THE RANK POLYNOMIALS OF LARGE RANDOM LATTICES

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ABSTRACT

The ergodic theory of subadditive stochastic processes is used to prove a limit theorem for the rank polynomials of expanding finite subsections of an arbitrary lattice from which vertices have been deleted at random. A similar result holds for the chromatic polynomials of these subsections.

1. *Introduction*

The (Whitney) rank polynomial of a graph is a function of two variables which, through the values which it and its derivatives take, contains information about the graph's structure; this includes, for example, details of its chromatic polynomial and the number of its connected components. It is an important quantity in the study of random phenomena associated with crystal lattices, such as percolation theory (see the papers by Temperley and Lieb [**11**] and Welsh [**12**]). Finite subsections of an infinite lattice are often too small to demonstrate such phenomena, and it becomes necessary to pass to the limit as the subsections expand to fill out the lattice. In this note I show the convergence of the rank polynomials of these subgraphs when suitably normalized; this is possible by using submultiplicative properties of such functions and an ergodic theorem for multidimensional subadditive stochastic processes. A previous paper (Grimmett [**7**]) contains similar results for the limiting behaviour of partition functions of interaction models (see Biggs [**4**]). That paper contains a more detailed account of the definition and properties of so called multidimensional lattices than is necessary here. For certain values of its parameters the rank polynomial is actually a partition function and the previous theorem may be applied; an alternative method is necessary for general non-negative values of the parameters. This method provides asymptotically sharp upper and lower bounds for the limit.

2. *Multidimensional lattices*

Firstly I shall define a multidimensional lattice and outline some of its relevant properties. Proofs of the statements which follow may be found in Grimmett [**7**]; the reader should refer to the books of N. Biggs [**3**, **4**] for general terminology. A *d-dimensional lattice* $\mathscr{L}$ is a locally finite infinite graph which admits as a group of automorphisms the product $\mathscr{A}$ of d infinite cyclic groups which act like $\mathbb{Z}^d$ and whose cyclic subgroups are generated by automorphisms which have no fixed vertices; furthermore we require that the set of vertices of $\mathscr{L}$ has only finitely many orbits under $\mathscr{A}$. Thus $\mathscr{A}$ is a free abelian group of rank d. It can be shown that there exists a connected finite subgraph K, called a *kernel*, of $\mathscr{L}$ containing exactly one vertex from each orbit of the vertices of $\mathscr{L}$ under $\mathscr{A}$; hence αK and βK are disjoint for any

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Received 21 October, 1977; revised 17 March, 1978.

distinct α, $\beta \in \mathscr{A}$ and

$$\mathscr{L} = \bigcup_{\alpha \in \mathscr{A}} \alpha K.$$

(The union of two subgraphs of $\mathscr{L}$ is the union of their vertex sets together with the edges inherited from $\mathscr{L}$.) Let $\sigma_1, \sigma_2, \ldots, \sigma_d$ be independent elements of $\mathscr{A}$ which generate $\mathscr{A}$. If $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ is an ordered set of integers, $G(\mathbf{n})$ is defined to be the subgraph of $\mathscr{L}$ given by

$$G(\mathbf{n}) = \bigcup_{-\mathbf{n} \leqslant \mathbf{i} \leqslant \mathbf{n}} \boldsymbol{\sigma}^{\mathbf{i}} K$$

where $\boldsymbol{\sigma}^{\mathbf{i}}$ is shorthand for the automorphism $\sigma_1^{i_1} \sigma_2^{i_2} \ldots \sigma_d^{i_d}$ for any vector $\mathbf{i} = (i_1, i_2, \ldots, i_d)$ of integers, and $\mathbf{m} \leqslant \mathbf{n}$ if and only if $m_j \leqslant n_j$ for each $j = 1, 2, \ldots, d$. Then $G(\mathbf{n})$ is the union of $|\mathbf{n}|$ copies of K where

$$|\mathbf{n}| = \prod_{j=1}^{d} (2n_j + 1).$$

Finally we shall need that the number $e(\mathbf{n})$ of edges of $\mathscr{L}$ with exactly one endpoint in $G(\mathbf{n})$ satisfies $e(\mathbf{n}) = o(|\mathbf{n}|)$ as $\mathbf{n} \to \infty$ (the limit here is as $n_j \to \infty$ for each $j = 1, 2, \ldots, d$); this is shown in Grimmett [7].

3. *The subadditive limit theorem*

I shall suppose that each vertex of $\mathscr{L}$ is coloured in the following random manner. A vertex is coloured *black* with probability p $(0 \leqslant p \leqslant 1)$ independently of the colours of all other vertices; an uncoloured vertex remains *white*. An edge is coloured *black* if it joins two black vertices. A realization of this colouring process is a set ω containing the black vertices of $\mathscr{L}$. This induces a random subgraph $G(\mathbf{n}; \omega)$ of $G(\mathbf{n})$ obtained by deleting all white vertices and their incident edges.

This random model underlies the theory of " site percolation " (see Shante and Kirkpatrick [9]). For some problems it may be preferable to colour the edges at random rather than the vertices; in this case one studies the covering lattice, or line graph, of $\mathscr{L}$. The deterministic case may be retrieved by setting $p = 1$.

I shall use the following ergodic theorem which follows from Smythe [10] and Grimmett [6]. Let X be a random process which associates a real quantity $X(G; \omega)$ with each finite subgraph G of $\mathscr{L}$ and realization ω of the colouring process; we call X *subadditive* if it satisfies the following conditions:

(a) for any $\alpha \in \mathscr{A}$, $X(G)$ and $X(\alpha G)$ are random variables with the same distributions;

(b) if G and H have no vertices in common, $X(G \cup H; \omega) \leqslant X(G; \omega) + X(G; \omega)$;

(c) $\gamma = \inf_{\mathbf{n}} E\big(X(G(\mathbf{n}))\big)/|\mathbf{n}|$ satisfies $\gamma > -\infty$.

Theorem 1. *If X is subadditive then*

$$X(G(\mathbf{n}))/|\mathbf{n}| \to \gamma \quad \text{in } L^1 \quad \text{as } \mathbf{n} \to \infty$$

and

$$P\Big(\limsup X(G(\mathbf{n}))/|\mathbf{n}| \leqslant \gamma\Big) = 1.$$

4. *The rank polynomial*

I am concerned with the rank polynomials of the black graphs $G(\mathbf{n}; \omega)$ as $\mathbf{n} \to \infty$. The (*Whitney*) *rank polynomial* (see Biggs [**1**; p. 64]) of a graph $G = (V, E)$ is defined by

$$W_G(x, y) = \sum_{E'} x^{r(G')} y^{c(G')}$$

where the summation is over all subsets E' of the edge set E of G, G' is the graph (V, E') and $r(G')$ and $c(G')$ are the rank and corank of G' respectively:

$$r(G') = |V| - k(G'), \quad c(G') = |E'| - |V| + k(G')$$

where $k(G')$ is the number of connected components of G'.

Hence W_G may be written as

$$W_G(x, y) = (x/y)^{|V|} \sum_{E'} (y/x)^{k(G')} y^{|E'|}.$$

If $y > -1$ and $x = y/n$ for some integer $n \geqslant 2$ then W_G is the partition function, up to a factor which may be treated separately, of a type of pair interaction model called a " resonant model " by Biggs [**4**]. Such models are characterized by the property that the contribution of the states of an edge to the partition function depends only on whether these states are like or unlike. The Ashkin–Teller–Potts model may be described in this way, as is evident from the form of its partition function.

The next lemma establishes submultiplicative properties of rank polynomials. As before, the union of two subgraphs of a graph G is the subgraph of G whose vertex set is the union of the vertex sets and whose edges are all those inherited from G.

LEMMA. *Let H and J be disjoint subgraphs of some graph G such that there are exactly e edges of G joining the vertex sets of H and J. Then if $y \geqslant x > 0$,*

$$W_H W_J (1+x)^e \leqslant W_{H \cup J} \leqslant W_H W_J (1+y)^e.$$

If $x \geqslant y > 0$ then the reverse inequalities hold.

Proof. Let $k(G)$ and $W(G)$ be the number of connected components and the rank polynomial of the graph G. Then

$$k(H)+k(J)-e \leqslant k(H \cup J) \leqslant k(H)+k(J) \tag{1}$$

where e is the number of edges joining H to J. The first inequality holds because removal of one of the edges which joins H to J increases $k(H \cup J)$ by at most one. Thus

$$W(H \cup J) = (x/y)^{v_1+v_2} \sum_{E_1 \subseteq E_H} \sum_{E_2 \subseteq E_J} \sum_{E_3 \subseteq E} (y/x)^k y^{|E_1|+|E_2|+|E_3|}$$

where v_1 and v_2 are the numbers of vertices in H and J, E_H, E_J and E are the sets of edges of H, J and those which join H and J, and k is the number of components of the subgraph $H \cup J$ with the same vertex set and with the edge set $E_1 \cup E_2 \cup E$. Then, using (1), we obtain if $y \geqslant x > 0$

$$\begin{aligned} W(H \cup J) &\leqslant W(H)\, W(J) \sum_{E_3} y^{|E_3|} \\ &= W(H)\, W(J)(1+y)^e \end{aligned}$$

because $(y/x)^k$ increases with k; also

$$W(H \cup J) \geqslant W(H)\, W(J) \sum_{E_3} y^{|E_3|}(y/x)^{-|E_3|}$$
$$= W(H)\, W(J)(1+x)^e$$

which proves the first part of the lemma.

Finally, suppose that $x \geqslant y > 0$. Then

$$W(H \cup J) \geqslant W(H)\, W(J) \sum_{E_3} y^{|E_3|}$$

because $(y/x)^k$ decreases as k increases; a similar reverse inequality completes the proof.

This provides the next limit theorem for the random variables $W_{\mathbf{n}}(x, y; \omega)$, being the rank polynomials of the subgraphs $G(\mathbf{n}; \omega)$ of a given multidimensional lattice $\mathscr{L}$.

THEOREM 2. *If $y \geqslant x > 0$ then $X_{\mathbf{n}} = \log W_{\mathbf{n}}$ satisfies*

$$X_{\mathbf{n}}/|\mathbf{n}| \to \mu \quad \textit{a.e. and in } L^1 \textit{ as } \mathbf{n} \to \infty$$

where μ is a constant given by

$$\mu = \inf_{\mathbf{r}} \Big((E(X_{\mathbf{r}}) + \tfrac{1}{2}p^2\, e(\mathbf{r}) \log(1+y))/|\mathbf{r}| \Big)$$
$$= \sup_{\mathbf{r}} \Big((E(X_{\mathbf{r}}) + \tfrac{1}{2}p^2\, e(\mathbf{r}) \log(1+x))/|\mathbf{r}| \Big).$$

If $x \geqslant y > 0$ then the result of the theorem holds with x and y interchanged in the definition of μ. The convergence is uniform on any subset of the parameter space of the form $B \geqslant x, y \geqslant A > 0$.

Proof. For any finite subgraph G of $\mathscr{L}$ and realization ω of the colouring process let

$$X'(G; \omega) = X(G; \omega) + \tfrac{1}{2}e(G; \omega) \log(1+y)$$
$$X''(G; \omega) = X(G; \omega) + \tfrac{1}{2}e(G; \omega) \log(1+x)$$

where $X(G; \omega) = \log W(G; \omega)$ is the logarithm of the rank polynomial of the black subgraph of G, and $e(G; \omega)$ is the number of black edges of $\mathscr{L}$ with exactly one end-point in this subgraph. Let $X_{\mathbf{n}}'(\omega) = X'(G(\mathbf{n}); \omega)$, $X_{\mathbf{n}}''(\omega) = X''(G(\mathbf{n}); \omega)$; then $E\,|X_{\mathbf{n}}'|/|\mathbf{n}|$ and $E\,|X_{\mathbf{n}}''|/|\mathbf{n}|$ are bounded uniformly in $\mathbf{n}$. To see this, note that by the lemma

$$(1+x)^{|E_G|} \leqslant W(G) \leqslant (1+y)^{|E_G|}$$

for any graph G with edge set E_G. Thus, by the lemma again, X' and $-X''$ are subadditive and the ergodic theorem yields

$$X_{\mathbf{n}}'/|\mathbf{n}| \to \mu_1 \quad \text{in} \quad L^1$$
$$X_{\mathbf{n}}''/|\mathbf{n}| \to \mu_2 \quad \text{in} \quad L^1$$

where

$$\mu_1 = \inf_{\mathbf{r}} E(X_{\mathbf{r}}')/|\mathbf{r}|$$
$$\mu_2 = \sup_{\mathbf{r}} E(X_{\mathbf{r}}'')/|\mathbf{r}|.$$

Now $\mu_1 = \mu_2$ because

$$E\,|X_{\mathbf{n}}' - X_{\mathbf{n}}''| = \tfrac{1}{2}\log\left((1+y)/(1+x)\right) E\big(e(G(\mathbf{n});\,\omega)\big) = o(|\mathbf{n}|).$$

The subadditive theory of Theorem 1 also implies that

$$P(\limsup X_{\mathbf{n}}'/|\mathbf{n}| \leqslant \mu_1) = 1$$

$$P(\liminf X_{n}''/|\mathbf{n}| \geqslant \mu_2) = 1$$

and the convergence a.e. of $X_{\mathbf{n}}$ follows immediately. The result for the case $x \geqslant y > 0$ holds similarly. The uniformity of the convergence is a consequence of the following remarks.

The theorem provides bounds for μ: if $x \leqslant y$ then

$$\tfrac{1}{2}p^2\, e(\mathbf{r}) \log(1+x) \leqslant \mu|\mathbf{r}| - E(X_{\mathbf{r}}) \leqslant \tfrac{1}{2}p^2\, e(\mathbf{r}) \log(1+y) \quad \text{for any } \mathbf{r}.$$

These bounds show that the convergence is uniform on any set of the parameters p, x and y subject to the previous constraint. In particular this demonstrates that μ is a continuous function of p, x and y. As a simple example of these bounds consider the plane square lattice with the two automorphisms which translate the vertices unit distances in the axial directions. The kernel is the origin alone. Taking $\mathbf{r} = (1, 1)$ gives

$$\tfrac{2}{3}p^2 \log(1+x) \leqslant \mu - \tfrac{1}{9}W \leqslant \tfrac{2}{3}p^2 \log(1+y)$$

where W is the mean rank polynomial of a block of nine vertices of the lattice arranged in a square configuration of size three by three.

5. *The edge percolation problem*

In this section I shall suppose that the randomness is associated with the edges of $\mathscr{L}$ rather than with the vertices. That is, instead of colouring the *vertices* black or white randomly, we colour each *edge* black with probability p independently of the colours of other edges. Associate with the edge subset E' of black edges the graph G' with the same vertices as before and the edges E'. There is a similar limit theorem for the rank polynomials of finite regions of such a random lattice as there is when the randomness is associated with the vertices because the covering lattice (or line graph) has properties analogous to those of the original lattice. The rank polynomials of finite regions of this new random lattice may be used directly in the study of percolation on the lattice as shown by Essam [5]. For example, it is well known that the expected number $E\big(k(G')\big)$ of components of a finite graph G when edges are deleted at random with probability $q = 1-p$, the subgraph G' remaining, is given by

$$E\big(k(G')\big) = \frac{d}{dz}\big(z^v q^e\, W_G(p/qz, p/q)\big)\bigg|_{z=1} \tag{2}$$

where G has v vertices and e edges, and the derivative is evaluated at $z = 1$. The method used in Grimmett [7] may be extended in the obvious way to show that, for random subgraphs ω of the lattice $\mathscr{L}$, $k_{\mathbf{n}}(\omega) = k\big(G(\mathbf{n};\,\omega)\big)$ satisfies

$$k_{\mathbf{n}}/|\mathbf{n}| \to \lambda(p) \qquad \text{a.e. and in } L^1 \text{ as } \mathbf{n} \to \infty,$$

where the limit λ is a continuous function of p. By the limit theorem of

§4, $V_{\mathbf{n}}(\omega) = \log W(G(\mathbf{n}; \omega))$ satisfies

$$V_{\mathbf{n}}/|\mathbf{n}| \to \gamma(x, y) \qquad \text{a.e. and in } L^1 \text{ as } \mathbf{n} \to \infty$$

if $x, y > 0$, where the convergence is uniform on any set of values of x and y bounded away from 0 and ∞. It is natural to conjecture a link between the two limits λ and γ through the relation (2) which holds for finite graphs.

CONJECTURE.

$$\lambda(p) = 1 + \frac{d}{dz}\gamma(p/qz, p/q)\bigg|_{z=1} .$$

6. *The chromatic polynomial*

The chromatic polynomial $P_G(\lambda)$ of a graph G is the number of ways of colouring the vertices of G, each with one of the λ available colours, so that neighbouring vertices have different colours; it satisfies

$$P_G(\lambda) = \lambda^v W_G(-\lambda^{-1}, -1) \tag{3}$$

where v is the number of vertices of G. Biggs [2] has used subadditivity to claim the existence of the number of colourings per vertex of the infinite square lattice, but his method is misguided; one may proceed with greater generality as follows.

I seek to show that the number of colourings per vertex of a finite subsection of the black subgraph ω of a lattice $\mathscr{L}$ converges as the subsection expands to fill $\mathscr{L}$. The limit theorem of §4 may not be applied to (3) because the parameters are negative. Indeed it is more natural to argue directly. Write $P(G)$ for $P_G(\omega)$ and $P_{\mathbf{n}}(\omega)$ for $P(G(\mathbf{n}; \omega))$. It is easy to see that $P(H \cup J) \leqslant P(H)\,P(J)$ since each colouring of $H \cup J$ induces colourings of H and of J, but not every pair of colourings of H and J may be combined to colour $H \cup J$. Thus P is subadditive, and so $Q_{\mathbf{n}}(\omega) = \log P_{\mathbf{n}}(\omega)$ satisfies

$$Q_{\mathbf{n}}/|\mathbf{n}| \to \nu \text{ in } L^1 \text{ as } \mathbf{n} \to \infty$$

$$P(\limsup Q_{\mathbf{n}}/|\mathbf{n}| \leqslant \nu) = 1$$

so long as

$$\nu = \inf_{\mathbf{r}} E(Q_{\mathbf{r}})/|\mathbf{r}| > -\infty .$$

If $\nu = -\infty$ then

$$E(Q_{\mathbf{n}})/|\mathbf{n}| \to -\infty,$$

and this will certainly be the case if $\lambda < \chi$, the chromatic number of $\mathscr{L}$. It may also be so when $\lambda = \chi$. For example, the chromatic number of the plane square lattice is two and the number of 2-colourings is also two.

To show convergence a.e. we need a supermultiplicative property for P. Let H and J be disjoint subgraphs of the finite graph G, and let δ be the maximum vertex degree of G. Let V be the subset of vertices of H which are joined to vertices of J by edges of G. I claim that if $\lambda > \delta + 1$

$$P(H \cup J) \geqslant P(H - V)\,P(J) \tag{4}$$

where $H - V$ is the subgraph of H obtained by deleting members of V and incident edges. For, any pair of colourings of $H - V$ and J may be extended to some colouring of $H \cup J$ by assigning suitable colours to vertices in V. This is always possible because each member of V is joined to at most δ vertices which have already been coloured, and there is always a spare colour. Let $v(\mathbf{n}; \omega)$ denote the cardinality of the set $V(\mathbf{n}; \omega)$ of vertices of $G(\mathbf{n}; \omega)$ which are joined in ω to a vertex not in $G(\mathbf{n}; \omega)$. The method used to prove the ergodic theorem may be adapted easily to yield

$$P(\liminf Q_{\mathbf{n}}'/|\mathbf{n}| \geqslant \nu') = 1 \tag{5}$$

where

$$\nu' = \sup_{\mathbf{r}} E(Q_{\mathbf{r}}')/|\mathbf{r}|$$

and

$$Q_{\mathbf{n}}'(\omega) = \log P\big(G(\mathbf{n}; \omega) - V(\mathbf{n}; \omega)\big).$$

But $Q_{\mathbf{n}}$ and $Q_{\mathbf{n}}'$ behave similarly for large $\mathbf{n}$ as is shown by the inequalities

$$0 \leqslant Q_{\mathbf{n}}(\omega) - Q_{\mathbf{n}}'(\omega) \leqslant v(\mathbf{n}) \log \lambda;$$

thus

$$\liminf Q_{\mathbf{n}}/|\mathbf{n}| = \liminf Q_{\mathbf{n}}'/|\mathbf{n}| \quad \text{a.e.}$$

because $v(\mathbf{n}) = o(|\mathbf{n}|)$. Hence, using Fatou's lemma (see Kingman and Taylor [**8**; p. 120]),

$$Q_{\mathbf{n}}/|\mathbf{n}| \to \nu' \quad \text{a.e. and in } L^1;$$

thus $\nu = \nu'$ satisfies

$$E(Q_{\mathbf{r}})/|\mathbf{r}| \geqslant \nu \geqslant E(Q_{\mathbf{r}}')/|\mathbf{r}| \geqslant E(Q_{\mathbf{r}} - v_{\mathbf{r}} \log \lambda)/|\mathbf{r}|$$

for all $\mathbf{r}$, where $v_{\mathbf{r}}(\omega) = v(\mathbf{r}; \omega)$. Thus

$$0 \leqslant E(Q_{\mathbf{r}})/|\mathbf{r}| - \nu \leqslant E(v_{\mathbf{r}}) \log \lambda$$

and it may be seen as before that the convergence is uniform in p.

It may be possible to extend this argument to the case when only δ colours are available. Certainly this is true for the special case of the plane square lattice, and I will use that example as an illustration. Unfortunately I need to return to the method of proof of Theorem 1. I shall consider the square lattice automorphism group which is generated by the two unit shifts parallel to the axes. The kernel is a single vertex, and $G(\mathbf{n})$ is the subgraph of $\mathscr{L}$ containing all vertices (x, y) which satisfy $|x| \leqslant n_1$, $|y| \leqslant n_2$. The argument becomes slightly simpler if, instead of copies of the graph $G(\mathbf{n})$, we consider copies of the graph $H(\mathbf{n})$ containing all vertices (x, y) with $0 \leqslant x \leqslant n_1$, $0 \leqslant y \leqslant n_2$. For any $\mathbf{n}$ and $\mathbf{r} \leqslant \mathbf{n}$, $H(\mathbf{n})$ may be decomposed into copies of $H(\mathbf{r})$ together with a remainder region around the extremes of $H(\mathbf{n})$; this remainder is small compared with $\mathbf{n}$ as $\mathbf{n} \to \infty$. The black subgraphs of these copies may be coloured in the prescribed manner with the λ colours available, but it will not in general be possible to combine these colourings to form an eligible colouring of $H(\mathbf{n})$. For this reason when the black subgraph of each copy of $H(\mathbf{r})$ is coloured initially we ignore those vertices of each copy which are on either the upper or rightwards borders. There exists at least one δ-colouring of the sections of $H(\mathbf{n})$ which are in none of the copies of $H(\mathbf{r}-\mathbf{1})$ $\big(\mathbf{1} = (1, 1)\big)$ thus obtained, such that the combined colouring of all the vertices of $H(\mathbf{n})$ colours no pair of neighbouring vertices alike; this is because vertices of these remaining sections may be coloured in such an order that at no stage is any uncoloured vertex adjacent to a complete set of four vertices which have already been coloured. More precisely, let $R_{\mathbf{n}}(\omega) = P\big(H(\mathbf{n}); \omega\big)$

and fix $\mathbf{r} = (r_1, r_2)$. For any $\mathbf{n}$ there exists $\mathbf{a}$ and $\mathbf{b}$ such that

$$n_j = a_j r_j + b_j \qquad (j = 1, 2;\ 0 \leqslant b_j < r_j).$$

Then, by (3) and the observations just made,

$$R_{\mathbf{n}}(\omega) \geqslant \prod_{\mathbf{0} \leqslant \mathbf{i} \leqslant \mathbf{a}} P(\boldsymbol{\tau}^{\mathbf{i}} H(\mathbf{r}-\mathbf{1});\ \omega)$$

where $\boldsymbol{\tau} = \boldsymbol{\sigma}^{\mathbf{r}}$. As in the proof of Theorem 1 we may conclude that, by the strong law of large numbers,

$$\liminf \big(\log R_{\mathbf{n}}/(n_1 n_2)\big) \geqslant E(\log R_{\mathbf{r}-\mathbf{1}})/(r_1 r_2) \quad \text{a.e.} \tag{6}$$

and it follows by Fatou's lemma that

$$\log R_{\mathbf{n}}/(n_1 n_2) \to v'' \quad \text{in } L^1$$

where

$$v'' = \sup_{\mathbf{r}} \big(E(\log H(\mathbf{r};\ .))/(r_1+1)(r_2+1)\big).$$

It is easy to deduce that $Q_{\mathbf{n}}(\omega) = \log P\big(G(\mathbf{n});\ \omega\big)$ satisfies

$$Q_{\mathbf{n}}/|\mathbf{n}| \to v'' \quad \text{in } L^1$$

and combining this with (5) and (6) yields

$$Q_{\mathbf{n}}/|\mathbf{n}| \to v \quad \text{a.e. and in } L^1 \text{ where } v = v''.$$

Biggs [3] has used transfer matrix methods to find alternative bounds for the limit v for this lattice.

7. *Summary of deterministic results*

Graph theorists are mostly interested in the deterministic case when $p = 1$, and it is convenient to summarize the results of this paper under the assumption that this holds. The theorem of §4 shows that the Whitney rank polynomial $W_{\mathbf{n}}(x, y)$ of $G(\mathbf{n})$ satisfies

$$W_{\mathbf{n}}^{|\mathbf{n}|^{-1}} \to \eta \qquad \text{as } \mathbf{n} \to \infty$$

if $x, y > 0$ where η is a continuous function of x and y. As a consequence of §5, the chromatic polynomial $P_{\mathbf{n}}(\lambda)$ of $G(\mathbf{n})$ satisfies

$$P_{\mathbf{n}}^{|\mathbf{n}|^{-1}} \to \kappa \qquad \text{as } \mathbf{n} \to \infty.$$

Acknowledgment. I am grateful to Norman Biggs for his comments on a draft version of this paper.

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