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LETTER TO THE EDITOR

On a conjecture of Hammersley and Whittington concerning bond percolation on subsets of the simple cubic lattice

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Abstract. We verify the truth of a conjecture of Hammersley and Whittington concerning bond percolation on certain subsets of the simple cubic lattice $\mathbb{Z}^3$. Let f and g be non-decreasing, non-negative functions on $[0, \infty)$ and let $\mathbb{Z}^3(f, g)$ denote the (f, g)-*wedge* of $\mathbb{Z}^3$, being the set of points (x, y, z) such that $0 \leq y \leq f(x)$, $0 \leq z \leq g(x)$ and $x \geq 0$. We show that the condition $(1+f(x))(1+g(x)) \to \infty$ as $x \to \infty$ is sufficient for the critical probability of the bond percolation process on $\mathbb{Z}^3(f, g)$ to be less than or equal to $\frac{1}{2}$.

We consider the bond percolation process on the simple cubic lattice $\mathbb{Z}^d$ in d dimensions, in which each edge is open with probability p. If A is a subset of $\mathbb{Z}^d$, the critical probability $\pi(A)$ of A is defined to be the infimum of the set of values of p for which A almost surely contains an infinite open cluster. It is generally impossible to ascertain the exact value of $\pi(A)$ by rigorous arguments, although the standard machinery of series expansions and Monte Carlo techniques may be brought to bear on the problem in some cases of interest. We may think of $\pi(A)$ as a measure of the 'effective dimensionality' of the bond percolation process on A, by comparing $\pi(A)$ with the critical probabilities of the complete lattices $\mathbb{Z}^n$ for $n = 1, 2, \ldots, d$. We note that $\pi(\mathbb{Z}) = 1$ and $\pi(\mathbb{Z}^2) = \frac{1}{2}$.

In the case of the two-dimensional square lattice $\mathbb{Z}^2$, a certain amount is known about the critical probabilities of a particular family of subsets. Let f be a non-negative function on $[0, \infty)$ and let $\mathbb{Z}^2(f)$ be the subset of $\mathbb{Z}^2$ containing all points (x, y) which satisfy $0 \leq y \leq f(x)$ and $x \geq 0$.

Theorem 1 (Grimmett 1983). If $f(x) = a \ln(x+1)$ where $0 \leq a < \infty$, then the critical probability $\nu(a)$ of $\mathbb{Z}^2(f)$ is a function ν: $[0, \infty) \to (\frac{1}{2}, 1]$ with the following properties:
$\nu(a)$ is a continuous function of a,
$\nu(a)$ is a strictly decreasing function of a,
$\nu(0) = 1$ and $\nu(a) \to \frac{1}{2}$ as $a \to \infty$.

This theorem implies, for example, that if f is non-decreasing then
(i) $\mathbb{Z}^2(f)$ is 'effectively one-dimensional' if $f(x)/\ln x \to 0$ as $x \to \infty$, and
(ii) $\mathbb{Z}^2(f)$ is 'effectively two-dimensional' if $f(x)/\ln x \to \infty$ as $x \to \infty$.

Hammersley and Whittington (1985) have discussed possible extensions of theorem 1 to the case of three dimensions. Let f and g be non-negative functions on $[0, \infty)$ and let $\mathbb{Z}^3(f, g)$ be the subset of $\mathbb{Z}^3$ containing all points (x, y, z) such that $0 \leq y \leq f(x)$, $0 \leq z \leq g(x)$ and $x \geq 0$. For each $k = 0, 1, 2, \ldots$, let $h(k)$ be the number of pairs (y, z)

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such that both (k, y, z) and $(k+1, y, z)$ lie in $\mathbb{Z}^3(f, g)$; that is to say, $h(k)$ is the number of edges in the x-direction from the slice $x=k$ to the slice $x=k+1$ in $\mathbb{Z}^3(f, g)$. Hammersley and Whittington present various results about the way in which the critical probability $\pi(f, g)$ of the bond percolation process on $\mathbb{Z}^3(f, g)$ depends on the asymptotic behaviour of $h(k)$ for large values of k. For example, they prove that

$$\pi(f, g) \geq 1 - \mathrm{e}^{-1/a}$$

if $h(k) \leq a \ln k$ for all large k; thus $\mathbb{Z}^3(f, g)$ is 'effectively one-dimensional', in the sense that $\pi(f, g) = 1$, whenever $h(k)/\ln k \to 0$ as $k \to \infty$. On the other hand, they conjecture that $\pi(f, g) \leq \frac{1}{2}$ if $h(k)/\ln k \to \infty$ as $k \to \infty$, and it is the purpose of this letter to show that this is true so long as f and g are non-decreasing functions.

Theorem 2. If f and g are non-decreasing functions such that $h(k) \geq a \ln k$ for all large k and some value of a satisfying $0 \leq a < \infty$, then $\pi(f, g) \leq \nu(a)$, where ν is the function given in theorem 1.

To see that this implies the conjecture of Hammersley and Whittington, just note that if, for all a, $h(k) \geq a \ln k$ for all large values of k, then

$$\pi(f, g) \leq \lim_{a \to \infty} \nu(a) = \tfrac{1}{2}.$$

That is to say, the 'effective dimension' of $\mathbb{Z}^3(f, g)$ is at least 2 if $h(k)/\ln k \to \infty$ as $k \to \infty$. It is likely that $\pi(f, g)$ depends on more than merely the asymptotic behaviour of h.

We note finally that, if $h(k)/\ln k \to a$ as $k \to \infty$ where $0 < a < \infty$, then the above results imply that

$$\max\{\pi(\mathbb{Z}^3), 1 - \mathrm{e}^{-1/a}\} \leq \pi(f, g) \leq \nu(a),$$

where $\pi(\mathbb{Z}^3)$ is the critical probability of bond percolation on $\mathbb{Z}^3$.

Proof of Theorem 2. We prove this theorem by a refinement of an argument of Hammersley and Whittington. We may assume that $f(k)$ and $g(k)$ are non-negative integers for each value of k, and thus

$$h(k) = (1 + f(k))(1 + g(k)),$$

since f and g are non-decreasing by the hypothesis of the theorem. For $k = 0, 1, 2, \ldots$, we define $\varphi(k)$ (respectively $\gamma(k)$) to be the greatest multiple of 2 not greater than $f(k)$ (respectively $g(k)$); more formally,

$$\varphi(k) = 2\,\mathrm{int}(\tfrac{1}{2}f(k)), \quad \gamma(k) = 2\,\mathrm{int}(\tfrac{1}{2}g(k)),$$

where $\mathrm{int}(x)$ denotes the integer part of x. Clearly $\pi(f, g) \leq \pi(\varphi, \gamma)$, and so it suffices to show that $\pi(\varphi, \gamma) \leq \nu(a)$. We define

$$\chi(k) = (1 + \varphi(k))(1 + \gamma(k)).$$

We shall prove the theorem for the case when $f(k) \to \infty$ and $g(k) \to \infty$ as $k \to \infty$; it is not difficult to adapt the proof if either f or g is bounded.

The principal step is to use the functions φ and γ to construct a path in the first quadrant of $\mathbb{Z}^2$ which starts at the origin $(0, 0)$ and visits each vertex (y, z), for $y, z = 0,$

1, 2, …, exactly once. We do this recursively as follows. We denote by $\alpha(-1)$ the path containing the origin $(0, 0)$ only and no edges. Having constructed $\alpha(-1)$, we add to this path to obtain a longer path $\alpha(0)$ which joins $(0, 0)$ to $(\varphi(0), \gamma(0))$ and which visits each vertex (y, z) with $0 \leq y \leq \varphi(0)$ and $0 \leq z \leq \gamma(0)$ exactly once and which visits no other vertex; we shall see in a moment how to do this. Having constructed a path $\alpha(k)$ for some $k \geq 0$, joining $(0, 0)$ to $(\varphi(k), \gamma(k))$ and visiting each vertex (y, z) with $0 \leq y \leq \varphi(k)$, $0 \leq z \leq \gamma(k)$ exactly once and no other vertex, we add the path sketched in figure 1 to obtain a longer path $\alpha(k+1)$ which joins $(0, 0)$ to $(\varphi(k+1), \gamma(k+1))$ and which visits each vertex in the enclosed rectangle exactly once. This recursive step is always possible since $\varphi(k+1) - \varphi(k)$ and $\gamma(k+1) - \gamma(k)$ are multiples of 2. Thus we obtain a nested sequence of paths $\alpha(-1) \subseteq \alpha(0) \subseteq \alpha(1) \subseteq \ldots \subseteq \alpha(k) \subseteq \alpha(k+1) \subseteq \ldots$.

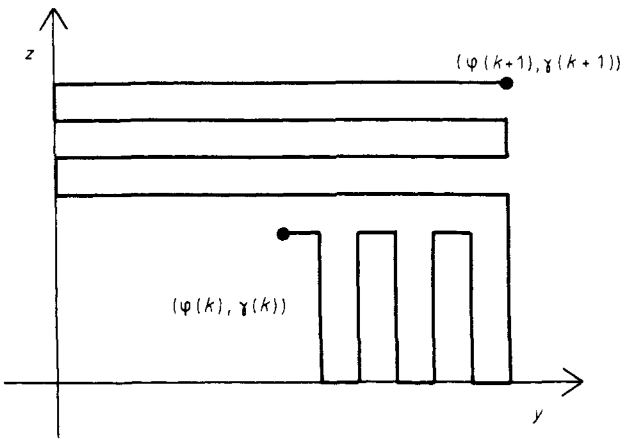

Figure 1. A sketch of the path joining $(\varphi(k), \gamma(k))$ to $(\varphi(k+1), \gamma(k+1))$ in the case when $\varphi(k+1) - \varphi(k) = 6$ and $\gamma(k+1) - \gamma(k) = 4$.

Next, we construct a subgraph S of $\mathbb{Z}^3(\varphi, \gamma)$ by including all vertices of $\mathbb{Z}^3(\varphi, \gamma)$ but deleting certain edges. We delete from $\mathbb{Z}^3(\varphi, \gamma)$ exactly those edges which join two vertices having the form (k, y, z), (k, y', z') for some k whenever (y, z) and (y', z') are not joined by an edge of $\alpha(k)$. We may now 'unroll' this subgraph S of $\mathbb{Z}^3(\varphi, \gamma)$ to see that S is isomorphic to the subgraph of $\mathbb{Z}^2$ containing all points (i, j) satisfying $0 \leq j \leq \chi(i)$ and $i \geq 0$. However,

$$\frac{\chi(i)}{h(i)} = \frac{(1+\varphi(i))(1+\gamma(i))}{(1+f(i))(1+g(i))}$$
$$\geq \frac{f(i)g(i)}{(1+f(i))(1+g(i))}$$
$$\to 1 \qquad \text{as } i \to \infty,$$

so that $\chi(i) \geq (a - \varepsilon)\ln i$ for all $\varepsilon > 0$ and all large i (depending on ε). Thus, by theorem 1, we have that the critical probability of S is at most $\nu(a)$, which gives in turn that $\pi(\varphi, \gamma) \leq \nu(a)$ as required. The proof is complete.

References

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