Journal of Theoretical Probability, Vol. 5, No. 3, 1992

On the Geometry of Random Cantor Sets and Fractal Percolation

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Received December 17, 1990; revised February 3, 1992

Random Cantor sets are constructions which generalize the classical Cantor set, "middle third deletion" being replaced by a random substitution in an arbitrary number of dimensions. Two results are presented here. (a) We establish a necessary and sufficient condition for the projection of a d-dimensional random Cantor set in $[0, 1]^d$ onto an e-dimensional coordinate subspace to contain an e-dimensional ball with positive probability. The same condition applies to the event that the projection is the entire e-dimensional unit cube $[0, 1]^e$. This answers a question of Dekking and Meester.[9] (b) The special case of "fractal percolation" arises when the substitution is as follows: The cube $[0, 1]^d$ is divided into M^d subcubes of side-length M^{-1}, and each such cube is retained with probability p independently of all other subcubes. We show that the critical value $p_c(M, d)$ of p, marking the existence of crossings of $[0, 1]^d$ contained in the limit set, satisfies $p_c(M, d) \to p_c(d)$ as $M \to \infty$, where $p_c(d)$ is the critical probability of site percolation on a lattice $\mathbb{L}^d$ obtained by adding certain edges to the hypercubic lattice $\mathbb{Z}^d$. This result generalizes in an unexpected way a finding of Chayes and Chayes,[4] who studied the special case when $d=2$.

KEY WORDS: Random Cantor sets; fractal percolation; critical probability.

1. INTRODUCTION

The classical Cantor set is obtained from the unit interval by the successive "deletion of middle thirds." In generalizing this construction, one may work with the unit cube $[0, 1]^d$ in d dimensions, and replace middle-third deletion by some fixed random substitution of smaller cubes. The ensuing random sets may be termed "random Cantor sets," and several authors have studied their geometrical properties (see Falconer,[11] Chapter 13, and the references below). Particular attention has been paid to properties of

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the projection of such a set onto a subspace of lower dimension, and to the existence of long paths in the set, particularly paths which traverse the unit cube $[0, 1]^d$.

Let $d \geqslant 1$, $M \geqslant 2$, and write C_0 for the unit cube $[0, 1]^d$ of $\mathbb{R}^d$. We construct a "random fractal" C in C_0 in the following way. Let $J = \{0, 1,..., M-1\}$, with $J^d = \{0, 1,..., M-1\}^d$, the set of d-vectors having entries $0, 1,..., M-1$. Let μ be a probability measure on the set $\mathscr{J}$ of subsets of J^d. The pair $(\mathscr{J}, \mu)$ generates a subset C_1 of C_0 given by

$$C_1 = \bigcup_{\mathbf{i} \in X} C(\mathbf{i})$$

where

$$C(\mathbf{i}) = \prod_{r=1}^{d} \left[\frac{i_r}{M}, \frac{i_r + 1}{M}\right] \qquad \mathbf{i} = (i_1, i_2,..., i_d) \in J^d$$

and X is a subset of J^d chosen at random according to the measure μ. Now C_1 is the union of $|X|$ cubes having side-lengths M^{-1}. We iterate the above random substitution on each of these cubes, each iteration being independent of all previous steps. That is to say, let $Y(\mathbf{i})$, $\mathbf{i} \in X$, be independent random subsets of J^d having measure μ, and independent of X. The next set C_2 in the construction is given as

$$C_2 = \bigcup_{\mathbf{i} \in X} \bigcup_{\mathbf{j} \in Y(\mathbf{i})} C(\mathbf{i}, \mathbf{j})$$

where

$$C(\mathbf{i}, \mathbf{j}) = \prod_{r=1}^{d} \left[\frac{i_r}{M} + \frac{j_r}{M^2}, \frac{i_r}{M} + \frac{j_r + 1}{M^2}\right]$$

and we have written $\mathbf{i} = (i_1, i_2,..., i_d)$, $\mathbf{j} = (j_1, j_2,..., j_d)$ as usual. Continuing ad infinitum, we obtain a decreasing sequence $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$ of random compact subsets of C_0, with limit

$$C = \bigcap_{n=0}^{\infty} C_n$$

and it is with the geometry of the compact set C that we are concerned.

As detailed in Dekking and Grimmett,[8] several authors have studied the Hausdorff dimension $\dim(C)$ of C, showing that

$$\dim(C) = \frac{\log E\,|X|}{\log M} \qquad \text{a.s. on } \{C \neq \varnothing\}$$

where E denotes expectation (and, later, P will denote probability). The exact measure function of C has been established by Graf, Mauldin, and Williams.[12] Dekking and Grimmett,[8] as well as Falconer[10] and Kesten,[17] have studied the set $\pi_e C$ obtained by projecting C onto the first e coordinates of $\mathbb{R}^d$, where $1 \leqslant e < d$. Chayes, Chayes, and Durrett[5] were concerned with the possible existence of long paths in C, particularly paths which traverse the unit cube $[0, 1]^d$ between opposite faces. More recently Chayes, Chayes, Grannan, and Swindle[6] have been interested in the existence of surfaces contained in C, when $d = 3$, surfaces which separate opposite faces of a three-dimensional box. Properties of projected sets, long paths, and surfaces may be set in one context as part of a "morphology" of random Cantor sets of the type proposed by Dekking and Meester.[9]

We present two results in this paper, the first of which is a necessary and sufficient condition for the existence in $\pi_e C$ of an e-dimensional ball with positive probability. We say that $\pi_e C$ *contains a ball* if there exists $\mathbf{x} \in [0, 1]^e$ and $\varepsilon > 0$ such that the e-dimensional ε-neighborhood of $\mathbf{x}$ is contained in $\pi_e C$. Here is some notation prior to the statement of the theorem. For $\mathbf{k} \in J^e = \{0, 1,..., M-1\}^e$, let

$$m(\mathbf{k}) = E\,|\{\mathbf{i} = (i_1,..., i_d) \in X\colon (i_1,..., i_e) = \mathbf{k}\}| \tag{1.1}$$

the mean number of cubes contributing to C_1 which project onto the subcube of $[0, 1]^e$ corresponding to the vector $\mathbf{k}$. Let

$$m = \min_{\mathbf{k} \in J^e} m(\mathbf{k}) \tag{1.2}$$

Theorem 1. Suppose that $1 \leqslant e < d$ and $P(C \neq \varnothing) > 0$.

(a) If $m > 1$ then

$$P(\pi_e C \text{ contains a ball} \mid C \neq \varnothing) = 1 \tag{1.3}$$

and

$$P(\pi_e C = [0, 1]^e) > 0 \tag{1.4}$$

(b) If $m < 1$ then

$$P(\pi_e C \text{ contains a ball}) = 0 \tag{1.5}$$

Note that $P(C = \varnothing)$ is the extinction probability of a branching process with family-size distributed as $|X|$. Therefore $P(C \neq \varnothing) > 0$ if and only if either $E\,|X| > 1$ or $|X| = 1$ a.s.

The case $m = 1$ is of course special. In this case, the conclusion of part (b) of Theorem 1 applies if and only if there exists $\mathbf{k} \in J^e$ with $m(\mathbf{k}) = 1$

such that $|\{\mathbf{i} \in X: (i_1, i_2, ..., i_e) = \mathbf{k}\}|$ is not a.s. equal to 1. [If such $\mathbf{k}$ exists, the proof of part (b) applies directly. If $m = 1$ but no such $\mathbf{k}$ exists, then $\pi_e C$ is unchanged by altering the substitution mechanism in the following way: For any $\mathbf{k}$ with $m(\mathbf{k}) = 1$, retain *all* cubes which project onto the subcube of $[0, 1]^e$ corresponding to the vector $\mathbf{k}$.]

Perhaps the simplest example of a random Cantor set of the above type is given by the following choice of the measure μ on $\mathscr{J}$: the product measure with density p on each of the M^d elements of J^d. In this case X is the random subset of J^d obtained by including each element with probability p independently of all other elements. The ensuing process is called *fractal percolation*; see Mandelbrot.[19] It was studied independently by Kesten, Guth, and Weinberg (see Guth and Weinberg[16]).

According to the Dekking–Meester[9] morphology, a random Cantor set with $d = 2$ is in phase

I. If $C = \varnothing$ a.s.
II. If $P(C \neq \varnothing) > 0$, and $\dim(\pi_1 C) = \dim(C)$ a.s.
III. If $\dim(\pi_1 C) < \dim(C)$ a.s. on $\{C \neq \varnothing\}$, but $\lambda(\pi_1 C) = 0$ a.s.
IV. If $0 < \lambda(\pi_1 C) < 1$ a.s. on $\{C \neq \varnothing\}$
V. If $P(\lambda(\pi_1 C) = 1) > 0$, but no percolation occurs a.s.
VI. If $P(\text{percolation occurs}) > 0$

Here, λ is one-dimensional Lebesgue measure, and *percolation* is said to occur if C contains a connected set having nonempty intersection with both the left- and right-hand sides of the unit square of $\mathbb{R}^2$. Dekking and Meester have remarked that two-dimensional fractal percolation is in phase III for no value of p; as p increases from 0 to 1, the phase-number of fractal percolation passes from I to VI but omits III. It is an immediate consequence of Theorem 1 above that fractal percolation can never be in phase IV. It is easily seen that $m = pM$ for this process (remember that $d = 2$ and $e = 1$), so that the process passes directly from phase II to phase V at the value $p = M^{-1}$ (it is in phase II when $p = M^{-1}$).

Our second result concerns fractal percolation in general dimension $d \geq 2$, in the limit as the rescaling factor M tends to infinity. As before, we say that *percolation occurs* in the set C if C contains a connected set having nonempty intersection with both the "left-hand side" $\{0\} \times [0, 1]^{d-1}$ and the "right-hand side" $\{1\} \times [0, 1]^{d-1}$ of C_0. We let $\theta(M, p) = P(\text{percolation occurs})$, and define

$$p_c(M, d) = \sup\{p: \theta(M, p) = 0\}$$

Chayes, Chayes, and Durrett[5] showed that $0 < p_c(M, d) < 1$ when $d \geq 2$.

For $d \geqslant 2$, let $\mathbb{L}^d$ be the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and with edge set given by the adjacency relation: $\mathbf{x} \sim \mathbf{y}$ if and only if $|x_i - y_i| \leqslant 1$ for all i *and* $x_i = y_i$ for at least one value of i. If $d = 2$, then $\mathbb{L}^d$ is the usual square lattice. For $d \geqslant 3$, $\mathbb{L}^d$ contains the d-dimensional hypercubic lattice $\mathbb{Z}^d$ as a strict sublattice, and is obtained from that lattice by an enhancement which permits connections between two vertices $\mathbf{x}$ and $\mathbf{y}$ whenever the unit cubes of $\mathbb{R}^d$ centered at $\mathbf{x}$ and $\mathbf{y}$ intersect in an e-dimensional subcube of $\mathbb{R}^d$, for some $e \geqslant 1$. We shall study site percolation on $\mathbb{L}^d$, and we denote by $p_c(d)$ the critical probability of this process. See Grimmett[14] for a general account of percolation theory.

Theorem 2. If $d \geqslant 2$ then, for all $M \geqslant 2$, (a) $\theta(M, p) = 0$ if $p < p_c(d)$, and (b) $\theta(M, p) \to 1$ as $M \to \infty$ if $p > p_c(d)$, implying that (c) $p_c(M, d) \to p_c(d)$ as $M \to \infty$.

This generalizes a conclusion of Chayes and Chayes[4] valid in the special case $d = 2$. Note, however, that $p_c(d) < p_c(\mathbb{Z}^d)$ for $d \geqslant 3$, where $p_c(\mathbb{Z}^d)$ is the critical probability of site percolation on the d-dimensional cubic lattice $\mathbb{Z}^d$ (see Menshikov[20] and Aizenman and Grimmett[1]). Thus, for $d \geqslant 3$, the limiting critical probability for fractal percolation is smaller than was anticipated by some. It is intuitively attractive that, for each d, $p_c(M, d)$ decreases with M, but this seems to be unproven at present.

We prove Theorem 1 in Section 2, and Theorem 2 in Section 3.

2. PROOF OF THEOREM 1

We consider only the case $d = 2$, $e = 1$; exactly similar arguments are valid for general $1 \leqslant e \leqslant d - 1$.

For $\mathbf{i} \in J^n = \{0, 1,..., M-1\}^n$, let $N(\mathbf{i})$ be the number of squares contributing to C_n which project onto the subinterval $[r(\mathbf{i}), r(\mathbf{i}) + M^{-n}]$ where

$$r(\mathbf{i}) = \sum_{j=1}^{n} i_j M^{-j} \tag{2.1}$$

We define $\mu_n = \min\{N(\mathbf{i}) : \mathbf{i} \in J^n\}$. We shall prove first that

$$P(\mu_n > 0 \text{ for all } n) > 0 \tag{2.2}$$

if

$$m(k) > 1 \qquad \text{for all } k \in J \tag{2.3}$$

where $m(k) = E\,|N(k)|$. Inequality (2.2) is equivalent to

$$P(\pi_1 C = [0, 1]) > 0$$

Suppose then that $m = \min_k m(k)$ satisfies $m > 1$, and pick η such that $1 < \eta < m$. A standard large-deviation estimate will be useful in proving (2.2), and we state this next. Fix $k \in J$, and let $Z_1(k)$, $Z_2(k)$,... be independent random variables having the same distribution as $N(k)$. By the choice of η, there exists α satisfying $0 < \alpha < 1$ such that

$$P(Z_1(k) + Z_2(k) + \cdots + Z_q(k) < q\eta) \leqslant \alpha^q \tag{2.4}$$

for all $q \geqslant 1$, and furthermore such an α may be chosen uniformly in k.

Let $\mathbf{i} = (i_1,..., i_n) \in J^n$ and $k \in J$, and write $(\mathbf{i}, k) = (i_1,..., i_n, k)$. Then $N(\mathbf{i}, k)$ has the same distribution as $Z_1(k) + Z_2(k) + \cdots + Z_{N(\mathbf{i})}(k)$, and therefore

$$P(N(\mathbf{i}, k) < \eta^{n+1} \mid N(\mathbf{i}) \geqslant \eta^n) \leqslant \alpha^{\eta^n}$$

Writing A_n for the event $\{\mu_n \geqslant \eta^n\}$, it follows that, for $r \leqslant n$,

$$\begin{aligned} P(\bar{A}_{n+1} \mid A_r \cap \cdots \cap A_n) &\leqslant \sum_{\mathbf{i} \in J^n} \sum_{k \in J} P(N(\mathbf{i}, k) < \eta^{n+1} \mid \mu_n \geqslant \eta^n) \\ &\leqslant M^{n+1} \alpha^{\eta^n} \end{aligned}$$

whence, for $r \leqslant s$,

$$P(A_r \cap A_{r+1} \cap \cdots \cap A_s) \geqslant P(A_r) \prod_{n=r}^{s-1} (1 - M^{n+1} \alpha^{\eta^n})$$

We may pick r such that

$$c = P(A_r) \prod_{n=r}^{\infty} (1 - M^{n+1} \alpha^{\eta^n})$$

satisfies $c > 0$, and with this choice it follows that

$$P(A_r \cap A_{r+1} \cap \cdots \cap A_s) \geqslant c \text{ for all } s \geqslant r$$

Hence

$$P(A_s \text{ for all } s \geqslant r) \geqslant c$$

implying in particular that (2.2) holds as claimed. This establishes (1.4).

Equation (1.3) is a consequence of (1.4). On the event $\{C \neq \varnothing\}$ it is the case that the number S_n of squares contributing to C_n satisfies $S_n \to \infty$ a.s. Each such square has probability

$$\varepsilon = P(\pi_1 C = [0, 1]) > 0$$

that its intersection with C is an interval (of length M^{-n}), and therefore

$$P(\pi_1 C \text{ contains no interval} \mid C \neq \varnothing) \leqslant P(S_n < N \mid C \neq \varnothing) + (1-\varepsilon)^N$$

which tends to 0 as $n \to \infty$ and $N \to \infty$ in that order.

Turning to part (b) of the theorem, suppose that there exists $k \in \{0, 1,..., M-1\}$ such that $m(k) < 1$. For any rational subinterval $[\alpha, \beta]$ of $[0, 1]$, there exists $n \geqslant 1$ and $\mathbf{i} \in \{0, 1,..., M-1\}^n$ such that $\alpha \leqslant r(\mathbf{i}) \leqslant r(\mathbf{i}) + M^{-n} \leqslant \beta$, where $r(\mathbf{i})$ is given by (2.1). Conditional on $N(\mathbf{i})$, the sequence $N(\mathbf{i})$, $N(\mathbf{i}, k)$, $N(\mathbf{i}, k, k)$,... is a subcritical branching process which is ultimately a.s. extinct. Therefore

$$P(\pi_1 C \supseteq [\alpha, \beta]) = 0$$

whence (1.5) follows, since there are only countably many intervals $[\alpha, \beta]$. □

3. PROOF OF THEOREM 2

We first state some preliminary results concerning site percolation on the lattice $\mathbb{L}^d$ defined in Section 1, beginning with some notation. Each vertex of $\mathbb{L}^d$ is declared to be *open* with probability π and *closed* otherwise, independently of all other vertices; we write P_π for the ensuing probability measure. Given two sets U and V of vertices, we write $U \leftrightarrow V$ if there exists an open path $\mathbf{x}(1), \mathbf{x}(2),..., \mathbf{x}(r)$ in $\mathbb{L}^d$ such that $\mathbf{x}(1) \in U$ and $\mathbf{x}(r) \in V$ [and $\mathbf{x}(i)$ is adjacent to $\mathbf{x}(i+1)$ for $1 \leqslant i < r$]; the event that some point of U is in an infinite open path is denoted by $\{U \leftrightarrow \infty\}$. We write $U \leftrightarrow V$ *in* W if such a path exists within the set W of vertices. The origin is denoted by $\mathbf{0}$, and the cube with vertex set $\{1, 2,..., M\}^d$ by B_M. Let

$$H_M = \{\mathbf{x} \in \mathbb{Z}^d \colon x_1 = M\}$$

be the hyperplane of distance M from the origin (remember that $\mathbb{Z}^d$ is the vertex set of $\mathbb{L}^d$). As usual x_i denotes the ith component of $\mathbf{x}$.

Lemma 1. There exists $\alpha(\pi)$ such that

$$P_\pi(\mathbf{0} \leftrightarrow H_M) \leqslant \alpha(\pi)^M \qquad \text{for all } M \tag{3.1}$$

and $\alpha(\pi) < 1$ if $\pi < p_c(d)$.

As in Section 1, $p_c(d)$ is the critical probability of site percolation on $\mathbb{L}^d$. Lemma 1 is proved exactly as is Theorem 5.1 of Grimmett.[14]

Let $\mathscr{L}$ be the set of bounding lines of the unit cube $[0, 1]^d$; that is, $\mathscr{L}$ contains all sets of the form

$$L_r(\mathbf{a}) = \{a_1\} \times \{a_2\} \times \cdots \times \{a_{r-1}\} \times [0, 1] \times \{a_{r+1}\} \times \cdots \times \{a_d\} \tag{3.2}$$

as r ranges over $\{1, 2,..., d\}$ and $\mathbf{a} = (a_1, a_2,..., a_d)$ ranges over $\{0, 1\}^d$. Each $L \in \mathscr{L}$ corresponds in a natural way to a line in the boundary of B_M: $L_r(\mathbf{a})$ corresponds to the set $\{\mathbf{x} \in B_M: x_i = \max\{1, a_i M\}$ for $1 \leqslant i \leqslant d, i \neq r\}$. For $L \in \mathscr{L}$, we write L_M for the corresponding subset of B_M. The principal lemma is the next one. Roughly speaking it asserts that, if $\pi > p_c(d)$, then any sufficiently large subset of any specified bounding line of B_M is joined (with large probability) by open paths to a large number of vertices in every other bounding line of B_M.

Lemma 2. Suppose $\pi > p_c(d)$ and $0 < \varepsilon < 1$. There exist positive integers $u = u(\pi, \varepsilon)$ and $v = v(\pi, \varepsilon)$ such that the following holds. For all $M \geqslant v$ and for all $L \in \mathscr{L}$,

$$P_\pi(|C_M \cap L_M| \geqslant u) \geqslant 1 - \varepsilon \tag{3.3}$$

and for all subsets U of L_M with $|U| \geqslant u$,

$$P_\pi(|C_M \cap U| \geqslant 1) \geqslant 1 - \varepsilon \tag{3.4}$$

where C_M is the largest open cluster of B_M.

Proof. Let $\pi > p_c(d)$ and $0 < \varepsilon < 1$. For M a positive integer, let $K = \lceil (\log M)^2 \rceil$. Let G_M be the set of open vertices of B_M which are joined by open paths both to the boundary ∂B_M of B_M and to some point in the subcube $B'_M = \{K+1, K+2,..., M-K\}^d$; as usual, the boundary ∂A of a set A of vertices is the set of vertices $\mathbf{a} \in A$ that are adjacent to some vertex not in A.

We shall prove first that, with large probability, G_M is the largest open cluster of B_M. To this end, we claim that

$$P_\pi(G_M \backslash I \neq \varnothing) \to 0 \qquad \text{as } M \to \infty \tag{3.5}$$

and

$$P_\pi(G_M \text{ is disconnected, and } G_M \backslash I = \varnothing) \to 0 \qquad \text{as } M \to \infty \tag{3.6}$$

where I is the set of vertices of $\mathbb{Z}^d$ in the (a.s. unique) infinite open cluster of $\mathbb{L}^d$. For (3.5), note that

$$P_\pi(G_M \backslash I \neq \varnothing) \leqslant |\partial B_M| \; P_\pi(\mathbf{0} \leftrightarrow \partial R_K, \mathbf{0} \not\leftrightarrow \infty)$$

where $R_K = \{-K,..., K\}^d$, since if $G_M \backslash I \neq \varnothing$ then some point $\mathbf{x}$ of ∂B_M is in a finite cluster of $\mathbb{Z}^d$ which intersects $\mathbf{x} + \partial R_K$. Hence

$$P_\pi(G_M \backslash I \neq \varnothing) \leqslant c M^{d-1} K^d e^{-\sigma(\pi) K}$$
$$\to 0 \qquad \text{as } M \to \infty$$

where $c, \sigma(\pi) > 0$; see Chayes, Chayes, and Newman,[7] Grimmett,[14] p. 122, and Grimmett and Marstrand,[15] henceforth referred to as [GM]. As for (3.6), if $\mathbf{x}, \mathbf{y} \in G_M \cap I$ but $\mathbf{x} \not\leftrightarrow \mathbf{y}$ in B_M, then there exist $\mathbf{a}, \mathbf{b} \in \partial B'_M$, both lying in I, such that $\mathbf{a} \not\leftrightarrow \mathbf{b}$ in B_M. Suppose first that $d \geqslant 3$. Using the argument of [GM] leading to their Eq. (5.1), we find that

$$P_\pi(G_M \text{ disconnected, and } G_M \backslash I = \varnothing) \leqslant |\partial B'_M|^2 \, e^{-\alpha(\pi) K}$$

for some $\alpha(\pi) > 0$. This tends to 0 as $M \to \infty$, and (3.6) is proved. [Note that Eq. (5.1) of [GM] is incorrect as given; this equation and its proof may be corrected in various ways, one of which involves setting $D(\mathbf{x}, M) = [\alpha, \beta]^d$ where

$$\alpha = -M + \min\{0, x_1, x_2,..., x_d\}, \; \beta = M + \max\{0, x_1, x_2,..., x_d\}$$

and $\mathbf{x} = (x_1, x_2,..., x_d) \in \mathbb{Z}^d$.] Second, consider the case $d = 2$, for which $\mathbb{L}^2$ is the usual square lattice. The region $B_M \backslash B'_M$ is an "annulus" of width K. With probability approaching 1 as $M \to \infty$, this annulus contains an open circuit with B'_M in its interior (see Grimmett[13]). The existence of such a circuit precludes the existence of such a pair $\mathbf{a}$, $\mathbf{b}$.

Now, G_M contains all points in $B'_M \cap I$. By the ergodic theorem,

$$P_\pi(|B'_M \cap I| > (1 - \tfrac{1}{2}\varepsilon)\, \theta(\pi)\, |B'_M|) \to 1 \qquad \text{as } M \to \infty$$

where $\theta(\pi) = P_\pi(\mathbf{0} \leftrightarrow \infty)$ as usual. Hence

$$P_\pi(|G_M| > (1 - \varepsilon)\, \theta(\pi)\, |B_M|) \to 1 \qquad \text{as } M \to \infty \tag{3.7}$$

Using Kesten and Zhang,[18] or otherwise, we deduce from (3.5), (3.6), and (3.7) that there exists v_1 such that

$$P_\pi(G_M \text{ is largest open cluster of } B_M) > 1 - \tfrac{1}{2}\varepsilon \quad \text{if } M \geqslant v_1 \tag{3.8}$$

Turning to the proof proper, we may take $L = [0, 1] \times \{0\}^{d-1}$ without loss of generality. In the light of (3.8) it suffices to prove that there exist u and v such that

$$P_\pi(|G_M \cap L_M| \geqslant u) \geqslant 1 - \tfrac{1}{2}\varepsilon \qquad \text{if } M \geqslant v \tag{3.9}$$

and, for any subset U of L_M with $|U| \geqslant u$,

$$P_\pi(|G_M \cap U| \geqslant 1) \geqslant 1 - \tfrac{1}{2}\varepsilon \qquad \text{if } M \geqslant v \tag{3.10}$$

In proving these two statements, we shall make use of the method of proof of Theorem A of [GM]. To review this proof in complete detail would occupy much space unnecessarily, and so we shall make use of the ideas therein without giving all the details. The proof in question related to site percolation on $\mathbb{Z}^d$, but the arguments apply equally in the case of site percolation on $\mathbb{L}^d$.

We shall require an intermediate result. Let $\alpha(\rho, r)$ be the P_ρ-probability that the vertex $\mathbf{1} = (1, 1,..., 1)$ is joined to the vertex $r\mathbf{1} = (r, r,..., r)$ by an open path contained within the cube B_r, using the adjacency relation of $\mathbb{Z}^d$ rather than $\mathbb{L}^d$ (this requirement is convenient but not essential for the argument which follows). We claim that

$$\alpha(\rho) = \inf_r \alpha(\rho, r) \tag{3.11}$$

satisfies

$$\alpha(\rho) \to 1 \qquad \text{as } \rho \to 1 \tag{3.12}$$

Here is a sketch proof of this. If $d = 2$, the claim follows by simple path-intersection properties of $\mathbb{Z}^2$. [Let $\beta(\rho, r)$ be the probability that, in north-east oriented percolation on $\{1, 2, ...\}^2$, the vertex $(1, 1)$ is the endpoint of an infinite path passing through some vertex (u, r) with $1 \leqslant u \leqslant r$. Then $\beta(\rho) = \inf_r \beta(\rho, r)$ satisfies $\beta(\rho) \to 1$ as $\rho \to 1$, by the FKG inequality (see Grimmett[14]). Also, $\alpha(\rho) \geqslant \beta(\rho)^2$.] The case $d = 3$ is more illuminating. The cube B_r in three dimensions may be "peeled" in such a way that the boundary ∂B_r is expressed in the shape sketched in Fig. 1. Now, by the FKG inequality or otherwise,

$$P_\rho(\mathbf{1} \leftrightarrow r\mathbf{1} \text{ in } B_r) \geqslant P_\rho(\mathbf{1} \leftrightarrow \mathbf{x} \text{ in } S_1)\, P_\rho(\mathbf{x} + (1, 0, 0) \leftrightarrow r\mathbf{1} \text{ in } S_2) \tag{3.13}$$

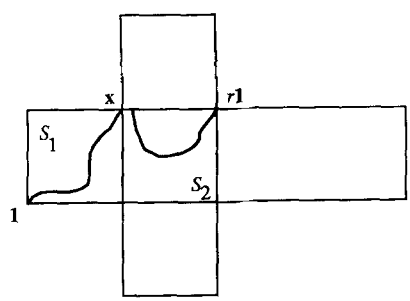

Fig. 1. If $\mathbf{1}$ is joined to $\mathbf{x}$, and $\mathbf{x} + (1, 0, 0)$ is joined to $r\mathbf{1}$, then $\mathbf{1}$ is joined to $r\mathbf{1}$.

where $\mathbf{x} = (1, r, r)$ and

$$S_1 = \{1\} \times \{1,..., r\}^2 \qquad S_2 = \{2,..., r\} \times \{1,..., r\} \times \{r\}$$

However, S_1 and S_2 are two-dimensional; using path-intersection properties of $\mathbb{Z}^2$ (as above) we deduce from (3.13) that

$$\alpha(\rho, r) \to 1 \qquad \text{as } \rho \to 1, \text{ uniformly in } r \tag{3.14}$$

as required.

The case of general d is similar. Simple peeling of the d-dimensional cube B_r is not sufficient, since ∂B_r is $(d-1)$-dimensional. However, it is easily seen that ∂B_r contains every member of some collection $\mathscr{C}$ of disjoint sets satisfying

a. $|\mathscr{C}| \leqslant d$.

b. Each set S in $\mathscr{C}$ is (graph) isomorphic to $\{1,..., s\} \times \{1,..., t\}$ for some s, $t \in \{r-2, r-1, r\}$.

c. If, for all $S \in \mathscr{C}$, every corner of S is in the same open cluster of S, then $\mathbf{1} \leftrightarrow r\mathbf{1}$ in B_r (actually in ∂B_r).

Using path-intersection properties of $\mathbb{Z}^2$ again, we obtain (3.14) as before.

Let N be a positive integer, and partition $\mathbb{Z}^d$ into N-cubes thus:

$$B_N(\mathbf{x}) = \mathbf{x}N + B_N \qquad \mathbf{x} = (x_1, x_2,..., x_d) \in \mathbb{Z}^d$$

In Section 4 of [GM], there is an inductive rule for declaring each $B_N(\mathbf{x})$ "occupied" or "unoccupied," in such a way that paths of occupied N cubes contain open paths of $\mathbb{L}^d$ joining the corresponding regions of $\mathbb{L}^d$. When examining a new N-cube in this inductive process, the probability this cube is occupied may be made as close to 1 as required by a prior choice of N sufficiently large. We claim that, as a consequence of this construction, there exists N' such that, if $N \geqslant N'$, then

$$\alpha(\pi, r, N) = P_\pi(\mathbf{1} \leftrightarrow B_N((r-1)\,\mathbf{1}) \text{ in } B_{rN} \mid B_N(\mathbf{0}) \text{ is open})$$

satisfies

$$\alpha(\pi, r, N) > 1 - \tfrac{1}{4}\varepsilon \qquad \text{for all } r \geqslant 1 \tag{3.15}$$

Here, we say that a set A is *open* if every vertex in A is open. Inequality (3.15) is obtained in the following way. Divide B_{rN} into its component N-cubes, and examine each of these cubes in the manner of [GM], beginning with $B_N(\mathbf{0})$, in order to determine their occupied/unoccupied states. For $\eta > 0$, we may find $N' = N'(\eta)$ such that, if $N \geqslant N'$, each new N-cube is occupied with probability at least $1-\eta$, conditional on $\{B_N(\mathbf{0})$ is open$\}$. The ensuing process on N-cubes stochastically dominates site percolation with density $1-\eta$ on the collection of N-cubes in B_{rN} (viewed as a subset of $\mathbb{Z}^d$ rather than of $\mathbb{L}^d$). It follows from (3.12) that, as $n \downarrow 0$ [and therefore $N'(\eta) \to \infty$], the chance that $B_N((r-1)\,\mathbf{1})$ is ultimately declared occupied tends to 1. If this last event occurs, there exists an open path of $\mathbb{L}^d$, contained in B_{rN}, and joining $\mathbf{1}$ to some point in $B_N((r-1)\,\mathbf{1})$. We therefore choose η sufficiently small, and N' sufficiently large, to obtain inequality (3.15).

Pick $N'=N'(\pi, \varepsilon)$ such that (3.15) holds for $N \geqslant N'$, and set $N=N'$. Assume $K=\lceil (\log M)^2 \rceil > N$, and define $r=2+\lceil (K+1)/N \rceil$. For $\mathbf{x} \in L_M$ let

$$\Gamma(\mathbf{x}, M)=\{\mathbf{x} \leftrightarrow \mathbf{x}-\mathbf{1}+B_N((r-1)\,\mathbf{1}) \text{ in } B(\mathbf{x}, rN)\}$$

where $B(\mathbf{x}, k)=\prod_{i=1}^{d}\{x_i,..., x_i+k-1\}=\mathbf{x}-\mathbf{1}+B_k$, and let

$$\Theta(\mathbf{x}, M)=\{B(\mathbf{x}, N) \text{ is open}\}$$

Now

$$P_\pi(\Theta(\mathbf{x}, M))=\pi^{N^d} \tag{3.16}$$

and

$$P_\pi(\Gamma(\mathbf{x}, M) \mid \Theta(\mathbf{x}, M)) \geqslant 1-\tfrac{1}{4}\varepsilon \tag{3.17}$$

by (3.15). Suppose $\Gamma(\mathbf{x}, M) \cap \Theta(\mathbf{x}, M)$ occurs and also $1 \leqslant x_1 \leqslant M-3K$. Then, for large M, $\mathbf{x}-\mathbf{1}+B_N((r-1)\,\mathbf{1})$ is contained in B'_M, implying that $\mathbf{x} \in G_M$.

For large M, we may find at least $\frac{1}{2}M/K$ points $\mathbf{x}$ satisfying $\mathbf{x} \in L_M$, $K+1 \leqslant x_1 \leqslant M-3K$, any two of which are separated by distance rN. The corresponding events $\Gamma(\mathbf{x}, M) \cap \Theta(\mathbf{x}, M)$ are independent. Each such event has probability at least $\pi^{N^d}(1-\frac{1}{4}\varepsilon)$, by (3.16) and (3.17), implying that, for all u and all large M, at least u of these events occur with probability at least $1-\frac{1}{2}\varepsilon$. This proves (3.9).

Finally we prove (3.10). Pick a positive integer s such that, out of s independent events each having probability π^{N^d}, at least one occurs with probability exceeding $1-\frac{1}{4}\varepsilon$. Suppose $u=2sN$ and U is a subset of L_M with $|U| \geqslant u$. Either U contains at least sN points in the "upper" half of L_M or U contains at least sN points in the "lower" half of L_M; we may, without loss of generality, assume the latter holds. At least s of these points are separated, pairwise, by a distance of at least N, and we pick a subset

$S=\{\mathbf{x}(i): 1 \leqslant i \leqslant s\}$ of U accordingly. The events $\Theta(\mathbf{x}(i), M)$, $1 \leqslant i \leqslant s$, are independent, and therefore

$$P_\pi(\Theta(\mathbf{x}(i), M) \text{ for some } 1 \leqslant i \leqslant s) > 1 - \tfrac{1}{4}\varepsilon$$

We examine $\mathbf{x}(i)$ in order of increasing first coordinates $\mathbf{x}(i)_1$, and determine the earliest $\mathbf{x}(i)$ such that $\Theta(\mathbf{x}(i), M)$ occurs; call this $\mathbf{x}(I)$. Conditional on I and $\Theta(\mathbf{x}(I), M)$, there is by (3.17) at least probability $1-\frac{1}{4}\varepsilon$ that $\Gamma(\mathbf{x}(I), M)$ occurs, implying that $\mathbf{x}(I) \leftrightarrow B'_M$ in B_M and therefore $\mathbf{x}(I) \in G_M$. Therefore

$$\begin{aligned} P_\pi(|U \cap G_M| \geqslant 1) &\geqslant P(I \text{ exists}, \mathbf{x}(I) \leftrightarrow B'_M \text{ in } B_M) \\ &\geqslant (1-\tfrac{1}{4}\varepsilon)^2 \geqslant 1 - \tfrac{1}{2}\varepsilon \end{aligned}$$

as required for (3.10). □

Let $\mathbf{x}(1), \mathbf{x}(2),\ldots$ be some fixed ordering of the vertices of $\mathbb{L}^d$. Let R be a random process on $\mathbb{L}^d$ taking values in $\{0, 1\}$. We think of the vertex $\mathbf{x}$ as *open* if $R(\mathbf{x})=1$ and *closed* if $R(\mathbf{x})=0$. Let $\mathscr{F}_n$ be the σ-field generated by $\{R(\mathbf{x}(i)): 1 \leqslant i \leqslant n\}$. The following lemma gives a condition for R to dominate a process in which the states of different vertices are independent. Recall that an event A is *increasing* if $\omega \in A$ implies $\omega' \in A$ whenever $\omega \leqslant \omega'$, with the obvious ordering by inclusion on subsets ω, ω' of $\mathbb{Z}^d$.

Lemma 3. Suppose that there exists $\pi \in [0, 1]$ such that, for all $n \geqslant 0$,

$$P(\mathbf{x}(n+1) \text{ is open} \mid \mathscr{F}_n) \geqslant \pi$$

Then, for every increasing event A,

$$P(A) \geqslant P_\pi(A)$$

where P_π is the product probability measure on $\mathbb{L}^d$ with intensity π.

The proof of this lemma uses standard techniques—see, for example, Barsky, Grimmett, and Newman.[2,3]

We use Lemma 3 to extend Lemma 2 in the following way.

Lemma 4. Suppose $\pi > p_c(d)$, $0 < \varepsilon < 1$, and q is a positive integer. Suppose that

$$P(\mathbf{x}(n+1) \text{ is open} \mid \mathscr{F}_n) \geqslant \pi \qquad \text{for all } n \geqslant 0 \tag{3.18}$$

There exist positive integers $u = u(\pi, \varepsilon, q)$ and $v = v(\pi, \varepsilon, q)$ such that the following holds for all $M \geqslant v$. Let $U(1)$, $U(2),\ldots$, $U(q)$ be subsets of vertices of B_M such that, for each r $(1 \leqslant r \leqslant q)$, first $|U(r)| \geqslant u$ and second there exists $L \in \mathscr{L}$ such that $U(r) \subseteq L_M$. Then

$$P(\exists \text{ an open cluster } C \text{ of } B_M \text{ such that } |C \cap L_M| \geqslant u \text{ for all } L \in \mathscr{L}, \text{ and } |C \cap U(r)| \geqslant 1 \text{ for all } r,\ 1 \leqslant r \leqslant q) \geqslant 1 - \varepsilon \tag{3.19}$$

As before, L_M denotes the set of vertices in B_M corresponding to the line L in the boundary of the unit cube $[0, 1]^d$.

Proof. By applying Lemma 2, with ε replaced by $\varepsilon/(d2^d + q)$, we see that (3.19) holds with P replaced by P_π (we can take C to be the largest open cluster of B_M). The result now follows by Lemma 3, noting that the event in (3.19) is increasing. □

Having stated the necessary results from percolation theory, we turn to the proof proper. With $J^d = \{0, 1,..., M-1\}^d$, we write $J^{d,n} = \{(i_1,..., i_n): i_j \in J^d\}$ for the set of n vectors with entries in J^d; we set $J^{d,0} = \{\varnothing\}$, the set containing exactly one element labeled as the empty set $\varnothing$. With $\underline{\mathbf{I}} = (\mathbf{i}_1,..., \mathbf{i}_n) = ((i_{1,1},..., i_{1,d}),..., (i_{n,1},..., i_{n,d}))$ we associate the subcube of $[0, 1]^d$ given by

$$C(\underline{\mathbf{I}}) = \mathbf{c}(\underline{\mathbf{I}}) + [0, M^{-n}]^d \tag{3.20}$$

where

$$\mathbf{c}(\underline{\mathbf{I}}) = \left(\sum_{j=1}^{n} M^{-j} i_{j,1},..., \sum_{j=1}^{n} M^{-j} i_{j,n} \right) \in \mathbb{R}^d \tag{3.21}$$

Such a cube $C(\underline{\mathbf{I}})$ is called a *level-n* cube, the cube $C_0 = C(\varnothing)$ being the level-0 cube.

Let $0 < p < 1$, and suppose that we are provided with a family $\{Z(\underline{\mathbf{I}}): \underline{\mathbf{I}} \in \mathscr{I}\}$ of independent random variables, each taking the value 1 with probability p and 0 otherwise. The index set $\mathscr{I}$ is taken to be

$$\mathscr{I} = J^{d,0} \cup J^{d,1} \cup J^{d,2} \cup \cdots \tag{3.22}$$

For each $\underline{\mathbf{I}} \in \mathscr{I}$ we define the indicator function $1(\underline{\mathbf{I}})$ by

$$1(\varnothing) = 1, \ 1(\underline{\mathbf{I}}) = Z(\mathbf{i}_1)\, Z(\mathbf{i}_2) \cdots Z(\mathbf{i}_n) \tag{3.23}$$

where $\underline{\mathbf{I}} = (\mathbf{i}_1,..., \mathbf{i}_n) \in J^{d,n}$. We call $C(\underline{\mathbf{I}})$ *passable* if $1(\underline{\mathbf{I}}) = 1$, and we write C_n for both the set of passable level-n cubes and for the union of this set. It is clear that $C_0, C_1,...$ has the required probability distribution, and we are primarily concerned with the limit set $C = \bigcap_{n=0}^{\infty} C_n$.

Percolation is said to occur in a subset S of $[0, 1]^d$ if S contains a connected subset having points in both $\{0\} \times [0, 1]^{d-1}$ and $\{1\} \times [0, 1]^{d-1}$. Note that

$$\{\text{percolation in } C\} = \bigcap_{n=0}^{\infty} \{\text{percolation in } C_n\} \tag{3.24}$$

an intersection of a decreasing family of events. Therefore

$$P(\text{percolation in } C) = \lim_{n \to \infty} P(\text{percolation in } C_n) \tag{3.25}$$

Percolation occurs in C_n if and only if C_n contains a sequence $C(1),..., C(r)$ of passable level-n cubes such that

a. $C(1)$ contains some point of $\{0\} \times [0, 1]^{d-1}$.
b. $C(r)$ contains some point of $\{1\} \times [0, 1]^{d-1}$.
c. $C(i) \neq C(j)$ if $i \neq j$.
d. $C(i) \cap C(i+1) \neq \varnothing$ for $1 \leqslant i < r$.

Such a sequence is called a *(level-n) traversal*. Two distinct level-n cubes C and C' may intersect in any of several ways, depending on the dimension of their intersection. That is, if $C \cap C' \neq \varnothing$ then $C \cap C'$ is congruent to one of $\{0\}$, $[0, 1],..., [0, 1]^{d-1}$, and we may correspondingly write $\dim(C \cap C') \in \{0, 1,..., d-1\}$. In general, traversals of $[0, 1]^d$ may include cube intersections of any of these d types. Given a traversal $\tau = (C(1),..., C(r))$, we write $\delta(\tau)$ for the minimal dimension of the cube intersections; that is

$$\delta(\tau) = \min\{\dim(C(i) \cap C(i+1)) : 1 \leqslant i < r\}$$

We say that *k-percolation occurs* in C_n if (a) percolation occurs in C_n, and (b) there exists a level-n traversal τ with $\delta(\tau) \geqslant k$.

Certainly

$$\{\text{percolation in } C\} = \bigcap_{n=0}^{\infty} \{0\text{-percolation in } C_n\}$$

since "percolation" and "0-percolation" are equivalent concepts. We make the further assertion that

$$P\left(\{\text{percolation in } C\} \mathbin{\triangle} \bigcap_{n=0}^{\infty} \{1\text{-percolation in } C_n\}\right) = 0 \tag{3.26}$$

where $\triangle$ denotes symmetric difference. The following argument explains this. Clearly

$$\begin{aligned}\{1\text{-percolation in } C_{n+1}\} &\subseteq \{1\text{-percolation in } C_n\}\\ &\subseteq \{0\text{-percolation in } C_n\}\end{aligned}$$

and therefore {1-percolation in C_n} is a decreasing sequence of events with intersection satisfying

$$\bigcap_{n=0}^{\infty} \{1\text{-percolation in } C_n\} \subseteq \{\text{percolation in } C\} \tag{3.27}$$

Equation (3.26) will have been proved once we have shown that

$$P(\text{percolation in } C, \text{ no 1-percolation in } C_n) = 0 \quad \text{for all } n \tag{3.28}$$

Now,

$$\begin{aligned}&\{\text{percolation in } C, \text{ no 1-percolation in } C_n\}\\ &\qquad = \{\text{percolation in } C, \text{ 0-percolation but no 1-percolation in } C_n\}\end{aligned}$$

If there is 0-percolation but no 1-percolation in C_n, then every level-n traversal uses at least one cube intersection having dimension 0, i.e., at least one cube intersection which is a single point. Let R_n be the total number of pairs C, C' of passable level-n cubes such that $C \cap C'$ is a single point, and write $\mathscr{P}_n$ for the set of such points. For $m \geqslant n$, let Q_m be the total number of pairs C, C' of passable level-m cubes such that

$$C \cap C' \neq \varnothing \qquad C \cap C' \subseteq \mathscr{P}_n$$

Thus Q_m is the number of such pairs whose intersection is a single point lying in $\mathscr{P}_n$. It is easily seen that

$$E(Q_m \mid R_n) = R_n (p^2)^{m-n} \leqslant M^{2dn} p^{2(m-n)}$$

and therefore

$$\begin{aligned}P(Q_m > 0 \mid R_n) &\leqslant M^{2dn} p^{2(m-n)}\\ &\to 0 \qquad \text{as } m \to \infty\end{aligned} \tag{3.29}$$

On the other hand, if there is 0-percolation but no 1-percolation in C_n, and in addition $Q_m = 0$ for some $m > n$, then it follows that there is no percolation in C_m. Using (3.29), we obtain (3.28), which in turn yields (3.26) as required. We deduce the following crucial lemma.

Lemma 5. It is the case that

$$P(\text{percolation in } C) = \lim_{n \to \infty} P(1\text{-percolation in } C_n)$$

The set C_n comprises passable level-n cubes, and it is natural and useful to "discretize" C_n in the following generic way; each cube in C_n is replaced by a vertex, and two such vertices are declared adjacent if and only if the two corresponding cubes intersect in a set of dimension 1 or greater. More precisely, we replace the level-n cube $C(\underline{\mathbf{I}})$ by a "level-n" vertex placed at the point $\mathbf{c}(\underline{\mathbf{I}})$; rather than introduce more notation, we label this vertex $\mathbf{c}(\underline{\mathbf{I}})$. Now $C(\underline{\mathbf{I}}) \cap C(\underline{\mathbf{J}})$ is a set of dimension 1 or greater if and only if $|\mathbf{c}(\underline{\mathbf{I}})_j - \mathbf{c}(\underline{\mathbf{J}})_j| \leqslant M^{-n}$ for $j = 1, 2,..., d$ and furthermore $\mathbf{c}(\underline{\mathbf{I}})_j = \mathbf{c}(\underline{\mathbf{J}})_j$ for at least one value of j. (Here $\mathbf{p}_j$ is the jth coordinate of $\mathbf{p} \in \mathbb{R}^d$.) Any two such vertices $\mathbf{c}(\underline{\mathbf{I}})$, $\mathbf{c}(\underline{\mathbf{J}})$ are declared to be adjacent. The ensuing graph is isomorphic to the box B_{M^n} of $\mathbb{L}^d$. We declare the level-n vertex at $\mathbf{c}(\underline{\mathbf{I}})$ to be passable if and only if the corresponding cube $C(\underline{\mathbf{I}})$ is passable. It is clear that there is 1-percolation in C_n if and only if the graph described above is traversed in the appropriate direction by a path of passable vertices.

Now

$$P(\text{1-percolation in } C_n \mid C_{n-1}) \leqslant P(\text{1-percolation in } C_n \mid C_{n-1} = [0, 1]^d)$$

However, if $C_{n-1} = [0, 1]^d$ then each level-n cube is passable with (conditional) probability p, independently of all other level-n cubes. Hence

$$P(\text{1-percolation in } C_n \mid C_{n-1}) \leqslant \kappa_p(M^n) \tag{3.30}$$

where $\kappa_p(m)$ is the P_p-probability that, in site percolation on $\mathbb{L}^d$, there is an open path traversing the cube B_m. We have from Lemma 1 that

$$\kappa_p(m) \leqslant m^{d-1} \alpha(p)^{m-1}$$

where $\alpha(p) < 1$ if $p < p_c(d)$, since such a traversal joins at least one of the m^{d-1} vertices on a given face of B_m to a hyperplane distance $m-1$ away. Therefore

$$P(\text{1-percolation in } C_n) \leqslant M^{n(d-1)} \alpha(p)^{M^n - 1}$$

whence

$$P(\text{percolation in } C) = 0 \qquad \text{if } p < p_c(d)$$

by Lemma 5. This proves part (a) of Theorem 2, and also that

$$p_c(M, d) \geqslant p_c(d) \qquad \text{for all } M \tag{3.31}$$

The principal part of the proof is to show that $\theta(M, p) \to 1$ as $M \to \infty$, if $p > p_c(d)$, and to this end we assume henceforth that $p > p_c(d)$.

We begin by making various choices which will be useful later. Pick π such that $p_c(d) < \pi < p$, let $q = 3^d$, and define ε by

$$p(1-\varepsilon) = \pi \tag{3.32}$$

With this choice of π and ε, we let $u = u(\pi, \varepsilon, q)$ and $v = v(\pi, \varepsilon, q)$ be determined by Lemma 4, and we suppose henceforth that $M \geqslant v$.

Fix $n \geqslant 1$. We shall show that, with probability greater than π/p, 1-percolation occurs in C_n, i.e., there is a sequence of cubes of C_n, each with (at least) an edge in common with the next, which joins the opposite faces $\{0\} \times [0, 1]^{d-1}$ and $\{1\} \times [0, 1]^{d-1}$ of C_0.

We consider the set of indices $\mathscr{I}^{(n)} \equiv J^{d,0} \cup J^{d,1} \cup \cdots \cup J^{d,n}$ (and the corresponding cubes $C(\underline{\mathbf{I}})$ for $\underline{\mathbf{I}} \in \mathscr{I}^{(n)}$) and order them in the following way. We first order $J^d = J^{d,1}$ in some way, say lexicographically by coordinates. We then declare $\underline{\mathbf{I}} = (\mathbf{i}_1,..., \mathbf{i}_a) < \underline{\mathbf{I}}' = (\mathbf{i}'_1,..., \mathbf{i}'_b)$ (where $0 \leqslant a, b \leqslant n$) if and only if *either* $\mathbf{i}_r < \mathbf{i}'_r$ (according to the order on J^d) where r ($\leqslant \min\{a, b\}$) is the least integer with $\mathbf{i}_r \neq \mathbf{i}'_r$ *or* if $a > b$ and $\mathbf{i}_r = \mathbf{i}'_r$ for $r = 1,..., b$. Thus $\varnothing$ is the "greatest" member of $\mathscr{I}^{(n)}$.

For convenience, we write $|\underline{\mathbf{I}}|$ for the "length" of $\underline{\mathbf{I}}$, i.e., $|\underline{\mathbf{I}}| = m$ if $\underline{\mathbf{I}} = (\mathbf{i}_1,..., \mathbf{i}_m) \in J^{d,m}$.

We examine the cubes $C(\underline{\mathbf{I}})$ in turn according to this ordering, and declare some of them to be "good" according to the rule given below.

Whether a cube $C(\underline{\mathbf{I}})$ is good or not is determined by the following inductive procedure. Let $\underline{\mathbf{I}} \in \mathscr{I}^{(n)}$, and assume that the goodness of $C(\underline{\mathbf{I}}')$ has been decided for all $\underline{\mathbf{I}}' < \underline{\mathbf{I}}$. We have the following possibilities:

a. $|\underline{\mathbf{I}}| = n$. Then $C(\underline{\mathbf{I}})$ is always declared good.

b. $0 \leqslant |\underline{\mathbf{I}}| = m < n$.

In the latter case we act as follows: We note that the subcubes $C(\underline{\mathbf{I}}, \mathbf{j})$ with $\mathbf{j} \in J^d$ have already been examined, since $(\underline{\mathbf{I}}, \mathbf{j}) < \underline{\mathbf{I}}$. Consider the set of level-$(m+1)$ subcubes of $C(\underline{\mathbf{I}})$ given by

$$\{C(\underline{\mathbf{I}}, \mathbf{j}) : \mathbf{j} \in J^d \text{ with } C(\underline{\mathbf{I}}, \mathbf{j}) \text{ good and } Z(\underline{\mathbf{I}}, \mathbf{j}) = 1\} \tag{3.33}$$

We declare $C(\underline{\mathbf{I}})$ to be good if this set of cubes has an edge-connected subset, $\mathscr{G}(\underline{\mathbf{I}})$ say, such that

i. Each edge of $C(\underline{\mathbf{I}})$ intersects at least u cubes of $\mathscr{G}(\underline{\mathbf{I}})$.

ii. For every good level-m cube $C(\underline{\mathbf{I}}')$ with $\underline{\mathbf{I}}' < \underline{\mathbf{I}}$ that has (at least) one edge in common with $C(\underline{\mathbf{I}})$, there is a cube of $\mathscr{G}(\underline{\mathbf{I}}')$ and a cube of $\mathscr{G}(\underline{\mathbf{I}})$ with a common edge.

(If there is more than one candidate for $\mathscr{G}(\underline{\mathbf{I}})$ we use some predetermined rule to choose one of them.)

This procedure determines whether $C(\underline{\mathbf{I}})$ is good for each $\underline{\mathbf{I}}$ in turn. We show first that if $C(\emptyset)$ is good then 1-percolation occurs in C_n, and second that this happens with probability at least π/p.

Assume then that $C(\emptyset)$ is good; we shall show, inductively, that for $1 \leqslant m \leqslant n$ the faces $\{0\} \times [0, 1]^{d-1}$ and $\{1\} \times [0, 1]^{d-1}$ are connected by some chain $C(\underline{\mathbf{I}}(1))$, $C(\underline{\mathbf{I}}(2))$,..., $C(\underline{\mathbf{I}}(k))$ of good edge-adjacent level-m cubes with $1(\underline{\mathbf{I}}(i)) = 1$ for $1 \leqslant i \leqslant k$. Clearly, if $C(\emptyset)$ is good, this is true for $m = 1$.

So take m with $1 \leqslant m < n$ and let $C(\underline{\mathbf{I}}(1))$,..., $C(\underline{\mathbf{I}}(k))$ be such a chain of good level-m cubes. For each i, $1 \leqslant i < k$, either $\underline{\mathbf{I}}(i) < \underline{\mathbf{I}}(i+1)$ or $\underline{\mathbf{I}}(i+1) < \underline{\mathbf{I}}(i)$. By condition (ii), there exist level-$(m+1)$ cubes of $\mathscr{G}(\underline{\mathbf{I}}(i+1))$ which are edge-adjacent to level-$(m+1)$ cubes of $\mathscr{G}(\underline{\mathbf{I}}(i))$. These level-$(m+1)$ cubes $C(\underline{\mathbf{J}})$ are all good and have $Z(\underline{\mathbf{J}}) = 1$, by (3.33) and the definition of the $\mathscr{G}(\underline{\mathbf{I}})$. It follows that there is an edge-adjacent chain of good level-$(m+1)$ cubes $C(\underline{\mathbf{J}})$ with $1(\underline{\mathbf{J}}) = 1$ which joins $\{0\} \times [0, 1]^{d-1}$ and $\{1\} \times [0, 1]^{d-1}$. (Of course, $\{0\} \times [0, 1]^{d-1}$ contains an edge of $C(\underline{\mathbf{I}}(1))$ and therefore the edge of a cube of $\mathscr{G}(\underline{\mathbf{I}}(1))$, and similarly $\{1\} \times [0, 1]^{d-1}$ contains an edge of a cube in $\mathscr{G}(\underline{\mathbf{I}}(k))$.)

It follows by induction that, if $C(\emptyset)$ is good, then 1-percolation occurs in C_n.

We now consider the probabilities of various events. For $\underline{\mathbf{I}} \in \mathscr{I}^{(n)}$, define the index $\underline{\mathbf{I}}^- \in \mathscr{I}^{(n)}$ by

$$\underline{\mathbf{I}}^- = \max\{\underline{\mathbf{I}}' : \underline{\mathbf{I}}' < \underline{\mathbf{I}} \text{ and } |\underline{\mathbf{I}}'| \leqslant |\underline{\mathbf{I}}|\} \tag{3.34}$$

[if there is no such index, $\underline{\mathbf{I}}^-$ is left undefined]. For each $\underline{\mathbf{I}}$ we let $\mathscr{F}(\underline{\mathbf{I}})$ denote the σ-field

$$\mathscr{F}(\underline{\mathbf{I}}) = \sigma(Z(\underline{\mathbf{I}}', \mathbf{j}) : |\underline{\mathbf{I}}'| \leqslant n-1, \underline{\mathbf{I}}' \leqslant \underline{\mathbf{I}},\ \mathbf{j} \in J^d)$$

[if $\underline{\mathbf{I}}^-$ is undefined as above, we take $\mathscr{F}(\underline{\mathbf{I}}^-)$ to be the trivial σ-field]. Note that $\mathscr{F}(\underline{\mathbf{I}})$ is generated by those Z that have been examined prior to deciding whether $C(\underline{\mathbf{I}})$ is good.

We prove by induction that

$$P(C(\underline{\mathbf{I}}) \text{ is good } | \mathscr{F}(\underline{\mathbf{I}}^-)) \geqslant \pi/p \tag{3.35}$$

for $\underline{\mathbf{I}} \in \mathscr{I}^{(n)}$. Assume that (3.35) has been established for all indices in $\mathscr{I}^{(n)}$ less than $\underline{\mathbf{I}}$ (when such indices exist). We have two cases:

a. $|\underline{\mathbf{I}}| = n$; then $P(C(\underline{\mathbf{I}})$ is good$) = 1$ and (3.35) is trivially true.

b. $0 \leqslant |\underline{\mathbf{I}}| = m < n$.

For case (b), given $\mathscr{F}(\underline{\mathbf{I}}^-)$, the goodness of $C(\underline{\mathbf{I}}')$ is determined (in particular) for all $\underline{\mathbf{I}}' < \underline{\mathbf{I}}$ with $|\underline{\mathbf{I}}'| = m$. Let

$$Q = \{\underline{\mathbf{I}}' : \underline{\mathbf{I}}' < \underline{\mathbf{I}} \text{ and } C(\underline{\mathbf{I}}') \text{ is a good level-}m \text{ cube with an edge in common with } C(\underline{\mathbf{I}})\}$$

For each $\underline{\mathbf{I}}' \in Q$, let $E(\underline{\mathbf{I}}')$ be some common edge of $C(\underline{\mathbf{I}})$ and $C(\underline{\mathbf{I}}')$. Since $C(\underline{\mathbf{I}}')$ is good, there are at least u level-$(m+1)$ subcubes in $\mathscr{G}(\underline{\mathbf{I}}')$ which intersect $E(\underline{\mathbf{I}}')$; call this set of subcubes $U(\underline{\mathbf{I}}')$.

To see whether $C(\underline{\mathbf{I}})$ is good, we look at $C(\underline{\mathbf{I}}, \mathbf{j}(k))$ where $\mathbf{j}(k)$ $(1 \leqslant k \leqslant M^d)$ are the vectors of J^d arranged in order. We have $(\underline{\mathbf{I}}, \mathbf{j}(k)) < \underline{\mathbf{I}}$, so by the induction hypothesis (3.35)

$$\begin{aligned}\pi/p &\leqslant P(C(\underline{\mathbf{I}}, \mathbf{j}(k)) \text{ is good} \mid \mathscr{F}((\underline{\mathbf{I}}, \mathbf{j}(k))^-)) \\ &= \begin{cases} P(C(\underline{\mathbf{I}}, \mathbf{j}(k)) \text{ is good} \mid \mathscr{F}(\underline{\mathbf{I}}, \mathbf{j}(k-1))) & \text{if } 2 \leqslant k \leqslant M^d \\ P(C(\underline{\mathbf{I}}, \mathbf{j}(1)) \text{ is good} \mid \mathscr{F}(\underline{\mathbf{I}}^-)) & \text{if } k = 1 \end{cases}\end{aligned}$$

so by independence

$$\begin{aligned}\pi &= p \times (\pi/p) \\ &\leqslant \begin{cases} P(C(\underline{\mathbf{I}}, \mathbf{j}(k)) \text{ is good}, Z(\underline{\mathbf{I}}, \mathbf{j}(k)) = 1 \mid \mathscr{F}(\underline{\mathbf{I}}, \mathbf{j}(k-1))) & \text{if } 2 \leqslant k \leqslant M^d \\ P(C(\underline{\mathbf{I}}, \mathbf{j}(1)) \text{ is good}, Z(\underline{\mathbf{I}}, \mathbf{j}(1)) = 1 \mid \mathscr{F}(\underline{\mathbf{I}}^-)) & \end{cases}\end{aligned}$$

We now apply Lemma 4, identifying the subcubes $C(\underline{\mathbf{I}}, \mathbf{j}(k))$ of $C(\underline{\mathbf{I}})$ with the vertices in B_M of the lattice $\mathbb{L}^d$. We deduce that, with probability at least $1 - \varepsilon = \pi/p$, the set (3.33) of level-$(m+1)$ subcubes of $C(\underline{\mathbf{I}})$ has an edge-connected subset [which we may take as $\mathscr{G}(\underline{\mathbf{I}})$] which (i) contains at least u cubes intersecting every edge of $C(\underline{\mathbf{I}})$, and (ii) for all $\underline{\mathbf{I}}' \in Q$, contains a cube that is edge-adjacent to a cube of $U(\underline{\mathbf{I}}')$. (Note that Q has cardinality at most $3^d = q$.) The induction step is therefore valid, and (3.35) follows.

Taking $\underline{\mathbf{I}} = \varnothing$, (3.35) becomes

$$P(1\text{-percolation in } C_n) \geqslant P(C(\varnothing) \text{ is good}) \geqslant \pi/p$$

It follows from Lemma 5 that

$$P(\text{percolation in } C) \geqslant \pi/p$$

This is valid for all $\pi \in (p_c(d), p)$ so long as $M = M(\pi)$ is sufficiently large. Taking the limit as $\pi \uparrow p$, we obtain part (b) of Theorem 2. Part (c) follows immediately. □

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