = \operatorname{var}(X_n) = c_2(F_n)$. Making use of the fact that

$$c_j(aA + bB) = a^j c_j(A) + b^j c_j(B)$$

for $a, b \in \mathbb{R}$ and independent random variables A and B, we obtain (in the obvious notation) that

$$c_1(Z_n) = c_1(F)$$

$$c_2(Z_n) = \frac{\sigma_n^2}{d_n^2} + \sigma_n^2 \to \sigma^2 \qquad \text{as} \quad n \to \infty$$

Furthermore, for $j \geqslant 3$,

$$c_j(Z_n) = \frac{c_j(F_n)}{d_n^j} + \left(\frac{\sigma_n}{\sigma}\right)^j c_j(F)$$

If $j \geqslant J$, then $c_j(Z_n) \to c_j(F)$. For $3 \leqslant j < J$,

$$|c_j(Z_n) - c_j(F)| \leqslant \left|\frac{c_j(F_n)}{d_n^j}\right| + \left|\left\{\left(\frac{\sigma_n}{\sigma}\right)^j - 1\right\} c_j(F)\right|$$

Now,

$$||c_j(F_n)| - |c_j(F_n) - c_j(F)|| \leqslant |c_j(F)|$$

and therefore

$$\lim_{n \to \infty} \sup_{3 \leqslant j < J} |c_j(Z_n) - c_j(F)| \leqslant 1$$

Similarly

$$|c_j(Z_n) - c_j(F)| \geqslant \left|\frac{c_j(F_n)}{d_n^j}\right| - \left|\left\{\left(\frac{\sigma_n}{\sigma}\right)^j - 1\right\} c_j(F)\right|$$

and therefore

$$\lim_{n \to \infty} \sup_{3 \leqslant j < J} |c_j(Z_n) - c_j(F)| = 1 \qquad (*)$$

In particular, there exists a constant K such that $|c_j(Z_n)| \leqslant K$ for all $3 \leqslant j < J$ and all n, whence there exists a subsequence $(n_m: m \geqslant 1)$ along which $\beta_j = \lim_{m \to \infty} c_j(Z_{n_m})$ exists for all $3 \leqslant j < J$. By the argument used in the case when Δ was assumed bounded, we deduce that $\beta_j = c_j(F)$ for all j, in contradiction of $(*)$. Therefore Δ is bounded, and the theorem is proved.

Proof (*Proposition* 3). The distribution function L is absolutely continuous with density function λ. It was shown by Gol'dberg$^{(7)}$ that there exist positive constants A_1, A_2, such that

$$0 < \lambda(x) < A_1 \exp\{-A_2[\log(|x| + 1)]^2\}, \qquad x \in \mathbb{R}$$

Gol'dberg noted the consequence that L has all its moments. The calculations in question may be adapted easily to show the existence of positive constants A_3, A_4, such that

$$\lambda(x) > A_3 \exp\{-A_4[\log(|x|+1)]^2\}, \qquad x \in \mathbb{R}$$

It follows immediately that

$$\left| \int_{-\infty}^{\infty} \frac{\log \lambda(x)}{1+x^2}\, dx \right| < \infty$$

The finiteness of this (Krein) integral implies that L is not specified by its moments (see Akhiezer,$^{(1)}$ p. 87). Note that λ has a similar tail behavior to that of the log-normal distribution.

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