ON THE DIFFERENTIABILITY OF THE NUMBER OF CLUSTERS PER VERTEX IN THE PERCOLATION MODEL

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ABSTRACT

The number $\lambda(p)$ of clusters per vertex of the vertex percolation process on the two-dimensional square lattice is once differentiable for all p and is infinitely differentiable except possibly on the interval $[p_T, p_H]$. Also $\lambda'(p)$ may be expressed in terms of the mean number of black clusters containing vertices adjacent to the white origin. Easy proofs are given of two theorems concerning the boundary sizes of the black cluster containing the origin and the infinite black cluster, when it exists. A central limit theorem is established for the latter quantity. Similar results may be established for certain other two-dimensional lattices.

1. *Introduction*

Exact values of critical percolation probabilities for lattices containing circuits remain unknown. Sykes and Essam [22] discussed an attractive, though nonrigorous, argument which gave precise indications about numerical values for certain two-dimensional processes. Recently, Seymour and Welsh [19] and Russo [18] have pursued, independently, a new approach to the problem. Their methods are very similar and show the existence of two critical points p_T and p_H, thought to be equal, with the property that, broadly speaking, percolation probabilities and mean cluster sizes are well behaved functions of the underlying probability p except possibly on the (perhaps trivial) interval $[p_T, p_H]$. The main purpose of this paper is to give a further justification of the argument of Sykes and Essam and to indicate that, in some sense, their assumptions may be equivalent to the assumption that $p_T = p_H$.

I am concerned here with the vertex percolation process on the two-dimensional square lattice. In Section 3 I give easy proofs of two theorems of Leath [13, 14] and Hankey [9], who compare the boundary size to the cluster size for the cluster containing the origin and the infinite cluster respectively. An estimate for the tail of the distribution of the size of a finite cluster provides a strong mixing condition which enables me to derive a central limit theorem for the contents of the infinite cluster and its boundary.

A partial justification of the argument of Sykes and Essam is contained in Grimmett [7]. That paper uses the theory of subadditive stochastic processes to show that the mean number $\lambda(p)$ of clusters per vertex exists. Since then Smythe and Wierman [20] have shown that λ has an explicit representation in terms of the mean value of the reciprocal of the size of the black cluster containing the origin. Sykes and Essam remark that, if we can show that λ is infinitely differentiable except at p_H, then knowledge of exact values for certain critical probabilities will follow. Here, I show that λ is once differentiable for all p and has all its derivatives except possibly on the interval $[p_T, p_H]$. There is a representation of its first derivative in terms of the mean number of black clusters which contain vertices adjacent to the white origin. This

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provides a rigorous link between critical behaviour of λ and of the percolation pair correlation functions.

In the study of percolation processes to date, most attention has been paid to edge and vertex percolation on the square lattice. It is likely that many of the known results, including the results of this paper, may be successfully reformulated in terms of pairs of matching lattices (see Sykes and Essam [**22**] and Essam [**6**] for definitions and discussion).

For a review of percolation theory see the monograph by Smythe and Wierman [**20**] and the references therein.

2. *Notation*

Let $\mathscr{L}$ be the plane square lattice, and let $\mathscr{L}^*$ be the lattice derived from $\mathscr{L}$ by adding all diagonals to all the faces of $\mathscr{L}$. In the following, I shall define certain quantities associated with $\mathscr{L}$; the corresponding quantities defined on $\mathscr{L}^*$ will be marked with an asterisk. Each vertex of $\mathscr{L}$ is coloured *black* with probability $p = 1-q$, where p takes a fixed value in $[0, 1]$. Denote by β_x the indicator function of the event that x is black. We colour vertices *white* if they are not coloured black; each vertex is coloured independently of all other vertex colourings. The colouring process decomposes $\mathscr{L}$ and $\mathscr{L}^*$ into the unions, ω and ω^* respectively, of black *clusters* (connected components). For each black vertex $x \in \mathscr{L}$, denote by γ_x the black cluster of $\mathscr{L}$ which contains x. Of course, $\gamma_x = \varnothing$, the empty set, if x is white.

If A is any nonempty subgraph of $\mathscr{L}$, its *boundary* ∂A is the set of vertices of $\mathscr{L}$ which are not members of A but which are adjacent to some vertex in A. The *internal boundary* ΔA is the set of vertices of A which are adjacent to some member of ∂A. If γ is a black cluster in $\mathscr{L}$, then all vertices in $\partial\gamma$ are white; contrary to remarks by other authors, $\partial\gamma$ may not be connected in $\mathscr{L}^*$. The *size* $|A|$ of any subgraph A of $\mathscr{L}$ is the number of vertices which it contains. The lattice $\mathscr{L}$ has a distinguished vertex called the *origin* and denoted by 0. If the origin is black then the black cluster γ_0 of $\mathscr{L}$ containing it will often be denoted by γ for notational simplicity.

Let $P_n(p) = P(|\gamma| \geqslant n)$. Then $P_n(p)$ is monotone nonincreasing as $n \to \infty$ and converges to a limit

$$P_n(p) \downarrow P_\infty(p) = P(|\gamma| = \infty)$$

called the *percolation probability*. Note that $P_1(p) = p$. The *critical probability* p_H is given by

$$p_H = \sup\{p : P_\infty(p) = 0\}.$$

Harris's argument [**10**] may be used to show that

$$\text{if} \quad P_\infty(p) > 0 \quad \text{then} \quad P_\infty^*(1-p) = 0, \tag{2.1}$$

where P_∞^* is the quantity defined as P_∞ but with respect to $\mathscr{L}^*$. This yields that $p_H + p_H^* \geqslant 1$; it is very likely that equality holds in this relation. Consider another critical probability:

$$p_T = \sup\{p : E|\gamma| < \infty\}.$$

It is clear that $p_T \leqslant p_H$ and $p_T^* \leqslant p_H^*$. Russo [**18**] has proved that

$$p_H + p_T^* = 1, \qquad p_H^* + p_T = 1\,; \tag{2.2}$$

he remarks that the method works for an arbitrary pair of matching lattices. An independent but almost identical proof of the corresponding result for bond percolation on the square lattice appears in Seymour and Welsh [**19**].

As a consequence of Harris [**10**], if $p > p_H$ then there exists almost surely a single infinite black cluster in $\mathscr{L}$; call this cluster σ.

For any pair x, y of vertices of $\mathscr{L}$, the *distance* $d(x, y)$ between x and y is the number of vertices in the shortest *path* $x = x_1, x_2, x_3, \ldots, x_{n-1}, x_n = y$ ($x_i \neq x_j$ for $i \neq j$) from x to y, excluding the starting point. The *length* of such a path is n.

Sometimes I will be concerned with the limits of sequences, the members of which are defined on finite regions of the lattice; the limits will be taken as these regions expand to fill out the lattice in every direction. For a precise specification of such expanding sequences for a general lattice see Grimmett [**8**]. For simplicity I shall restrict my attention here to an increasing sequence of square boxes

$$L_N = \{(x, y) \in \mathscr{L} : |x| \leqslant N,\, |y| \leqslant N\}$$

which expand to fill out $\mathscr{L}$ as $N \to \infty$. The box L_N contains $V(N) = (2N+1)^2$ vertices. More general results will be available for other expanding sequences $\{L_N\}$ of connected regions subject to certain conditions such as

$$|\partial L_N| = o(|L_N|)\,;$$

see Neaderhouser [**15, 16**] for an example of the use of such regions.

3. *Cluster boundary size and a central limit theorem*

Many authors have considered the problem of comparing $|\partial\gamma|$ with $|\gamma|$; a general discussion of the results to date is provided by Stauffer [**21**]. Of principal physical interest is the limiting behaviour of $n^{-1}E(|\partial\gamma| \mid |\gamma| = n)$. Kunz and Souillard [**12**] give a rigorous argument to show that this ratio converges to qp^{-1} when $p > p_H$. Here, I give very easy rigorous proofs of two theorems of Leath [**13, 14**] and Hankey [**9**] concerning the boundary sizes of γ and the infinite cluster σ respectively. The method which I use may already be known; certainly D. J. A. Welsh (personal communication) has used it independently to obtain these results.

THEOREM 3.1. *For any* L_N,

$$p^{-1}E|\gamma \cap L_N| - q^{-1}E|\partial\gamma \cap L_N| = 1\,.$$

Note that if $p < p_T$ then all the moments of γ are finite and, letting $N \to \infty$ the theorem asserts that $p^{-1}E|\gamma| - q^{-1}E|\partial\gamma| = 1$. Leath gives a 'proof' that this holds for $p < p_H$ by differentiating the relation

$$P_\infty(p) = p - \sum p^{|\gamma|}q^{|\partial\gamma|},$$

where the summation is over all γ with $1 \leqslant |\gamma| < \infty$, and assuming the uniform convergence of the term by term derivative. This can be justified for $p < p_T$. I will give an alternative proof.

Next I study the asymptotic behaviour of the numbers $\Sigma_N = |\sigma \cap L_N|$ and $\partial\Sigma_N = |\partial\sigma \cap L_N|$ of vertices of the infinite cluster σ and its boundary $\partial\sigma$ which are in L_N, for values of p greater than p_H.

THEOREM 3.2. *If $p > p_H$ then*

$$\lim_{N \to \infty} \partial\Sigma_N/\Sigma_N = qp^{-1} \text{ a.e. and in any mean.}$$

It follows from the proof of this theorem that 0 is a boundary vertex of the infinite cluster with probability $qp^{-1}P_\infty(p)$.

THEOREM 3.3. *The random vector $N^{-1}(\Sigma_N - E\Sigma_N, \partial\Sigma_N - E\partial\Sigma_N)$ converges in distribution as $N \to \infty$ to a bivariate normal distribution.*

Similar limit theorems (see Cox and Grimmett [3]) may be established for the number of vertices of L_N which are joined by black paths to ΔL_N; such results have been conjectured by Kunz and Souillard [12]. It is also possible to prove laws of the iterated logarithm by applying the results contained in Neaderhouser [16].

The proof of Theorem 3.3 requires a lemma which I shall use in the next section also. When $p > p_H$ there exists almost surely a single infinite black cluster σ in $\mathscr{L}$. Consider the complementary graph $\mathscr{L}\setminus\sigma$ comprising all vertices and edges of $\mathscr{L}$ not in σ without reference to the vertex colours. The origin 0 is almost surely in some maximal finite subset H of this graph which is connected in $\mathscr{L}^*$ and whose boundary ∂H in $\mathscr{L}$ is black; H and ∂H are empty sets if $0 \in \sigma$. Russo calls H the *hole* containing the origin. Let

$$q_n(p) = P(|H| = n), \qquad Q_n(p) = P(|H| \geqslant n)$$

and put

$$p_n(p) = P(|\gamma| = n).$$

A recent estimate of Kesten [11] allows us to establish the following upper bound for the p_n.

LEMMA 3.4. *There exist $A > 0$ and $\rho \in (0, 1)$ such that if $p > p_H$ then*

$$p_n(p) \leqslant Q_n(p) \leqslant A\rho^{\sqrt{n}},$$

and if $p < p_T$ then

$$p_n(p) \leqslant A\rho^{\sqrt{n}}.$$

Seymour, Welsh and Russo use an argument which implies that for $p > p_H$, $Q_n(p)$ decays faster than any power law in n. It is possible that the p_n decay exponentially for $p < p_T$. Kunz and Souillard [12] show that the decay is exponential for p not greater than some $p' < p_T$. For $p > p_H$ they show that the p_n cannot decay exponentially and they make a conjecture which amounts to

$$p_n \sim A\rho^{\sqrt{n}}, \quad \text{for some } A, \rho,$$

for two-dimensional lattices. They show that for p greater than some $p'' > p_H$ there exist upper and lower bounds for p_n of this form.

Proof of Theorem 3.1. Let ξ_x and η_x be the indicator functions of the events that $x \in \gamma$ and $x \in \partial\gamma$ respectively. Then

$$E|\partial\gamma \cap L_N| = E\left(\sum_{x \in L_N} \eta_x\right) = \sum_{x \in L_N} P(\eta_x = 1) = \sum_{\substack{x \in L_N \\ x \neq 0}} qp^{-1}P(\xi_x = 1),$$

because recolouring x black instead of white would have the effect of including x in γ. Thus

$$E|\partial\gamma \cap L_N| = qp^{-1}\left(\sum_{x \in L_N} E(\xi_x) - p\right) = qp^{-1}(E|\gamma \cap L_N| - p).$$

Proof of Theorem 3.2. Let ν_x and χ_x be the indicator functions of the events that $x \in \partial\sigma$ and $x \in \sigma$ respectively. Then

$$(2N+1)^{-2}|\partial\sigma \cap L_N| = (2N+1)^{-2} \sum_{x \in L_N} \nu_x$$

$$\to P(\nu_0 = 1) = qp^{-1}P_\infty(p) \text{ a.e. and in any mean,}$$

by an ergodic theorem (see for example Dunford [5]) and the observation that if 0 had been black instead of white then it would have been included in the infinite cluster by virtue of its adjacency to a member of σ. Also

$$(2N+1)^{-2} \sum_{x \in L_N} \chi_x \to P(\chi_0 = 1) = P_\infty(p) \text{ a.e. and in any mean.}$$

Convergence of the ratio of these two quantities follows immediately, noting that $0 \leqslant \dfrac{|\partial\sigma \cap L_N|}{|\sigma \cap L_N|} \leqslant 4$ whenever the denominator is nonzero.

Proof of Lemma 3.4. Kesten [11] shows that, for bond percolation on the square lattice $\mathscr{L}$,

$$P(\exists \text{ a black path of length exceeding } n \text{ from } 0) \leqslant 2\rho_1^n$$

for some fixed $\rho_1 \in (0, 1)$ whenever $p < p_T^B$, the critical bond probability for $\mathscr{L}$. His argument may be rewritten in terms of vertex percolation on the covering lattice of $\mathscr{L}$; the same argument may be used to establish corresponding results for vertex percolation on $\mathscr{L}$ and $\mathscr{L}^*$. Thus, for some $a > 0$ and $\rho \in (0, 1)$,

$$p_n(p) \leqslant P(\exists \text{ a black path of length } a\sqrt{n} \text{ from } 0) \leqslant 2\rho^{\sqrt{n}}$$

if $p < p_T$. Now suppose $p > p_H$. Then for some $b > 0$,

$$\begin{aligned}\{|H| = n\} &\subseteq \{|H| = n, \partial H \text{ contains a white circuit in } \mathscr{L}^* \text{ of length } > b\sqrt{n}\} \\ &\subseteq \{|H| = n, \exists \text{ a white path in } \mathscr{L}^* \text{ of length } \geqslant b\sqrt{n} \\ &\qquad \text{starting from a vertex distance } \leqslant n \text{ from } 0\}.\end{aligned}$$

But $q < p_T^*$ and so

$$q_n(p) \leqslant A_2 n^2 \rho_2^{\sqrt{n}} \text{ for some } A_2 > 0 \text{ and } \rho_2 \in (0, 1),$$

showing that

$$p_n(p) \leqslant Q_n(p) \leqslant A_3 \rho_3^{\sqrt{n}} \text{ for some } A_3 > 0 \text{ and } \rho_3 \in (0, 1).$$

Proof of Theorem 3.3. Let $\theta_x = 1 - \chi_x - \nu_x$ be the indicator function of the event that $x \notin \sigma \cup \partial\sigma$, and let

$$\Sigma_N = \sum_{x \in L_N} \chi_x, \qquad \Theta_N = \sum_{x \in L_N} \theta_x .$$

I propose to use Theorem 3.2 of Neaderhouser [**16**] and the Cramèr-Wold device to show that pair (Σ_N, Θ_N), when suitably normalized, is asymptotically bivariate normal. The theorem for the pair $(\Sigma_N, \partial\Sigma_N)$ will follow because

$$\Sigma_N + \partial\Sigma_N + \Theta_N = V(N) = (2N+1)^2 .$$

Firstly I need to show a strong mixing condition (condition (M) of Neaderhouser [**16**]). Let A and B be disjoint finite vertex sets of $\mathscr{L}$ and let $m = \min\{d(a,b) : a \in A, b \in B\}$ be the distance between them. Let E_1 and E_2 be events defined in terms of $\{\chi_x, \theta_x : x \in A\}$ and $\{\chi_x, \theta_x : x \in B\}$ respectively. I claim that E_1 and E_2 satisfy the mixing condition

$$|P(E_1 E_2) - P(E_1)P(E_2)| \leqslant |A|\,|B|\alpha(m)$$

where

$$\int_1^\infty (\alpha(m))^{1/5} m^3 dm < \infty .$$

For, let

$$A^+ = \{x \in \mathscr{L} : d(x,a) < \tfrac{1}{2}m \text{ for some } a \in A\},$$

$$B^+ = \{x \in \mathscr{L} : d(x,b) < \tfrac{1}{2}m \text{ for some } b \in B\}.$$

Introduce a new black vertex v_∞ specified to be adjacent to each vertex in $\Delta A^+ \cup \Delta B^+$ and to no other. For any $x \in A^+ \cup B^+$ let χ_x^+ be the indicator function of the event that x is in the black cluster γ_∞ containing v_∞ and let θ_x^+ be the indicator function of the event that x is neither in γ_∞ nor in $\partial\gamma_\infty$. I claim that the events

$$F_A = \{\chi_x = \chi_x^+,\ \theta_x = \theta_x^+,\ \text{for all } x \in A\},$$

$$F_B = \{\chi_x = \chi_x^+,\ \theta_x = \theta_x^+,\ \text{for all } x \in B\}$$

have very large probability when m is large. For, if F_A does not occur then some vertex a of $A \cup \partial A$ is in a finite black cluster of $\mathscr{L}$ which intersects ΔA, no member of which is within distance $\frac{1}{2}m-1$ of a. Thus the probability that F_A fails satisfies

$$P(\bar{F}_A) \leqslant |A \cup \partial A|\pi_{\frac{1}{2}m-2} \leqslant |A|k\rho^{\sqrt{m}}$$

for some $k > 0$ and $\rho \in (0, 1)$, by Lemma 3.4, where

$$\pi_m(p) = \sum_{n \geqslant m} p_n(p) = P_m(p) - P_\infty(p)$$

is the probability that the origin is in a finite black cluster of size at least m. Similarly

$$P(\bar{F}_B) \leqslant |B|k\rho^{\sqrt{m}}.$$

Let E_1^+ and E_2^+ be events defined exactly as E_1 and E_2 but in terms of the χ_x^+ and θ_x^+ instead of the χ_x and θ_x; for example, if $E_1 = \{\chi_x = 1\}$ then $E_1^+ = \{\chi_x^+ = 1\}$. Now $E_1 \,\Delta\, E_1^+ \subseteq \bar{F}_A$ and $E_2 \,\Delta\, E_2^+ \subseteq \bar{F}_B$, and elementary set operations together with the observation that E_1^+ and E_2^+ are independent may be used to show that

$$|P(E_1E_2) - P(E_1)P(E_2)| \leqslant \tfrac{5}{2}(P(\bar{F}_A) + P(\bar{F}_B))$$

$$\leqslant 10|A|\,|B|k\rho^{\sqrt{m}}.$$

(Here, Δ denotes symmetric difference.) Hence the mixing condition holds with $\alpha(m) = 10k\rho^{\sqrt{m}}$.

Let $s_N^2 = \operatorname{var}(\Sigma_N)$, $t_N^2 = \operatorname{var}(\Theta_N)$, $\rho_N = \operatorname{cov}(\Sigma_N, \Theta_N)$. Expand these three quantities to obtain

$$s_N^2/V(N) \to s^2, \qquad t_N^2/V(N) \to t^2, \qquad \rho_N/V(N) \to \rho \qquad \text{as } N \to \infty,$$

where

$$s^2 = \sum_x \operatorname{cov}(\chi_0, \chi_x), \qquad t^2 = \sum_x \operatorname{cov}(\theta_0, \theta_x), \qquad \rho = \sum_x \operatorname{cov}(\chi_0, \theta_x).$$

Note that $s^2 > 0$, $t^2 > 0$, $\rho < 0$ since $\{\chi_x, -\theta_x : x \in \mathscr{L}\}$ are positively correlated, being increasing functions on the set-theoretic lattice of colour configurations with the partial order inherited from the inclusion relation on the set of black vertices.

Let X and Y be a pair of random variables with the bivariate normal distribution with zero means, variances s^2 and t^2, and covariance ρ. I claim that $V(N)^{-1/2}(\Sigma_N - E\Sigma_N, \Theta_N - E\Theta_N) \to (X, Y)$ in distribution. It is sufficient, by the Cramèr-Wold device (see Billingsley [1; p. 49]), to show that

$$V(N)^{-1/2}\big(u(\Sigma_N - E\Sigma_N) + v(\Theta_N - E\Theta_N)\big) \to uX + vY \text{ in distribution}$$

for all real u and v. Let

$$\delta_x = u\chi_x + v\theta_x, \qquad \Delta_N = \sum_{x \in L_N} \delta_x = u\Sigma_N + v\Theta_N .$$

Then Δ_N is the sum of identically distributed uniformly bounded random variables whose joint distributions are invariant under lattice shifts. Also

$$\operatorname{var}(\Delta_N)/V(N) \to q(u, v) = u^2 s^2 + 2uv\rho + v^2 t^2 .$$

Neaderhouser's Theorem (3.2) provides the required central limit theorem for Δ_N whenever the quadratic form $q(u, v)$ is nonzero. It is easy to see that $q(u, v) = 0$ if and only if the correlation $\rho(st)^{-1}$ between X and Y takes the value -1 and $us = vt$. But if this holds then $\operatorname{var}(\Delta_N)/V(N) \to 0$ and so $(\Delta_N - E\Delta_N)V(N)^{-1/2} \to uX + vY$ in distribution, trivially, since the right hand side is almost surely zero.

4. *The number of clusters per vertex*

The black vertices of $\mathscr{L}$ form connected clusters. Let $K(N)$ be the number of black clusters in the finite graph L_N. In Grimmett [7] it is shown that

$$K(N)/V(N) \to \lambda(p) \text{ a.e. and in any mean as } N \to \infty \tag{4.1}$$

where $V(N) = (2N+1)^2$ is the number of vertices in L_N and $\lambda(p)$ is a constant called the *number of clusters per vertex* of the percolation process. The proof made use of the fact that $\{K(N)\}$ is part of a two-dimensional subadditive stochastic process. It is not strongly subadditive (see Nguyen [17]) but, with slight modifications, it is also superadditive; this allows us to show that the convergence takes place a.e. as well as in any mean.

The quantity $\lambda(p)$ was used by Sykes and Essam in their attractive heuristic calculation of certain critical probabilities. They showed that

$$\lambda(p) = \lambda^*(1-p) + \phi(p) \tag{4.2}$$

where $\phi(p)$ is a finite polynomial in p and λ^* is the quantity analogous to λ but defined on $\mathscr{L}^*$. Under the assumption that λ (λ^*) has all its derivatives except at p_H (p_H^*) it follows that $p_H + p_H^* = 1$. In this section I am concerned with exploring the existence of the derivatives of λ. Theorem 4.3 provides an exact representation of $\lambda'(p)$.

Firstly I give a short and easy proof of the existence of λ which does not use subadditivity, and which yields the explicit representation of λ which was found by Smythe and Wierman [**20**]. There is evidence (see Kunz and Souillard [**12**; p. 81]) that this representation may be known to others; they give no indication of proof. It is interesting to observe that K is one of the few known subadditive processes whose limit has an explicit known representation.

THEOREM 4.1. *The limit $\lambda(p)$ exists in* (4.1) *and is given by*

$$\lambda(p) = E(\beta_0 |\gamma_0|^{-1}) = pE(|\gamma|^{-1} \mid 0 \textit{ is black}) .$$

THEOREM 4.2. *The function λ is infinitely differentiable for all p except possibly on the interval $[p_T, p_H]$.*

THEOREM 4.3. *The function λ is differentiable for all p with derivative*

$$\lambda'(p) = 1 - q^{-1}\mu(p)$$

where $\mu(p) = E((1-\beta_0)\kappa)$ and κ is the number of black clusters, including the infinite cluster if it exists, which contain vertices which are adjacent to the white origin.

Proof of Theorem 4.1. For any L_N and any black $x \in L_N$ let $g(x) = |\gamma_x|$ be the size of the black cluster of $\mathscr{L}$ containing x and $g(x, N)$ be the size of the black cluster of L_N containing x. Define

$$D(x, N) = \begin{cases} g(x, N)^{-1} - g(x)^{-1} & \text{if } x \text{ is black}, \\ 0 & \text{if } x \text{ is white}. \end{cases}$$

Consider

$$S(N) = \sum_{x \in L_N} D(x, N).$$

I intend to show that $S(N)$ is small compared with $V(N)$. Roughly speaking this is because if $D(x, N) > 0$ then either x is not far away from ∂L_N or x is near the centre of L_N. In the first case the contribution to $S(N)$ is small because there are not many such x; in the second case $g(x, N)$ will be quite large, ensuring that $D(x, N) \leqslant g(x, N)^{-1}$ will be small.

More precisely, let $n > 0$, and let $R_1 \subseteq L_N$ be the set of vertices of L_N which are joined to ∂L_N by paths of length not exceeding n; $R_2 = L_N \setminus R_1$. Then

$$S(N) = \textstyle\sum_1 D(x, N) + \sum_2 D(x, N)$$

where $\sum_i$ is over all $x \in R_i$. But

$$\textstyle\sum_1 D(x, N) \leqslant |R_1|, \qquad \sum_2 D(x, N) \leqslant |R_2| n^{-1}$$

because $D(x, N) \leqslant 1$ for all $x \in R_1$ and $D(x, N) \leqslant g(x, N)^{-1} \leqslant n^{-1}$ if $x \in R_2$ and $D(x, N) > 0$. Thus

$$S(N)/V(N) \leqslant (|R_1| + n^{-1}|R_2|)/V(N) \to n^{-1} \qquad \text{as } N \to \infty$$

since $|R_1| = o(|\partial L_N|)$ and $|R_2| = V(N) - |R_1|$. But n was arbitrary and it follows that $S(N)/V(N) \to 0$ a.e. and in any mean.

Now write $S(N)$ in another way:

$$S(N) = \sum_{\substack{x \in L_N \\ x \text{ black}}} g(x, N)^{-1} - \sum_{\substack{x \in L_N \\ x \text{ black}}} . g(x)^{-1} \tag{4.3}$$

$$= K(N) - \sum_{x \in L_N} \beta_x g(x)^{-1} \tag{4.4}$$

since any cluster in L_N of size $g(x, N)$ accounts for exactly $g(x, N)$ contributions of size $g(x, N)^{-1}$ to the first sum in (4.3).

The final sum of (4.4) is the sum of a family of identically distributed random variables which is stationary under the operations of translation of the lattice. Thus, by an ergodic theorem (see for example Dunford [5]),

$$V(N)^{-1} \sum_{x \in L_N} \beta_x g(x)^{-1} \to E(\beta_0 g(0)^{-1}) \text{ a.e. and in any mean.} \tag{4.5}$$

We see that $K(N)/V(N)$ converges also to the limit in (4.5) as required.

Proof of Theorem 4.2. From Theorem 4.1, $\lambda(p)$ may be written

$$\lambda(p) = \sum_{\substack{n \geqslant 1 \\ b \geqslant 1}} n^{-1} p^n (1-p)^b a(n, b), \tag{4.6}$$

where $a(n, b)$ is the number of clusters γ containing 0 with $|\gamma| = n$ and $|\partial\gamma| = b$. Successive differentiation term by term of (4.6) yields expression involving sums of the form

$$S_{\alpha,\beta} = \sum_{n, b \geqslant 1} b^\alpha n^\beta p^n (1-p)^b a(n, b)$$

for integers α, β with $\alpha \geqslant 1$, $\beta \geqslant -1$. The tail of $S_{\alpha,\beta}$ is

$$\sum_{\substack{n \geqslant N \\ b \geqslant B}} b^\alpha n^\beta p^n (1-p)^b a(n, b) \leqslant 4 \sum_{n \geqslant N} n^{\alpha+\beta} p_n,$$

by the upper bound of the inequalities for $|\partial\gamma|$:

$$\sqrt{|\gamma|} \leqslant |\partial\gamma| \leqslant 4|\gamma|. \tag{4.7}$$

But, if $\varepsilon > 0$ and $p < p_T$ then this sum is bounded above by

$$4 \sum_{n \geqslant N} n^{\alpha+\beta} P_n(p_T - \varepsilon)$$

which converges to 0 as $N \to \infty$ by Lemma 3.4; thus $S_{\alpha,\beta}$ converges to 0 uniformly on $[0, p_T - \varepsilon]$ and we may deduce that $\lambda(p)$ has all its derivatives on $[0, p_T)$. A similar argument shows that all its derivatives exist on $(p_H, 1]$ by the bound $p_n \leqslant Q_n(p_H + \varepsilon)$ for $p \geqslant p_H + \varepsilon$. Alternatively one might use the previous argument to show that $\lambda^*(p)$ has all its derivatives on $[0, p_T^*)$ and then use relation (4.2).

Proof of Theorem 4.3. From (4.6) the term by term derivative of λ is

$$p^{-1} \sum_{n, b \geqslant 1} p^n (1-p)^b a(n, b) - q^{-1} \sum_{n, b \geqslant 1} b n^{-1} p^n (1-p)^b a(n, b). \tag{4.8}$$

We wish to show that both these series converge uniformly on suitable subintervals of [0, 1]. The tails of the two series may be considered simultaneously. For, by (4.7),

$$\sum_{\substack{n \geqslant N \\ b \geqslant B}} bn^{-1}p^n(1-p)^b a(n,b) \leqslant 4 \sum_{\substack{n \geqslant N \\ b \geqslant B}} p^n(1-p)^b a(n,b),$$

which is, except for the factor 4, the tail of the first series of (4.8). This satisfies

$$S_{N,B} = \sum_{\substack{n \geqslant N \\ b \geqslant B}} p^n(1-p)^b a(n,b) \leqslant P_N(p) - P_\infty(p). \tag{4.9}$$

Now, if $P_\infty(p_H) = 0$ then P_∞ is a continuous function of p (see Russo [**18**; p. 47]). Since the P_N are monotone decreasing to a continuous limit we may apply Dini's theorem (see for example Dieudonné [**4**]) to deduce that $P_N(p) \to P_\infty(p)$ uniformly in p. Thus the tail $S_{N,B} \to 0$ uniformly and we deduce that λ is differentiable with the term by term derivative

$$\lambda'(p) = p^{-1}(p - P_\infty(p)) - q^{-1}E(|\partial\gamma|/|\gamma| \mid 0 < |\gamma| < \infty)(p - P_\infty(p)). \tag{4.10}$$

We have supposed that $P_\infty(p_H) = 0$; this would certainly hold if $\mathscr{L}$ were self-matching by Harris's argument [**10**]. For the case of the square lattice it is not known whether or not the statement is valid. However, we know that

$$\lambda(p) = \lambda^*(1-p) + \phi(p). \tag{4.2}$$

At least one of $P_\infty(p_H)$, $P_\infty^*(p_H^*)$ is zero ((2.1)). Suppose that $P_\infty(p_H) \neq 0$; then $P_\infty^*(p_H^*) = 0$ and $\lambda^*(p)$ is differentiable for all p. Thus, by (4.2), $\lambda(p)$ is differentiable also.

Finally we wish to provide an alternative form of (4.10), which we may write as

$$\lambda'(p) = 1 - q^{-1}E(|\partial\gamma|/|\gamma|)$$

in the light of Theorem 3.2, where $|\partial\gamma|/|\gamma|$ is interpreted as zero if $|\gamma| = 0$ and as $\lim_{N\to\infty} |\partial\gamma \cap L_N|/|\gamma \cap L_N|$ if $|\gamma| = \infty$. I claim that

$$E(|\partial\gamma|/|\gamma|) = \mu(p).$$

For any $x \in \mathscr{L}$ define

$$\alpha(x) = \begin{cases} 0 & \text{if } x \text{ is white}, \\ |\partial\gamma_x|/|\gamma_x| & \text{if } x \text{ is black and } \gamma_x \text{ finite}, \\ qp^{-1} & \text{if } \gamma_x \text{ is an infinite cluster}. \end{cases}$$

Then

$$\begin{aligned} S(N) = \sum_{x \in L_N} \alpha(x) &= qp^{-1} \sum_{x \in L_N} \chi_x + \textstyle\sum_1 |\partial\delta| + \sum_2 \dfrac{|\delta \cap L_N|}{|\delta|}|\partial\delta| \\ &= S_1(N) + S_2(N) + S_3(N) \end{aligned}$$

where $\sum_1$ sums over all black finite clusters δ contained entirely in L_N and $\sum_2$ sums over all finite black δ which intersect both L_N and ∂L_N. Now, the $\alpha(x)$ $(x \in \mathscr{L})$ are identically distributed, and so

$$E(S(N))/V(N) = E(\alpha(0)) = E(|\partial\gamma|/|\gamma|).$$

Similarly

$$E(S_1(N))/V(N) = qp^{-1}P_\infty(p).$$

It remains to show that

$$E(S_2(N)+S_3(N))/V(N) \to \mu(p)-qp^{-1}P_\infty(p). \tag{4.11}$$

To see this note first that

$$S_2(N)+S_3(N) \leqslant \sum_{x \in L_N} (1-\beta_x)\kappa_x + 4\sum\nolimits_2 |\delta \cap L_N|$$

where κ_x is the number of finite black clusters in $\mathscr{L}$ with vertices adjacent to the white vertex x. Let $\varepsilon > 0$ and fix $n > 0$ such that $\pi_n = \sum_{m=n}^{\infty} p_m < \varepsilon$; this is always possible whatever the value of p. Now,

$$E(S_2(N)+S_3(N))/V(N) \leqslant \mu(p)-qp^{-1}P_\infty(p)+4V(N)^{-1}\sum_{x \in L_N} P(A(x,N))$$

where $A(x, N)$ is the event that x is joined to ∂L_N by a black path contained in a finite black cluster. Write $L_N = R_1 \cup R_2$ as before. The last sum is

$$V(N)^{-1}\Bigg(\sum_{x \in R_1} P(A(x,N)) + \sum_{x \in R_2} P(A(x,N))\Bigg) \leqslant V(N)^{-1}(|R_1|+|R_2|\pi_n)$$

$$\to \pi_n < \varepsilon \text{ as } N \to \infty.$$

Thus

$$\limsup_{N \to \infty} E(S_2(N)+S_3(N))/V(N) \leqslant \mu(p)-qp^{-1}P_\infty(p).$$

Finally

$$S_2(N)+S_3(N) \geqslant S_2(N) \geqslant \sum_{x \in R_2} (1-\beta_x)(\kappa_x - \bar{\kappa}_x),$$

where $\bar{\kappa}_x$ is the number of finite black clusters adjacent to x which intersect ∂L_N. Therefore

$$E(S_2(N)+S_3(N)) \geqslant |R_2|\Big(\mu(p)-qp^{-1}P_\infty(p)-E((1-\beta_x)\bar{\kappa}_x)\Big).$$

But $E((1-\beta_x)\bar{\kappa}_x) \leqslant 4\pi_n \leqslant 4\varepsilon$. Divide by $V(N)$ and let $N \to \infty$ to get the result.

Note. R. T. Smythe tells me that M. Vahidi has proved a result like Theorem 4.2. G. Brånvall [2] has independently proved limit theorems for $\lambda(p)$ and demonstrated the existence of its derivatives outside $[p_T, p_H]$; he has also found central limit theorems which are related to Theorem 3.3. Since this paper was written, H. Kesten has shown that $p_H = p_T = \frac{1}{2}$ for bond percolation on the square lattice.

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