RANDOM ELECTRICAL NETWORKS ON COMPLETE GRAPHS

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ABSTRACT

A random electrical network is a graph G with edges which are electrical connections, whose resistances form a family of independent identically distributed random variables. We consider the case when G is a complete graph on $n+2$ vertices and a typical edge-resistance R has distribution

$$P(R = \infty) = 1 - \frac{\gamma(n)}{n}, \quad P(R \leqslant x) = \frac{\gamma(n)}{n} F(x) \qquad \text{for } 0 \leqslant x < \infty,$$

where $0 \leqslant \gamma(n) \leqslant n$ and F is a fixed distribution function concentrated on $[0, \infty)$. It turns out that if $\gamma(n) \to \infty$, then the effective resistance R_n between two specified vertices of G satisfies, as $n \to \infty$,

$$\gamma(n) R_n \to 2 \left\{ \int_{[0,\infty)} x^{-1} dF(x) \right\}^{-1} \text{ in probability}.$$

We only give a complete proof of this if $\gamma(n) > n^{\beta}$, for some positive number β. We state theorems which assert that, if $\gamma(n) \to \gamma \in [0, \infty)$ as $n \to \infty$, then $\gamma_c = 1$ is a critical value of γ in that

$$\text{if } \gamma \leqslant 1 \text{ then } P(R_n = \infty) \to 1,$$

$$\text{if } \gamma > 1 \text{ then } R_n \text{ converges to } R' + R'' \text{ in distribution},$$

where R' and R'' are independent random variables, each of which is distributed as the electrical resistance between the root and 'infinity' in the family tree of a branching process whose offspring distribution is Poisson, mean γ, and each of whose edges has a random electrical resistance which is independent of all other edge-resistances and has distribution function F.

1. *Introduction*

An electrical network is a graph G whose edges are wires joining pairs of terminals. In this paper we consider *random* electrical networks, being graphs G for which the resistances $R(e)$ of edges e are such that $\{R(e): e \in G\}$ is a family of independent, identically distributed random variables. Much work has been done on the distribution of the effective electrical resistance of the network between two disjoint sets, A_0 and A_1, of vertices of G. Quite explicit results are known when G is a Bethe lattice (or Cayley tree), see [12]. Perhaps the most interesting results arise out of the case when G is a section of a periodic 'crystalline' lattice. Suppose, for example, that G is a large cube contained in $\mathbb{Z}^d$ and A_0 and A_1 are opposite faces of this cube. If one allows $P(R(e) = \infty) = q$ to be strictly positive, then there is a strong link with percolation theory, and critical phenomena occur as q varies in $[0, 1]$; see [7] and Chapter 11 of [11]. We are not aware of work of a similar nature for other types of graph G. We are interested here in the case when G is a complete graph (see [2] for

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graph theory definitions), and *we assume henceforth that* $G = K_{n+2}$, the complete graph on $n+2$ points; we label the points of K_{n+2} as $\{0, 1, 2, \ldots, n, \infty\}$. We assume that each edge e of G has a random resistance $R(e)$ whose distribution is of the form

$$(1.1)\qquad \begin{cases} P(R(e) \leqslant x) = \dfrac{\gamma(n)}{n} F(x) & \text{for } 0 \leqslant x < \infty, \\[2ex] P(R(e) = \infty) = 1 - \dfrac{\gamma(n)}{n}, \end{cases}$$

where F is a fixed distribution function concentrated on $[0, \infty)$ and $\gamma(n)$ is a sequence of numbers such that $0 \leqslant \gamma(n) \leqslant n$. Note that we allow $R(e) = \infty$ with positive probability; of course this is equivalent to allowing e to be removed from the network, since if $R(e) = \infty$ then no current passes through e. Also $R(e) = 0$ can occur with positive probability if $F(0) > 0$; if e has endpoints i and j then taking $R(e) = 0$ amounts to inserting a short-circuit between i and j.

We may think of such electrical networks as arising in the following way. Let G_n^* be a random graph with vertex set $\{0, 1, 2, \ldots, n, \infty\}$ and edge set defined as follows: each pair of vertices is joined by an edge independently of all other pairs and with probability $n^{-1}\gamma(n)$. The electrical network obtained by endowing the edges of G_n with independent random resistances with common distribution function F has the same distribution as the above electrical network on G. Results about the effective resistance of G relate to the strength of connectedness of G_n. See [6] for a review of random graphs.

Denote by R_n the (random) effective resistance of G between the vertices 0 and ∞. We are able to prove the following Theorems 1–3 about the asymptotic behaviour of R_n for large n.

THEOREM 1. *If* $\gamma(n) \to \infty$ *as* $n \to \infty$ *then*

$$(1.2)\qquad \gamma(n)R_n \to 2\left(\int_{[0,\infty)} \frac{1}{x}\, dF(x)\right)^{-1} \text{ in probability},$$

where the limit is interpreted as 0 *if the integral is equal to* ∞.

We note that it is easy to define all the random variables $\{R_n : n \geqslant 1\}$ on *one* probability space, so that convergence in probability of the sequence $\{\gamma(n)R_n\}$ becomes meaningful. When the R_n are defined on different probability spaces one has, strictly speaking, only convergence in distribution in (1.2).

THEOREM 2. *If*

$$(1.3)\qquad \lim_{n\to\infty} \gamma(n) = \gamma \leqslant 1$$

then

$$(1.4)\qquad \lim_{n\to\infty} P(R_n = \infty) = 1.$$

In the case when $\gamma(n)$ converges to a constant γ (as in Theorem 2) there is a critical value for γ. As Theorem 2 shows, R_n a.s. becomes infinite if $\gamma \leqslant 1$. For $\gamma > 1$, R_n has a limit distribution which is no longer concentrated on $\{\infty\}$ (even though $P(R_n = \infty)$ always has a positive limit). To describe the limit distribution of R_n we must introduce a (one-type) Bienaymé–Galton–Watson branching process $\{Z_n : n \geqslant 0\}$ in which the offspring distribution is a Poisson distribution with mean γ (see Chapter I of [8]). The more traditional name for this process is a Galton–Watson process, see [9]. Associated with such a process is a random family tree T (see Chapter VI.2 of [8] or Chapter 1.2 of [10]). This is a rooted labelled tree whose root is labelled $\langle 0 \rangle$; this corresponds to the 0-th generation or the progenitor of the branching process. The other vertices of T (of which there may be none) have labels of the form $\langle i_1, \ldots, i_n \rangle$ with $i_j \in \{1, 2, \ldots\}$, $n \geqslant 1$. The label $\langle i_1, \ldots, i_n \rangle$ stands for an individual of the n-th generation which is a child of $\langle i_1, \ldots, i_{n-1} \rangle$ in the $(n-1)$-th generation (or of $\langle 0 \rangle$ if $n = 1$). Not all $\langle i_1, \ldots, i_n \rangle$ with $i_1, \ldots, i_n \geqslant 1$ occur as vertices of T; only those which correspond to individuals which are actually born or 'realized' occur as vertices of T. If $\langle i_1, \ldots, i_{n-1} \rangle$ is a vertex of T, it represents an actual individual in the $(n-1)$-th generation. If this individual has $N = N(i_1, \ldots, i_{n-1})$ children, then $\langle i_1, \ldots, i_{n-1}, i_n \rangle$ is a vertex of T if and only if $1 \leqslant i_n \leqslant N$, and $\langle i_1, \ldots, i_{n-1}, j \rangle$ is called (for $1 \leqslant j \leqslant N$) the j-th child of $\langle i_1, \ldots, i_{n-1} \rangle$ (in some arbitrary ordering of these children). In our process, given that $\langle x_1 \rangle, \ldots, \langle x_k \rangle$ are individuals of the $(n-1)$-th generation, $N(x_1), \ldots, N(x_k)$ are conditionally independent, each with a Poisson distribution with mean γ. Now T has an edge between each pair of vertices $\langle i_1, \ldots, i_{n-1} \rangle \in T$ and $\langle i_1, \ldots, i_{n-1}, i_n \rangle$ for $1 \leqslant i_n \leqslant N(i_1, \ldots, i_{n-1})$, and between $\langle 0 \rangle$ and $\langle i_1 \rangle$ for $1 \leqslant i_1 \leqslant N(0)$; T has no other edges. We write $e(i_1, \ldots, i_n)$ for the edge between $\langle i_1, \ldots, i_{n-1} \rangle$ and $\langle i_1, \ldots, i_n \rangle$, and $e(i_1)$ for the edge between $\langle 0 \rangle$ and $\langle i_1 \rangle$. For $n \geqslant 1$, we call the collection of all vertices $\langle i_1, \ldots, i_n \rangle$ the n-th *generation* of T and denote it by T_n; $T_0 = \{\langle 0 \rangle\}$. Then $Z_n = |T_n|$, the cardinality of T_n. In particular $T_n = \emptyset$ if and only if $Z_n = 0$, or equivalently if and only if the branching process dies out before the n-th generation. The subtree of T consisting of all vertices in $T_0 \cup T_1 \cup \ldots \cup T_n$ together with all the edges between these vertices is denoted by $T_{[n]}$. Finally we introduce electrical resistances. Let $\{R(e) : e \in T\}$ be a family of independent, identically distributed random variables each with distribution function F (thus $P(R(e) = \infty) = 0$ for $e \in T$). We write $R(T_{[n]})$ for the resulting resistance of the network $T_{[n]}$ between $\langle 0 \rangle$ and T_n (formally this is defined by first identifying all vertices in T_n—or short-circuiting them—and then finding the resistance between $\langle 0 \rangle$ and the single vertex obtained by this identification). We note that $R(T_{[n]}) = \infty$ if and only if the branching process is extinct by the n-th generation (or $T_n = \emptyset$). The limit

$$R(T) := \lim_{n \to \infty} R(T_{[n]})$$

exists always by the monotonicity property (2.4) given below.

THEOREM 3. *If*

$$\lim_{n \to \infty} \gamma(n) = \gamma > 1 \tag{1.5}$$

then, as $n \to \infty$, *the distribution of* R_n *converges to that of* $R'(\gamma) + R''(\gamma)$, *where* $R'(\gamma)$ *and* $R''(\gamma)$ *are independent random variables each with the distribution of* $R(T)$, *defined* *above, with the mean of the Poisson offspring distribution equal to the value γ given by* (1.5). *In particular, the atom at ∞ of the limit distribution of R_n is equal to $2q(\gamma)-q(\gamma)^2$ (< 1) where $q(\gamma)$ is the probability of extinction of a branching process with Poisson offspring distribution with mean γ (and $q(\gamma)$ is the smaller root of the equation $q = \exp(-\gamma(1-q))$).*

We shall not give the complete proofs of Theorems 1, 2 and 3 in this paper, since they are very long. The only result which we prove with complete rigour is the following weaker form of Theorem 1.

THEOREM 4. *If there exists $\beta > 0$ such that*

$$\gamma(n) > n^{\beta} \quad \text{for all } n \tag{1.6}$$

then, as $n \to \infty$,

$$\gamma(n)R_n \to 2\left(\int\limits_{[0,\infty)} \frac{1}{x}\,dF(x)\right)^{-1} \text{ in probability}, \tag{1.7}$$

where the limit is interpreted as 0 if the integral is equal to ∞.

Condition (1.6) requires that $\gamma(n) \to \infty$ at least as fast as some power of n. We shall indicate how more technical arguments may be used to remove this condition and obtain the general result of Theorem 1.

Theorems 2 and 3 show that the value 1 is a critical value for γ. If $\gamma \leqslant 1$, then all the mass in the distribution of R_n escapes to ∞ as $n \to \infty$, while for $\gamma > 1$, R_n has a limit distribution which puts some mass on $[0, \infty)$ (but also has an atom at ∞). This is clearly related to the threshold phenomenon for the spread of epidemics discussed by von Bahr and Martin-Löf [**1**] (especially in Section 5). Both [**1**] and the current paper use the existence of embedded random trees which behave like the family trees of branching processes.

The basic idea for the proofs of Theorems 2 and 3 is to look at the graphs of the vertices of K_{n+2} which are connected to 0 and ∞, respectively, by paths of finite resistance. These graphs more or less resemble two independent trees, each with the same distribution as T, given above. In addition there is a large number of edges with finite resistances joining pairs of vertices, one from each tree, but these are incident only to vertices which are far away from 0 and ∞, respectively. It is easy to see from a correct formulation of this statement that, with high probability, R_n is at least as large as the sum of the resistances of these two trees, which has precisely the distribution of $R'(\gamma)+R''(\gamma)$. Considerably more work is needed to show that there are enough interconnections between the trees to imply that R_n is not much larger than the sum of the resistances of the two trees. We shall omit almost all of this work, but shall sketch the necessary arguments in outline in Section 4. Theorem 2 is now intuitively clear: $R_n = \infty$ if and only if there is no path with finite resistance joining 0 to ∞. With large probability, this occurs if either of the trees discussed above becomes extinct after only finitely many generations; thus if $\gamma \leqslant 1$ then a.s. $R_n = \infty$ for all large n (cf. Theorem I.6.1 of [**8**]).

In Section 2 we remind the reader of the elementary theory of electrical networks. Section 3 contains the proof of Theorem 4; Section 4 contains sketch proofs of

Theorems 2 and 3 and of the part of Theorem 1 which is not covered by Theorem 4. The details of these proofs may be obtained from the authors.

2. *Preliminaries*

Let G be a graph with associated edge-resistances $\{r(e) : e \in G\}$, and let A_0 and A_1 be disjoint sets of vertices of G; we shall assume that $0 < r(e) < \infty$ for all $e \in G$. Elementary consequences of Kirchhoff's laws and Ohm's law are the following rules for combining resistors in series or parallel (see, for instance, Sections I.25.5 and II.22.3 of [4]). If $v_1, v_2, \ldots, v_k$ is a sequence of distinct vertices such that v_i is connected to v_{i+1} by an edge e_i of resistance r_i, and the degree of each of $v_2, v_3, \ldots, v_{k-1}$ in G is 2, then the r_i $(1 \leqslant i \leqslant k)$ are in *series* and we may replace the path $e_1, e_2, \ldots, e_{k-1}$ by a single edge with resistance $r_1+r_2+\ldots+r_{k-1}$ without affecting the effective resistance between any pair v' and v'' of vertices outside $\{v_2, v_3, \ldots, v_{k-1}\}$. If $e_1, e_2, \ldots, e_k$ are distinct edges with resistances $r_1, r_2, \ldots, r_k$, all connecting the vertices v_1 and v_2, then these resistances are in *parallel* and we may replace $e_1, e_2, \ldots, e_k$ by a single edge between v_1 and v_2 with resistance

$$(2.1) \qquad \left(\frac{1}{r_1} + \ldots + \frac{1}{r_k}\right)^{-1}.$$

From these rules and Kirchhoff's laws and Ohm's law one can prove that in any *finite* graph G the resistance between any two vertices is a continuous function of the resistances of the individual edges of G (see, for instance, Chapter 11 of [11]).

To compute the resistance of G between A_0 and A_1 we introduce the *potential function* $V(v)$, where v ranges over the vertices of G. We impose a unit potential difference between A_0 and A_1 by requiring that V has the boundary values

$$V(v) = \begin{cases} 0, & \text{if } v \in A_0, \\ 1, & \text{if } v \in A_1. \end{cases}$$

To produce these boundary values physically, one has to connect A_0 and A_1 to a voltage source external to the network. Off $A_0 \cup A_1$, V is determined by Kirchhoff's laws and Ohm's law:

$$(2.2) \qquad V(v) = \left\{\sum \frac{1}{r(e)}\right\}^{-1} \sum \frac{V(w(e))}{r(e)}, \qquad v \notin A_0 \cup A_1,$$

where the sums run over all edges e of G which are incident to v, and $w(e)$ denotes the endpoint of e other than v. Thus $V(v)$ is a weighted average of the potentials of the neighbours of v. The existence and uniqueness of V has been known for over a century (for connected graphs G). The resistance between A_0 and A_1 is given by

$$(2.3) \qquad \left\{\sum \frac{V(w(e))}{r(e)}\right\}^{-1},$$

where the sum runs over all edges e with exactly one endpoint in A_0, and $w(e)$ denotes the other endpoint of e (see Chapter 11 of [11]). |

Minor modifications of the above argument are necessary if the edge-resistances of G may take the value 0; this involves contracting subsets S of the vertex set of G into single composite vertices whenever every pair of vertices in S is linked by a path of edges, each having zero resistance. We do not describe the details here, noting only that the resistance between A_0 and A_1 is 0 if and only if there is a path from some vertex in A_0 to some vertex in A_1 which uses edges with zero resistance only.

Very intuitive is the following monotonicity property. Let G, A_0 and A_1 be as above, and let $\{r'(e)\}$, $\{r''(e)\}$ be two assignments of resistances to the edges of G. Denote the corresponding resistances between A_0 and A_1 in G by $r'(A_0, A_1)$ and $r''(A_0, A_1)$ respectively. Then

$$r'(e) \leqslant r''(e) \qquad \text{for all } e$$

implies that

$$r'(A_0, A_1) \leqslant r''(A_0, A_1). \tag{2.4}$$

Unfortunately the proof is not all that simple (see [5, 3] and, for the case when $r'(e)$ and $r''(e)$ may be zero or infinite, Chapter 11 of [11]). Note that this monotonicity property states in particular that short-circuiting some pairs of vertices or inserting additional edges (no matter what their resistances may be) can only decrease the resistance between A_0 and A_1; also, removal of any edges can only increase this resistance.

We call an edge *conducting* (respectively *insulating*) if its resistance is finite (respectively infinite). A path using the edges $e_1, e_2, \ldots, e_k$ is called *conducting* if the resistance of e_i is finite for $1 \leqslant i \leqslant k$.

We denote the integer part of x by $\lfloor x \rfloor$, and the smallest integer not smaller than x by $\lceil x \rceil$.

3. *Proof of Theorem* 4

We divide the proof into several lemmas, the first of which is easy to prove. Note that its proof does not require condition (1.6).

LEMMA 5. *Suppose that* $\gamma(n) \to \infty$ *as* $n \to \infty$. *If* $\varepsilon > 0$ *then*

$$P\left(\gamma(n)R_n \geqslant 2(1-\varepsilon)\left\{\int_{[0,\infty)} \frac{1}{x}\, dF(x)\right\}^{-1}\right) \to 1 \qquad \text{as } n \to \infty. \tag{3.1}$$

Proof. We may assume that

$$h := \int_{[0,\infty)} \frac{1}{x}\, dF(x) < \infty \tag{3.2}$$

and hence that $F(0) = 0$, since if (3.2) does not hold then (3.1) holds trivially for all $\varepsilon > 0$. Let G_n be the random edge-subgraph of K_{n+2} which contains the conducting edges only; thus each edge of K_{n+2} is included in G_n with probability $n^{-1}\gamma(n)$. We identify (or short-circuit) all the vertices of G_n except for 0 and ∞, and denote the

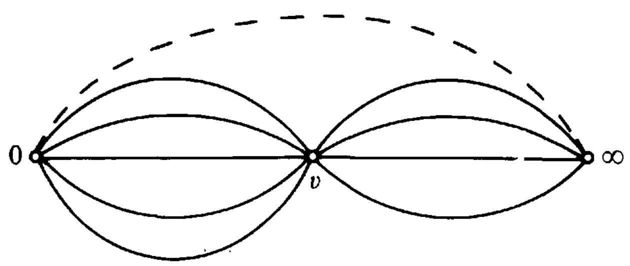

FIG. 1. The graph G_n with the vertices $\{1, 2, ..., n\}$ contracted to a single point v

vertex obtained by this identification by v. The network obtained by this identification looks something like Figure 1. Between 0 and v (respectively ∞ and v) there are τ^0 (respectively τ^∞) edges in parallel, and the resistances of these edges have distribution function F; there is one further edge joining 0 directly to ∞, with probability $n^{-1}\gamma(n)$. Since the new graph is constructed from G_n by short-circuiting, it has a resistance between 0 and ∞ which is no more than R_n.

Now, τ^0 and τ^∞ are independent with the binomial distribution, parameters n and $n^{-1}\gamma(n)$. Suppose that $\varepsilon > 0$ and $\eta > 0$. By the weak law of large numbers, we can choose N such that

$$P\left(\left|\frac{\tau^0}{\gamma(n)} - 1\right| > \varepsilon \text{ or } \left|\frac{\tau^\infty}{\gamma(n)} - 1\right| > \varepsilon\right) \leqslant \tfrac{1}{2}\eta \qquad \text{for } n > N\,.$$

Off an event with probability less than $\frac{1}{2}\eta$ we have for $n > N$ that

$$\gamma(n)(1-\varepsilon) \leqslant \tau^0, \tau^\infty \leqslant \gamma(n)(1+\varepsilon)\,. \tag{3.3}$$

Let $R_1, ..., R_{\tau^0}$ be the resistances of the τ^0 edges in G_n between 0 and v. Note that, given τ^0, the R_i $(i \leqslant \tau_0)$ are conditionally independent, each with the conditional distribution F. This follows from the fact that the conditional distribution of $R(e)$, given $R(e) < \infty$, is F. Let $R(0, v)$ and $R(v, \infty)$ be the effective resistances, respectively, of the blocks of conducting edges between 0 and v, and between v and ∞. By (2.1) we have for $n > N$ on the event (3.3)

$$\frac{1-\varepsilon}{\tau^0}\sum_{i=1}^{\tau^0}\frac{1}{R_i} \leqslant \{\gamma(n)R(0,v)\}^{-1} \leqslant \frac{1+\varepsilon}{\tau^0}\sum_{i=1}^{\tau^0}\frac{1}{R_i}\,,$$

and by the weak law of large numbers

$$P\left((1-\varepsilon)h \leqslant \frac{1}{\tau^0}\sum_{i=1}^{\tau^0}\frac{1}{R_i} \leqslant (1+\varepsilon)h \,\middle|\, \tau^0, \tau^\infty\right) \geqslant 1 - \tfrac{1}{4}\eta$$

on the set (3.3), for sufficiently large n. Consequently

$$P((3.3) \text{ occurs and } (1-\varepsilon)^2 h \leqslant \{\gamma(n)R(0,v)\}^{-1} \leqslant (1+\varepsilon)^2 h) \geqslant 1 - \tfrac{3}{4}\eta\,.$$

A similar argument for $R(v, \infty)$ shows that

$$(3.4)\quad P\Big(2(1+\varepsilon)^{-2}h^{-1} \leqslant \gamma(n)\big(R(0,v)+R(v,\infty)\big) \leqslant 2(1-\varepsilon)^{-2}h^{-1}\Big) \geqslant 1-\eta$$

for n sufficiently large. The relation (3.4) falls short of (3.1) only in as much as we have neglected the possible existence of an edge $e(0, \infty)$ of G_n joining 0 to ∞ directly. Indeed, the monotonicity principle (2.4) and the parallel law (2.1) imply that

$$(3.5)\qquad \frac{1}{R_n} \leqslant \frac{1}{R\big(e(0,\infty)\big)} + \frac{1}{R(0,v)+R(v,\infty)},$$

where $R\big(e(0, \infty)\big)$ is the resistance of $e(0, \infty)$; $1/R\big(e(0, \infty)\big)$ is interpreted as zero if $e(0, \infty)$ is not present in G_n. But (3.4) shows that with probability near one

$$(3.6)\qquad R(0,v)+R(v,\infty) \leqslant \frac{K}{\gamma(n)} \to 0 \qquad \text{as } n \to \infty$$

for some $K < \infty$, and $1/R\big(e(0, \infty)\big)$ is finite with probability one. Equation (3.1) follows easily from (3.4)–(3.6).

It is more difficult to prove the other half of Theorem 4. Suppose that (1.6) holds but (1.7) does not hold. Then it follows that (1.7) does not hold when the limit is taken down some subsequence $\{n_l : l \geqslant 1\}$ satisfying either

$$(3.7)\qquad \frac{1}{n_l}\gamma(n_l) \to \alpha \in (0, 1] \qquad \text{as } l \to \infty$$

or

$$(3.8)\qquad \frac{1}{n_l}\gamma(n_l)^{k-2} \to 0 \quad \text{and} \quad \frac{1}{n_l}\gamma(n_l)^{k} \to \infty \qquad \text{as } l \to \infty,$$

for some k with $3 \leqslant k \leqslant \lceil \beta^{-1} \rceil + 1$. It therefore suffices to prove (1.7) along subsequences $\{n_l\}$ which satisfy either (3.7) or (3.8). In order to avoid writing too many subscripts, we prove (1.7) separately under the assumption that (3.7), respectively (3.8), holds for the full sequence of positive integers (that is, we assume that we may take $n_l = l$). The reader may easily convince himself that the same proof works for subsequences.

Until further notice we assume that F is concentrated on $\{1\}$, so that

$$(3.9)\qquad F(1-) = 0, \qquad F(1) = 1;$$

each edge-resistance $R(e)$ now has distribution

$$P\big(R(e) = 1\big) = \frac{\gamma(n)}{n}, \qquad P\big(R(e) = \infty\big) = 1 - \frac{\gamma(n)}{n}.$$

Next we prove a result which, taken with Lemma 5, implies that Theorem 4 holds whenever (3.7) and (3.9) hold. This is a special case of the problem, and requires a special argument.

LEMMA 6. *If F satisfies* (3.9) *and*

$$\frac{\gamma(n)}{n} \to \alpha \in (0, 1] \qquad \text{as } n \to \infty$$

then, for all $\varepsilon > 0$,

$$P(\gamma(n)R_n \leqslant 2(1+\varepsilon)) \to 1 \qquad \text{as } n \to \infty . \tag{3.10}$$

Proof. Let B_0 (respectively B_∞) be the subset of the vertices $\{1, 2, ..., n\}$ of G which are connected to 0 (respectively ∞) by a conducting edge and let $C = B_0 \cap B_\infty$, $C_0 = B_0 \backslash C$, $C_\infty = B_\infty \backslash C$, as in Figure 2. Denote by H_n the network consisting of the vertices 0, ∞ and the vertices in C together with the edges between 0 and C and between C and ∞. Denote the resistance between 0 and ∞ in H_n by $R(H_n)$. It is easy to see that

$$\frac{1}{n}|C| \to \alpha^2 \text{ in probability} \qquad \text{as } n \to \infty .$$

Since the potential in the network H_n at each vertex in C is equal to $\frac{1}{2}$ (when 0 and ∞ are given the potentials 0 and 1, respectively) it follows from (2.3) that

$$nR(H_n) \to \frac{2}{\alpha^2} \text{ in probability} . \tag{3.11}$$

Next consider the network J_n consisting of the vertices 0, ∞ and the vertices in $C_0 \cup C_\infty$, and all conducting edges between these vertices excluding the possibly

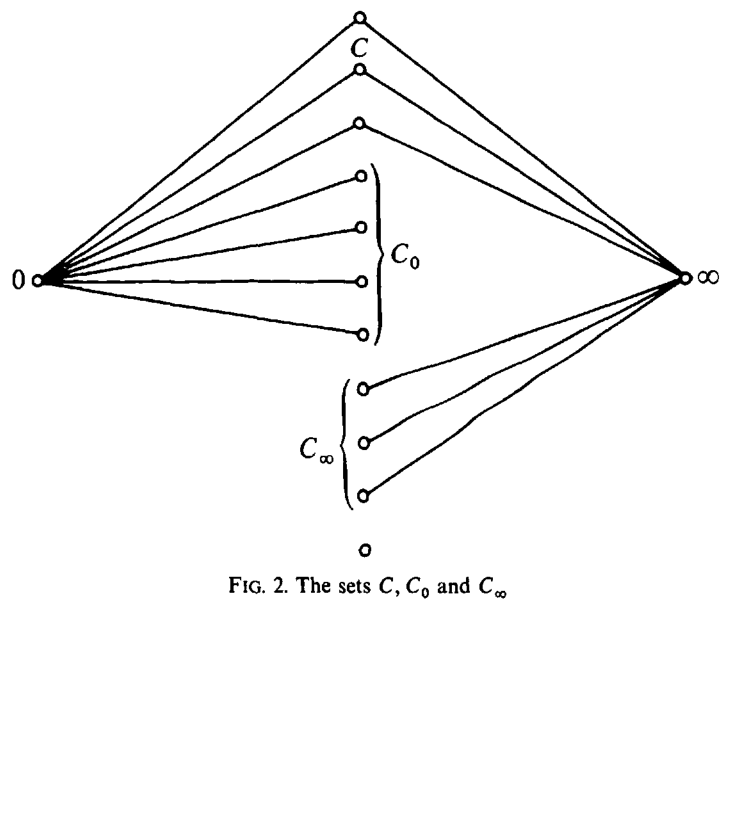

FIG. 2. The sets C, C_0 and C_∞

conducting edge between 0 and ∞. Denote the resistance between 0 and ∞ in J_n by $R(J_n)$. Now H_n and J_n form parallel connections between 0 and ∞. They are disjoint subnetworks of G, and hence

$$\frac{1}{R_n} \geqslant \frac{1}{R(H_n)} + \frac{1}{R(J_n)}. \tag{3.12}$$

Moreover, by (2.3),

$$\frac{1}{R(J_n)} = \sum_{x \in C_0} V(x),$$

where V is the potential function in the network J_n when 0 and ∞ are given the potentials 0 and 1 respectively. Consequently

$$E\left(\frac{1}{R(J_n)}\right) = \sum_{x=1}^{n} E(V(x) \mid x \in C_0) P(x \in C_0). \tag{3.13}$$

The main point is to show that

$$E(V(x) \mid x \in C_0) \to \tfrac{1}{2} \qquad \text{for } 1 \leqslant x \leqslant n. \tag{3.14}$$

Once this is shown, we easily obtain (3.10) since (3.12)–(3.14), together with

$$P(x \in C_0) = \frac{\gamma(n)}{n}\left(1 - \frac{\gamma(n)}{n}\right) \to \alpha(1-\alpha), \tag{3.15}$$

imply that

$$\liminf_{n \to \infty} E\left(\frac{1}{nR_n}\right) \geqslant \tfrac{1}{2}\alpha^2 + \tfrac{1}{2}\alpha(1-\alpha) = \tfrac{1}{2}\alpha. \tag{3.16}$$

But Lemma 5 gives that, for all $\varepsilon > 0$,

$$P\left(\frac{1}{nR_n} > \tfrac{1}{2}\alpha + \varepsilon\right) \to 0$$

whilst it is easy to see, by considering the case when *all* the edges of K_{n+2} have unit resistance, that

$$P\left(\frac{1}{nR_n} \leqslant \frac{3}{2}\right) = 1;$$

thus (3.16) implies that

$$P\left(\frac{1}{nR_n} \geqslant \tfrac{1}{2}\alpha - \varepsilon\right) \to 1 \qquad \text{for all } \varepsilon > 0,$$

which is equivalent to (3.10).

For the proof of (3.14) set

$$\delta(x,y)=\begin{cases}1, & \text{if } x \text{ and } y \text{ are connected by a conducting edge},\\ 0, & \text{otherwise},\end{cases}$$

and

$$d(x)=\sum_{y\in C_0\cup C_\infty}\delta(x,y);$$

$d(x)$ is the number of vertices in $C_0\cup C_\infty$ connected to x by conducting edges. By Kirchhoff's law (see (2.2)) we have for $x\in C_0$

$$(d(x)+1)V(x)=\sum_{y\in C_0}V(y)\delta(x,y)+\sum_{y\in C_\infty}V(y)\delta(x,y). \tag{3.17}$$

Put

$$d_0(x)=\sum_{y\in C_0}\delta(x,y),\qquad d_\infty(x)=\sum_{y\in C_\infty}\delta(x,y).$$

Then $d(x)=d_0(x)+d_\infty(x)$, since C_0 and C_∞ are disjoint. If we now sum (3.17) over $x\in C_0$, take expectations, and use obvious symmetry properties, we obtain

$$\begin{aligned}E|C_0|\{E(d_\infty(x)V(x)\mid x\in C_0)+E(V(x)\mid x\in C_0)\}\\ =E|C_\infty|E(d_0(y)V(y)\mid y\in C_\infty).\end{aligned} \tag{3.18}$$

But $0\leqslant V(x)\leqslant 1$ by the maximum principle (see, for instance, Chapter 11 of [**11**]), and trivially

$$\frac{1}{n}E|C_0|=\frac{1}{n}E|C_\infty|=\frac{\gamma(n)}{n}\left(1-\frac{\gamma(n)}{n}\right)\to\alpha(1-\alpha),$$

$$\frac{d_\infty(x)}{n}\to\alpha^2(1-\alpha),\quad \frac{d_0(y)}{n}\to\alpha^2(1-\alpha)\qquad\text{in probability}.$$

Thus (3.18), after division by n^2, shows that

$$E(V(x)\mid x\in C_0)-E(V(y)\mid y\in C_\infty)\to 0.$$

Finally, by interchanging the roles of 0 and ∞, one obtains

$$E(V(x)\mid x\in C_0)=1-E(V(y)\mid y\in C_\infty),$$

and (3.14) follows.

The next lemma contains most of the remaining work.

LEMMA 7. *Suppose that F is given by* (3.9) *and that*

$$\frac{\gamma(n)^{k-2}}{n}\to 0,\qquad \frac{\gamma(n)^k}{n}\to\infty \tag{3.19}$$

for some fixed k satisfying $3 \leqslant k \leqslant \lceil \beta^{-1} \rceil + 1$. *Then, for every* $\rho > 0$ *there exists* $N(\rho)$ *such that*

$$(3.20) \qquad P(\gamma(n) R_n \leqslant 2(1+\rho)) > 1-\rho \qquad \textit{for } n > N(\rho).$$

Before proving this we require a preliminary lemma concerning the binomial distribution.

LEMMA 8. *Let X be binomially distributed with parameters m and* $p > 0$. *For* $\eta > 0$ *there exists* $\delta(\eta)$ *depending on* η *alone and satisfying* $0 < \delta(\eta) < 1$, *such that*

$$P(Y \leqslant (1-\eta)mp) \leqslant \delta(\eta)^{mp}.$$

Proof. We may assume that $0 < p < 1$. By Markov's inequality, for $x > 1$,

$$\begin{aligned} P(Y \leqslant (1-\eta)mp) &= P(-Y \geqslant -(1-\eta)mp) \\ &\leqslant x^{(1-\eta)mp}(1-p+px^{-1})^m \\ &= (g(x))^{mp}, \end{aligned}$$

where

$$g(x) = x^{1-\eta}(1-p+px^{-1})^{1/p}.$$

Set $x = x_0$, where

$$x_0 = \frac{1-p(1-\eta)}{(1-p)(1-\eta)} > 1,$$

to find that

$$\begin{aligned} g(x_0) &= \frac{1}{(1-\eta)^{1-\eta}} \left\{ \left(1 - \frac{p\eta}{1-p(1-\eta)} \right)^{1-p(1-\eta)} \right\}^{1/p} \\ &\leqslant \frac{e^{-\eta}}{(1-\eta)^{1-\eta}} \qquad \text{since } 1-x \leqslant e^{-x} \text{ if } x \geqslant 0 \\ &= \delta(\eta), \quad \text{say}. \end{aligned}$$

It is not hard to verify that $0 < \delta(\eta) < 1$ if $0 < \eta < 1$.

Proof of Lemma 7. In the following proof we sometimes use non-integer-valued quantities when integers are required; it will be clear that this aberration makes no essential difference to the argument. Let $0 < \rho < 1$ and define $\varepsilon = \rho^2/12$. For $i \geqslant 1$, $N_i = N_i(\varepsilon)$ will denote a positive integer which depends on ε alone. Define $C, D > 0$ by

$$(3.21) \qquad C = C(\varepsilon) = \frac{\varepsilon^2}{144},$$

$$(3.22) \qquad D = D(\varepsilon) = \left(\frac{16^3}{\varepsilon^3(1-\varepsilon)^k} \right)^2,$$

with k as in (3.19). Choose N_1 such that

$$\gamma(n)^{k-2} \leqslant Cn, \quad \gamma(n)^k \geqslant Dn, \qquad \text{for } n > N_1. \tag{3.23}$$

Let G_n be the (random) edge-subgraph of G containing only the conducting edges of G. We shall construct a subgraph J of G_n which consists of two finite rooted trees connected by a number of edges between the vertices in the last 'generations' of these trees. The form of J will be such that its resistance can be estimated. If, at any stage in the construction of J, we fail to find an appropriate set of vertices then we stop the construction and say that it has *failed*; we shall show that the probability of failure is very small. First, we find the first $\mu(n) = (1-\varepsilon)\gamma(n)$ vertices in $\{1, \dots, n\}$ which are connected by a conducting edge to 0. We denote these vertices by $a_1(j)$, $1 \leqslant j \leqslant \mu(n)$, and write A_1 for the collection $\{a_1(j) : 1 \leqslant j \leqslant \mu(n)\}$. The vertex $a_1(1)$ is simply the smallest $i \in \{1, \dots, n\}$ which is adjacent to 0 in G_n, and $a_1(j+1)$ is the smallest $i > a_1(j)$ which is adjacent to 0 in G_n. Next we construct sets $A_2(1), \dots, A_2(\mu(n))$ as follows. Having found $A_2(1), \dots, A_2(r-1)$, we define $A_2(r)$ to be the subset of $\{1, 2, \dots, n\} \setminus \left\{A_1 \cup \left(\bigcup_{i=1}^{r-1} A_2(i)\right)\right\}$ consisting of the $\mu(n)$ vertices of lowest index in this set which are connected to $a_1(r)$ by a conducting edge. We write $A_2(r) = \{a_2(r, j) : 1 \leqslant j \leqslant \mu(n)\}$ and $A_2 = A_2(1) \cup \dots \cup A_2(\mu(n))$. We continue in exactly the same way, obtaining disjoint sets $A_1, A_2, \dots, A_{k-2}$ such that, for $2 \leqslant s \leqslant k-2$,

$$A_s = \bigcup_{\substack{1 \leqslant i_1 \leqslant \mu(n) \\ \vdots \\ 1 \leqslant i_{s-1} \leqslant \mu(n)}} A_s(i_1, \dots, i_{s-1}),$$

$$A_s(i_1, \dots, i_{s-1}) = \{a_s(i_1, \dots, i_s) : 1 \leqslant i_t \leqslant \mu(n),\ 1 \leqslant t \leqslant s\},$$

$$a_{s-1}(i_1, \dots, i_{s-1}) \text{ is adjacent in } G_n \text{ to each } j \in A_s(i_1, \dots, i_s).$$

Finally, we construct A_{k-1} in the same way as the previous A-sets except for the fact that

$$A_{k-1} = \bigcup_{\substack{1 \leqslant i_1 \leqslant \mu(n) \\ \vdots \\ 1 \leqslant i_{k-2} \leqslant \mu(n)}} A_{k-1}(i_1, \dots, i_{k-2}),$$

where this time the $A_{k-1}(\mathbf{i})$ each have cardinality

$$v(n) = \sqrt{\frac{n}{\gamma(n)^{k-2}}}. \tag{3.24}$$

We think of $\{0\}$, $A_1, \dots, A_{k-1}$ as the generations of a tree emanating from the root 0 in which a member of the r-th generation has $g(r)$ descendants in the $(r+1)$-th generation, where

$$g(r) = \begin{cases} \mu(n), & \text{if } r < k-2, \\ v(n), & \text{if } r = k-2. \end{cases}$$

Having constructed $A_1, \ldots, A_{k-1}$, we now find disjoint sets $B_1, \ldots, B_{k-1}$ defined in terms of sets $B_s(i_1, \ldots, i_{s-1})$ in the same way as the A-sets except that the B-sets are disjoint from the A-sets and they originate from the vertex ∞ in place of 0; we think of $\{\infty\}$, $B_1, \ldots, B_{k-1}$ as the generations of a tree emanating from ∞. Note that

$$|A_{k-1}| = |B_{k-1}| = \mu(n)^{k-2}\nu(n). \tag{3.25}$$

We define J to be the following subgraph of G_n: J has vertex set $A \cup B$ where

$$A = \{0\} \cup A_1 \cup \ldots \cup A_{k-1}, \qquad B = \{\infty\} \cup B_1 \cup \ldots \cup B_{k-1},$$

and vertices i and j in $A \cup B$ are joined by an edge of J if and only if one of the following three cases applies:

(a) $i = a_{s-1}(i_1, \ldots, i_{s-1})$ for some $2 \leqslant s \leqslant k-1$ and $j \in A_s(i_1, \ldots, i_{s-1})$, or $i = 0$ and $j \in A_1$,

(b) $i = b_{s-1}(i_1, \ldots, i_{s-1})$ for some $2 \leqslant s \leqslant k-1$ and $j \in B_s(i_1, \ldots, i_{s-1})$, or $i = \infty$ and $j \in B_1$,

(c) $i \in A_{k-1}$, $j \in B_{k-1}$, and i and j are joined by an edge in G_n.

If the construction of J is successful then J is a subgraph of G_n. Indeed, the $a_r(i_1, \ldots, i_r)$ and $b_r(i_1, \ldots, i_r)$ have been chosen such that each of the edges of J corresponding to case (a) or (b) above is a conducting edge. Note also that the vertex set A, together with the edges of J between the vertices in A, forms a rooted tree with root at 0. Similarly B (with its edges) is a tree with root at ∞. See Figure 3 for a sketch of J. We note also that, given the sets $A_1, \ldots, A_{k-1}, B_1, \ldots, B_{k-1}$, the events $E(i,j) = \{i \text{ and } j \text{ are adjacent in } J\}$, with i and j ranging over A_{k-1} and B_{k-1} respectively, are conditionally independent, each one occurring with the conditional probability $n^{-1}\gamma(n)$. This follows from the method of construction of J. At each stage, the decision whether to include a given vertex l in some A_r (or B_r) does not depend on the resistance of any edge from l to vertices other than those in A_{r-1} (or B_{r-1} respectively). In order to find A_{k-1} and B_{k-1} we never have to use the resistance of any of the edges in K_{n+2} between A_{k-1} and B_{k-1}.

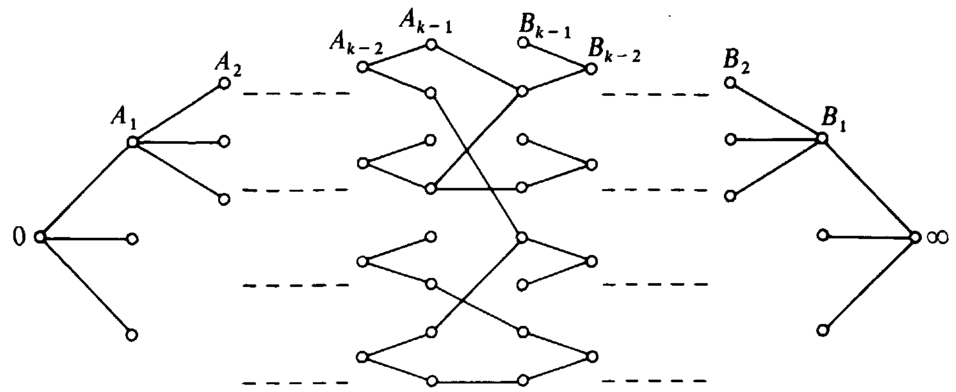

FIG. 3. The graph J

We now estimate the probability that the construction fails. In the language of family trees, the construction of $A \cup B$ requires

$$\phi(n) := 2(1+\mu(n)+\ldots+\mu(n)^{k-2})$$

families, and

$$|A \cup B| = 2(1+\mu(n)+\ldots+\mu(n)^{k-2}+\mu(n)^{k-2}\nu(n));$$

for $n > N_2(> N_1)$ we have by (3.24), (3.23) and (3.21) that

$$\phi(n) \leqslant 4\gamma(n)^{k-2},$$

$$\frac{|A \cup B|}{n} \leqslant 4\frac{\gamma(n)^{k-2}}{n} + 2\sqrt{\frac{\gamma(n)^{k-2}}{n}} \leqslant 4C+2\sqrt{C} \leqslant \tfrac{1}{2}\varepsilon.$$

Thus, conditional on having reached the stage of constructing the family of any given vertex in $A\backslash A_{k-1}$ or $B\backslash B_{k-1}$, the number of vertices in G_n available for that family is, in distribution, at least binomially distributed with parameters $n(1-\frac{1}{2}\varepsilon)$ and $n^{-1}\gamma(n)$. Thus the probability that the construction of J fails satisfies

$$P(\text{failure}) \leqslant \phi(n)P(Y \leqslant (1-\varepsilon)\gamma(n))+2\gamma(n)^{k-2}P(Y \leqslant \nu(n)),$$

where Y is binomially distributed with parameters $n(1-\frac{1}{2}\varepsilon)$ and $n^{-1}\gamma(n)$. But, from (3.24) and (3.23),

$$\frac{\nu(n)}{\gamma(n)} = \sqrt{\frac{n}{\gamma^k}} \leqslant \frac{1}{\sqrt{D}} \leqslant \tfrac{1}{2} \leqslant 1-\varepsilon$$

giving that

$$\text{(3.26)} \qquad \begin{aligned} P(\text{failure}) &\leqslant 6\gamma(n)^{k-2}P(Y \leqslant (1-\varepsilon)\gamma(n)) \\ &\leqslant 6\gamma(n)^{k-2}\delta(\tfrac{1}{2}\varepsilon)^{\gamma(n)/2} \qquad \text{by Lemma 8} \\ &\leqslant \varepsilon \qquad \text{for all } n > N_3\ (> N_2). \end{aligned}$$

Thus, with probability at least $1-\varepsilon$, the construction of J is successful.

We now argue more or less as in the proof of Lemma 6. Until further notice all expectations and probabilities are calculated conditional on the vertex set $A \cup B$ and relate only to the random disposition of edges joining A_{k-1} to B_{k-1}. Let V be the potential function on J with boundary conditions $V(0) = 0$, $V(\infty) = 1$. Let $a \in A_{k-1}$, $b \in B_{k-1}$, and write $(0, a_1, a_2, \ldots, a_{k-2}, a)$ and $(\infty, b_1, b_2, \ldots, b_{k-2}, b)$ for the unique paths in J which contain exactly k vertices and join 0 to a and ∞ to b respectively. Let $d_B(a)$ (respectively $d_A(b)$) be the number of vertices in B (respectively A) which are adjacent to a (respectively b) in J. By Chebyshev's inequality and the independence of the events $E(i,j)$ mentioned above,

$$P\left(\left|\frac{d_B(a)}{M} - 1\right| > \frac{\varepsilon}{16}\right) \leqslant \frac{256}{\varepsilon^2 M}, \qquad E\left|\frac{d_B(a)}{M} - 1\right| \leqslant \frac{1}{\sqrt{M}},$$

where, from (3.25) and (3.23),

$$M = E(d_B(a)) = E(d_A(b)) = \frac{1}{n}\gamma(n)\mu(n)^{k-2}\nu(n) \geqslant (1-\varepsilon)^k\sqrt{D}\,, \tag{3.27}$$

giving from (3.22) that

$$P\left(\left|\frac{d_B(a)}{M} - 1\right| > \frac{\varepsilon}{16}\right) \leqslant \frac{\varepsilon}{16}, \tag{3.28}$$

$$E\left|\frac{d_B(a)}{M} - 1\right| \leqslant \frac{\varepsilon}{16}, \tag{3.29}$$

together with similar inequalities in which $d_B(a)$ is replaced by $d_A(b)$. Arguing exactly as in equations (3.17) and (3.18), we find that

$$|E(d_B(a)V(a) - d_A(b)V(b))| = |E(V(a_{k-2}) - V(a))| \leqslant 1$$

(recall that $0 \leqslant V(\cdot) \leqslant 1$ by the maximum principle). Writing $I(F)$ for the indicator function of the event F, we have that, with $\eta = \varepsilon/16$,

$$\begin{aligned} \frac{1}{M} &\geqslant E\left(\frac{d_A(b)}{M}V(b)\right) - E\left(\frac{d_B(a)}{M}V(a)\right) \\ &\geqslant (1-\eta)E\{V(b)I(d_A(b) \geqslant (1-\eta)M)\} \\ &\quad -(1+\eta)E\{V(a)I(d_B(a) \leqslant (1+\eta)M)\} \\ &\quad - E\left\{\frac{d_B(a)}{M}I(d_B(a) > (1+\eta)M)\right\}. \end{aligned} \tag{3.30}$$

Also

$$\begin{aligned} E(V(b)) &\leqslant E\{V(b)I(d_A(b) \geqslant (1-\eta)M)\} + P(d_A(b) < (1-\eta)M) \\ &\leqslant E\{V(b)I(d_A(b) \geqslant (1-\eta)M)\} + \eta\,, \end{aligned} \tag{3.31}$$

$$E(V(a)) \geqslant E\{V(a)I(d_B(a) \leqslant (1+\eta)M)\} \tag{3.32}$$

and

$$\begin{aligned} E\left\{\frac{d_B(a)}{M}I(d_B(a) > (1+\eta)M)\right\} &\leqslant E\left|\frac{d_B(a)}{M} - 1\right| + P\left(\frac{d_B(a)}{M} > 1+\eta\right) \\ &\leqslant 2\eta\,. \end{aligned} \tag{3.33}$$

From (3.30)–(3.33)

$$E(V(b) - V(a)) \leqslant \frac{1}{M} + 5\eta\,;$$

a similar inequality holds with a and b interchanged, giving that

$$|E(V(b)-V(a))| \leqslant \frac{1}{M} + 5\eta .$$

By the obvious symmetry

$$E(V(a)) = 1 - E(V(b))$$

and so

$$|E(V(a)) - \tfrac{1}{2}| \leqslant \frac{1}{2M} + 3\eta .$$

Next consider a_{k-2}. Again by (2.2) and symmetry,

$$(1+\nu(n))E(V(a_{k-2})) = \nu(n)E(V(a)) + E(V(a_{k-3})),$$

giving that

$$|E(V(a_{k-2}) - V(a))| \leqslant \frac{1}{\nu(n)},$$

and hence

$$\begin{aligned} |E(V(a_{k-2})) - \tfrac{1}{2}| &\leqslant |E(V(a_{k-2}) - V(a))| + |E(V(a)) - \tfrac{1}{2}| \\ &\leqslant \frac{1}{\nu(n)} + \frac{1}{2M} + 3\eta . \end{aligned}$$

Iterate for $a_{k-3}, a_{k-4}, \ldots, a_1$ to find that

$$|E(V(a_1)) - \tfrac{1}{2}| \leqslant \frac{1}{\nu(n)} + \frac{k-3}{\mu(n)} + \frac{1}{2M} + 3\eta .$$

Hence the resistance $R(J_n)$, of J_n between 0 and ∞, satisfies, by (2.3) and (3.27),

$$\begin{aligned} E\left(\frac{1}{\gamma(n)R(J_n)}\right) &\geqslant (1-\varepsilon)E(V(a_1)) \\ &\geqslant \tfrac{1}{2}(1-\varepsilon)\left(1 - \frac{2k}{\mu(n)} - \frac{2}{\nu(n)} - \frac{1}{(1-\varepsilon)^k\sqrt{D}} - \frac{\varepsilon}{2}\right) \\ &\geqslant \tfrac{1}{2}(1-2\varepsilon) \qquad \text{for } n > N_4 \ (> N_3); \end{aligned}$$

we have used (3.22) and (3.19) here, and assumed that n is large enough to ensure that

$$\frac{2k}{\mu(n)} + \frac{2}{\nu(n)} < \frac{\varepsilon}{4}.$$

Now we can use the monotonicity principle (2.4) to deduce that

$$E\left(\frac{1}{\gamma(n)R_n}\right) \geqslant \tfrac{1}{2}(1-2\varepsilon).$$

Next we remove the condition that J has been successfully constructed, obtaining for the *unconditioned* mean value

$$E\left(\frac{1}{\gamma(n)R_n}\right) \geqslant E\left(\frac{1}{\gamma(n)R_n}\,\middle|\, J \text{ successful}\right) P(J \text{ successful}) \tag{3.34}$$

$$\geqslant \tfrac{1}{2}(1-2\varepsilon)(1-\varepsilon) \qquad \text{by (3.26)}$$

$$\geqslant \tfrac{1}{2}(1-3\varepsilon).$$

Set $X_n = (\gamma(n)R_n)^{-1}$ and observe that, by (3.9), all conducting edges have resistance 1, and hence by (3.5),

$$X_n \leqslant \frac{1}{\gamma(n)}\left(1+\tfrac{1}{4}(\tau^0+\tau^\infty)\right)$$

where, as before, τ^0 (respectively τ^∞) is the number of conducting edges between 0 (respectively ∞) and $\{1, 2, ..., n\}$. Therefore, we may pick $N_5 > N_4$ such that

$$E\{X_n I(X_n > \tfrac{1}{2}(1+\varepsilon))\} < \varepsilon \qquad \text{for } n > N_5.$$

Routine arguments yield that

$$E(X_n) \leqslant \tfrac{1}{2}(1-\tfrac{1}{2}\rho)P(X_n \leqslant \tfrac{1}{2}(1-\tfrac{1}{2}\rho)) + \tfrac{1}{2}(1+\varepsilon)P(\tfrac{1}{2}(1-\tfrac{1}{2}\rho) < X_n \leqslant \tfrac{1}{2}(1+\varepsilon))$$
$$+E\{X_n I(X_n > \tfrac{1}{2}(1+\varepsilon))\}$$
$$\leqslant -\tfrac{1}{4}(\rho+2\varepsilon)P(X_n \leqslant \tfrac{1}{2}(1-\tfrac{1}{2}\rho)) + \tfrac{1}{2}(1+\varepsilon)+\varepsilon, \qquad \text{if } n > N_5,$$

giving from (3.34) that

$$P(X_n \leqslant \tfrac{1}{2}(1-\tfrac{1}{2}\rho)) \leqslant \frac{12\varepsilon}{\rho+2\varepsilon} < \frac{12\varepsilon}{\rho} = \rho,$$

and hence

$$P(\gamma(n)R_n \geqslant 2(1+\rho)) \leqslant P\left(\frac{1}{\gamma(n)R_n} \leqslant \tfrac{1}{2}(1-\tfrac{1}{2}\rho)\right) < \rho.$$

This proves the lemma with $N(\rho) = N_5(\varepsilon)$.

As remarked before, Lemmas 6 and 7 together prove Theorem 4, when F has the form (3.9), because (1.6) implies that each sequence of integers has a subsequence satisfying (3.7) or (3.8). It therefore only remains to reduce the case of general F to the special case of (3.9). We now give this reduction.

Proof of Theorem 4. As we noted before Lemma 6, it suffices to prove the remaining part of Theorem 4 for the cases when either

$$\frac{1}{n}\gamma(n) \to \alpha \in (0, 1] \qquad \text{as } n \to \infty, \tag{3.35}$$

or

(3.36) $$\frac{1}{n}\gamma(n)^{k-2} \to 0\,, \quad \frac{1}{n}\gamma(n)^{k} \to \infty \qquad \text{as } n \to \infty\,,$$

for some k with $3 \leqslant k \leqslant \lceil \beta^{-1} \rceil + 1$. Suppose first that

(3.37) $$F(0) = \delta > 0\,.$$

(Actually one can avoid consideration of this special case by truncating F at the lower end, say by moving its mass on $[0, \varepsilon)$ to ε, and afterwards letting $\varepsilon \downarrow 0$.) If (3.35) holds, then one easily sees that, for large n, there exists with probability close to 1 a vertex $i \in \{1, \ldots, n\}$ which is connected to 0 and to ∞ by edges with zero resistance. Thus, in this case,

(3.38) $$P(R_n = 0) \to 1 \qquad \text{as } n \to \infty\,.$$

If (3.36) holds, then we compare R_n to the resistance $\tilde{R}_n$ between 0 and ∞ obtained by setting each non-zero resistance equal to ∞. By the monotonicity principle

$$R_n \leqslant \tilde{R}_n\,.$$

One easily verifies that the distribution of $\tilde{R}_n$ is the same as the resistance between 0 and ∞ in K_{n+2} in our original problem, when one replaces the distribution of $R(e)$ by

$$P\big(R(e) = \infty\big) = 1 - \frac{\delta\gamma(n)}{n}, \qquad P\big(R(e) = 0\big) = \frac{\delta\gamma(n)}{n}\,.$$

If (3.36) holds, one can construct J exactly as in Lemma 7, the only difference being that $\gamma(n)$ is replaced by $\delta\gamma(n)$ and that now each conducting edge has resistance 0 instead of resistance 1. Thus, if the construction of J is successful for the modified problem, then 0 and ∞ are connected by a path of zero resistance, and hence $\tilde{R}_n = 0$. Thus, under (3.37) and (3.36)

$$\begin{aligned} P(R_n = 0) &\geqslant P(\tilde{R}_n = 0) \\ &\geqslant P\,(\text{construction of } J \text{ is successful in the modified problem}) \\ &\to 1 \qquad \text{as } n \to \infty \end{aligned}$$

by (3.26). Hence

(3.39) $$P(R_n = 0) \to 1$$

as soon as (3.37) holds. Since this conclusion is stronger than (1.7) we may henceforth assume that

$$F(0) = 0\,.$$

We wish to show that, for all $\varepsilon > 0$,

$$(3.40)\qquad P\left(\gamma(n)R_n \geqslant 2(1+\varepsilon)\left\{\int_{[0,\infty)} \frac{1}{x}\,dF(x)\right\}^{-1}\right) \to 0 \qquad \text{as } n \to \infty\,,$$

since, together with Lemma 5, this implies that Theorem 4 holds. To obtain (3.40) for general F we modify the resistances of the edges of K_{n+2} as follows. Choose $\eta > 0$ and an integer M. If $R(e) \in [j\eta, (j+1)\eta)$, set $\bar{R}(e) = (j+1)\eta$, for $0 \leqslant j < M$; if $R(e) \geqslant M\eta$, set $\bar{R}(e) = \infty$. Let G_{nj} be the network formed by the edges with resistances $\bar{R}(e) = (j+1)\eta$ and the vertices which are endpoints of these edges. These networks form edge-disjoint connections between 0 and ∞. Let R_{nj} be the resistance between 0 and ∞ in G_{nj}. Then (2.1) and (2.4) imply that

$$(3.41)\qquad \frac{1}{R_n} \geqslant \sum_{j=0}^{M-1} \frac{1}{R_{nj}}\,.$$

In addition, all conducting edges in G_{nj} have the (non-random) resistance $(j+1)\eta$, so that Lemmas 6 and 7 apply with $\gamma(n)$ replaced by $\gamma(n)F\{[j\eta, (j+1)\eta)\}$ (in the obvious notation), and it follows from Lemmas 5–7 that

$$\gamma(n)R_{nj} \to \frac{2(j+1)\eta}{F\{[j\eta, (j+1)\eta)\}} \qquad \text{in probability}\,.$$

Combined with (3.41) this yields, for each η, $\varepsilon > 0$ and integer M,

$$P\left(\frac{1}{\gamma(n)R_n} \leqslant \tfrac{1}{2}(1-\varepsilon)\sum_{j=0}^{M-1} \frac{1}{(j+1)\eta} F\{[j\eta, (j+1)\eta)\}\right) \to 0$$

as $n \to \infty$. Equation (3.40) now follows by letting $M \to \infty$, $\eta \downarrow 0$ and $\varepsilon \downarrow 0$, in that order. This argument works when $F(0) > 0$ also, but does not show the stronger conclusion (3.39).

4. *The remaining results*

We defer discussion of the part of Theorem 1 which is not covered by Theorem 4, and begin by considering the case when

$$(4.1)\qquad \gamma(n) \to \gamma \in [0, \infty)\,, \qquad \text{as } n \to \infty\,,$$

and

$$(4.2)\qquad F(\varepsilon) = 0 \qquad \text{for some } \varepsilon > 0\,.$$

Condition (4.2) implies that each edge-resistance is a.s. bounded away from 0. The first major step in proving Theorems 2 and 3 is to show that the subgraph G_n of G, containing the conducting edges of G only, may be approximated by two disjoint trees, rooted at 0 and ∞, together with certain interconnections between these trees.

For each vertex j, define $d_0(j)$ (respectively $d_\infty(j)$) to be the number of edges in the shortest path of G_n joining j to 0 (respectively ∞), and write

$$A_i = \{j : d_0(j) = i\}, \qquad B_i = \{j : d_\infty(j) = i\}.$$

For fixed $k \geqslant 1$, the graph formed from the vertex set $A_{[k]} = \{0\} \cup A_1 \cup \ldots \cup A_k$ together with the edges inherited from G_n tends in distribution as $n \to \infty$ to the family tree of the first k generations of a branching process whose offspring distribution is the Poisson distribution with mean γ. A similar remark holds with $A_{[k]}$ replaced by $B_{[k]} = \{\infty\} \cup B_1 \cup \ldots \cup B_k$ together with the edges inherited from G_n. Furthermore, $P(A_{[k]} \cap B_{[k]} = \varnothing) \to 1$ as $n \to \infty$. It follows that R_n is at least the resistance between 0 and ∞ in the network obtained from $A_{[k]} \cup B_{[k]}$ when the vertices in $A_k \cup B_k$ are short-circuited. In the notation of Section 1, for $x \geqslant 0$,

$$\text{(4.3)} \qquad \limsup_{n\to\infty} P(R_n \leqslant x) \leqslant P(R'_{[k]} + R''_{[k]} \leqslant x) \to P\big(R'(\gamma) + R''(\gamma) \leqslant x\big) \quad \text{as } k \to \infty,$$

where $R'_{[k]}$ and $R''_{[k]}$ are independent random variables, each distributed like $R(T_{[k]})$. It is more difficult to show the converse inequality to (4.3); the argument proceeds roughly as follows. Let $1 < \delta < \gamma$ and define

$$k = k(n) = \left\lfloor \frac{3 \log n}{4 \log \gamma} \right\rfloor.$$

One shows that $A_{[k(n)]}$ and $B_{[k(n)]}$ contain, with probability close to 1, tree subgraphs of G_n which are independent and disjoint, and which are distributed like the first $k(n)$ generations of a branching process whose offspring distribution is the Poisson distribution with mean δ. Furthermore, with probability close to 1, there exist sufficiently many edges of G_n between $A_{k(n)}$ and $B_{k(n)}$ to ensure that the effective resistance between 0 and ∞ of this subnetwork is not very much greater than $R'(\delta) + R''(\delta)$. When correctly formulated this yields, for $x \geqslant 0$,

$$\text{(4.4)} \qquad \liminf_{n\to\infty} P(R_n \leqslant x) \geqslant P\big(R'(\delta) + R''(\delta) \leqslant x\big)$$

and the result follows by letting $\delta \uparrow \gamma$ and using a continuity argument. There are special arguments which allow us to remove condition (4.2), and which show that

$$\lim_{n\to\infty} P(R_n = \infty) = 1 - (1 - q(\gamma))^2,$$

where $q(\gamma)$ is the extinction probability of a branching process whose offspring distribution is the Poisson distribution with mean γ.

The unproven part of Theorem 1 asserts that

$$\text{(4.5)} \qquad \text{if } \gamma(n) \to \infty \text{ then } \gamma(n) R_n \to 2\left\{ \int_{[0,\infty)} \frac{1}{x}\, dF(x) \right\}^{-1} \quad \text{in probability}.$$

The proof of Lemma 5 requires only that $\gamma(n) \to \infty$, and so (4.5) is shown as soon as we know that, if $\gamma(n) \nrightarrow \infty$, then for $\varepsilon > 0$,

$$P\left(\gamma(n)R_n \leqslant 2(1+\varepsilon)\left\{\int\limits_{[0,\infty)} \frac{1}{x}\,dF(x)\right\}^{-1}\right) \to 1\,. \tag{4.6}$$

This may be shown in very much the same way as (4.4). In the paragraph before (4.4), we replace $k(n)$ by

$$k(n) = \left\lfloor \frac{3\log n}{4\log\gamma(n)} \right\rfloor$$

and show that, for each $0 < \varepsilon < 1$, the corresponding graphs on $A_{[k(n)]}$ and $B_{[k(n)]}$ contain, with probability close to 1, disjoint copies of the family tree of the first $k(n)$ generations of a branching process each of whose families has size exactly $\lfloor(1-\varepsilon)\gamma(n)\rfloor$. Furthermore there exist sufficiently many interconnections between these two trees for the resistance between 0 and ∞, in the network to which they give rise, to be close to that obtained by short-circuiting all the vertices in $A_{k(n)} \cup B_{k(n)}$ (very much as in the proof of Lemma 7) and (4.6) follows.

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