revised 26 April 2002

Rigidity of the interface in percolation and random-cluster models

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Abstract: We study conditioned random-cluster measures with edge-parameter p and cluster-weighting factor q satisfying $q \geq 1$. The conditioning corresponds to mixed boundary conditions for a spin model. Interfaces may be defined in the sense of Dobrushin, and these are proved to be 'rigid' in the thermodynamic limit, in three dimensions and for sufficiently large values of p. This implies the existence of non-translation-invariant (conditioned) random-cluster measures in three dimensions. The results are valid in the special case $q = 1$, thus indicating a property of three-dimensional percolation not previously noted.

Keywords: Random-cluster model, percolation, Ising model, Potts model, interface, Dobrushin boundary condition.

Mathematics Subject Classification (2000): 60K35, 82B20, 82B43.

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1. Introduction

Dobrushin's proof [13] of the existence of non-translation-invariant Gibbs states for the three-dimensional Ising model was the starting point for the study of interfaces in disordered spin systems. We show in the current paper that such results are valid for all ferromagnetic random-cluster models on $\mathbb{Z}^3$, including percolation. This generalization of Dobrushin's theorem is achieved by defining a family of conditioned measures, and by showing the stiffness of the ensuing interface.

The random-cluster model has since its introduction [14, 15, 16] around 1970 provided a beautiful mechanism for the study of Ising and Potts models, as well as being an object worthy of study in its own right. Many (but not all) central results for ferromagnetic Ising/Potts systems are best proved in the context of random-cluster models; the stochastic geometry of such models may be exploited the better to understand the behaviour of correlations in the original system. The spectrum of random-cluster models extends to percolation (and beyond), and one sees thus that percolative techniques have direct application to Ising and Potts models. The reader is referred to [20] for more information concerning the history of random-cluster models, and to [2, 4, 9, 18, 21] for examples of them in action.

The question addressed here concerns the stiffness of interfaces. In the case of the Ising model, Dobrushin introduced the boundary condition on the box $\Lambda = [-L, L]^3$ having $+1$ on the upper half of the boundary and -1 on its complement. He then studied the interface separating the two regions behaving respectively as the $+1$ phase and the -1 phase. He showed for sufficiently low temperatures that this interface deviates only locally from the horizontal plane through the equator of the box. This effect is seen in all dimensions of three or more, but not in two dimensions, for which case the interface may be thought of as a line with Gaussian fluctuations (see [17, 19, 27]).

This problem may be cast in the more general setting of the random-cluster model on the box Λ subject to the following boundary condition and to a certain conditioning. The vertices on the upper hemisphere of Λ are wired together into a single composite vertex labelled Λ^+. The vertices on the complement of the upper hemisphere are wired into a single composite vertex labelled Λ^-. Let $\mathcal{D}$ be the event that no open path of Λ exists joining Λ^- to Λ^+, and let ϕ_Λ be the random-cluster measure on Λ with edge-parameter p and cluster-weighting factor q, with the above boundary condition *and conditioned on the event* $\mathcal{D}$. It is a geometrical fact that there exists an interface separating two regions of Λ, each of which is in the wired phase. It follows by the results of [13] that, when $q = 2$ and p is sufficiently large, this interface deviates only locally from the horizontal plane through the equator of Λ. The purpose of this paper is to prove that this is so for all $q \geq 1$ and all sufficiently large p. In doing so we shall work directly with the random-cluster model. The geometry of the interfaces for this model is notably different from that of a spin model since the configurations are indexed by edges rather than by vertices, and this leads to some new difficulties.

Extensions of our results to dimensions d satisfying $d \geq 4$ are, to quote from [13], "obvious", though the proofs may involve some extra complications. It is striking that our results are valid for high-density percolation on $\mathbb{Z}^3$, being the case $q = 1$. That is, conditional on the existence of a surface (suitably defined) of dual plaquettes spanning the equator of Λ, this surface deviates only locally from the flat plane. A corresponding question for supercritical percolation in two dimensions has been studied in depth in [7], where it is shown effectively that the (one-dimensional) interface converges when re-scaled to a Brownian bridge.

We have spoken above of interfaces which 'deviate only locally' from a plane, and we shall make this expression more rigorous in Section 9, where our principal Theorem 2 is presented. We include in Section 3 a weaker version of Theorem 2 which does not make use of the notation developed later in the work.

Our theorems are proved under the assumption that $q \geq 1$ and p is sufficiently large. It is a major open question to determine whether or not such results are valid under the weaker assumption that p exceeds the critical value $p_c(q)$ of the random-cluster model with cluster-weighting factor q (see [22]). The answer may be expected to depend on the value of q and the number d of dimensions. Since the percolation measure (when $q = 1$) is a conditioned product measure, it may be possible as with other problems to gain insight into the existence or not of a 'roughening transition' by concentrating on the special case of percolation. It is of interest that much of the argument of this paper is valid also when $q < 1$ and p is sufficiently large, but we shall not specify the details. Also, it may be possible to extend some of the conclusions of this paper to measures with certain other boundary conditions, such as that generated with free boundary conditions and conditioned on $\mathcal{D}$, but we shall not pursue this here.

As described above, the measures studied here are obtained by conditioning on a certain event $\mathcal{D}$. When p is large, $\mathcal{D}$ has probability of order $\exp(-\alpha L^2)$ where $\alpha = \alpha(p, q)$, and thus we are in the realm of the large-deviation theory of the process. See [9, 12].

We introduce random-cluster measures in the next section, followed by a summary of our main results in Section 3. Necessary properties of random-cluster measures are developed in Section 4. Interfaces are defined in Section 5, where we prove some geometrical lemmas of independent interest which we believe will find applications elsewhere. In Section 6 we study the probability of having a configuration that is compatible with a given interface, under the appropriate conditioned measure. We present in Section 7 a microscopic geometrical description of the random-cluster interfaces using a terminology based on that introduced for the Ising model in [13]. This is followed in Section 8 by an exponential bound for the probability of finding local perturbations of a flat interface, and in Section 9 by the statement and proof of our main theorems.

The methods of this paper are inspired by those of [13] subject to some serious variations. Dobrushin [13] studied the Ising model, and his arguments were later simplified in part by van Beijeren [3]. We have been unable to extend the methods of [3], which may be special to the Ising model. Related results may be found in [5, 6, 11, 25, 26] and the references therein. We have found the first of the latter references to be particularly useful in the present work. It should be noted that, in order to study interfaces for spin systems rigorously, certain lemmas concerning their geometry are required; see [25, 30] for example.

The Pirogov–Sinai theory of contours has enabled ([30, 32, 33]) a study of Potts models and random-cluster models for large q, when $p = p_c(q)$, the critical point. It seems now to be accepted that the random-cluster model is especially well adapted to the study of contours and interfaces. However, it appears that certain pivotal facts, implicit in earlier work, and concerning the relationship between interfaces and random-cluster measures, have never been proven. Specifically, certain key results in three dimensions concerning the 'external boundary' of a set of connected edges, and the 'internal boundary' of a cavity of plaquettes of $\mathbb{Z}^3$, are missing from the literature. These are akin to the well known fact, proved in [28], that the external boundary of a finite cluster of $\mathbb{Z}^2$ contains, in its dual representation, a circuit separating the cluster from infinity. One of the targets of the current paper is to state and prove the necessary geometrical facts; see Propositions 5 and 6.

Since finishing this work, we have received the preprint [10], which uses Pirogov–Sinai theory to study the rigidity of interfaces for sufficiently large q and with p equal to the critical point of the random-cluster model. It is proved there that there is a rigid interface at a first-order transition for large q, with the boundary condition a mixture of the wired and the free.

2. Conditioned random-cluster measures

Let $\mathbb{Z}^3$ be the set of all vectors $x = (x_1, x_2, x_3)$ of integers, termed *vertices*, and let

$$|x-y| = \sum_{i=1}^{3} |x_i - y_i|, \ \ \|x-y\| = \max\Big\{|x_i - y_i| : 1 \le i \le 3\Big\} \quad \text{for } x, y \in \mathbb{R}^3.$$

We place an *edge* between vertices x and y if and only if $|x-y| = 1$, and we denote by $\mathbb{L} = (\mathbb{Z}^3, \mathbb{E})$ the resulting lattice. We write $x \sim y$ if $|x-y| = 1$, and we write $\langle x, y\rangle$ for the corresponding edge. We sometimes think of the edge $e = \langle x, y\rangle$ as the closed straight-line segment with endpoints x and y. For $E \subseteq \mathbb{E}$, we write $V(E)$ for the set of vertices in $\mathbb{Z}^3$ that belong to at least one of the edges in E. We shall sometimes abuse notation by referring to the graph $(V(E), E)$ as the graph E. The L^∞ distance between two edges e_1, e_2 is defined to be the distance between their centres, and is denoted $\|e_1, e_2\|$.

A *path* in a subgraph $G = (V, E)$ of $\mathbb{L}$ is an alternating set of distinct vertices and bonds $x = z_0, \langle z_0, z_1\rangle, z_1, \dots, \langle z_{n-1}, z_n\rangle, z_n = y$ using only edges $\langle z_i, z_{i+1}\rangle \in E$. Such a path is said to connect x and y and to have length n. The graph G is called *connected* if every pair of vertices is connected by some path. A *connected component* of G is a maximal connected subgraph of G. We shall occasionally speak of a set $\Lambda \subseteq \mathbb{Z}^3$ of vertices as being *connected*, by which we mean that Λ induces a connected subgraph of $\mathbb{L}$.

For $x \in \mathbb{Z}^3$, we denote by $\tau_x : \mathbb{Z}^3 \to \mathbb{Z}^3$ the translate given by $\tau_x(y) = x + y$. The translate τ_x acts on edges and subgraphs of $\mathbb{L}$ in the natural way. For sets A, B of edges or vertices of $\mathbb{L}$, we write $A \simeq B$ if $B = \tau_x A$ for some $x \in \mathbb{Z}^3$. Note that two edges e, f satisfy $\{e\} \simeq \{f\}$ if and only if they are parallel (in which case we write $e \simeq f$).

We write S^{c} for the complement of a set S. The *upper* and *lower boundaries* of a set Λ of vertices are defined as

$$\partial^+\Lambda = \{x \in \Lambda^{\mathrm{c}} : x_3 > 0,\ x \sim z \text{ for some } z \in \Lambda\},$$
$$\partial^-\Lambda = \{x \in \Lambda^{\mathrm{c}} : x_3 \leq 0,\ x \sim z \text{ for some } z \in \Lambda\},$$

and the *boundary* of Λ is denoted $\partial\Lambda = \partial^+\Lambda \cup \partial^-\Lambda$. For positive integers L, M we define the box $\Lambda_{L,M} = [-L, L]^2 \times [-M, M]$, and write $E_{L,M}$ for the set of all edges having at least one endvertex in $\Lambda_{L,M}$. [We abuse notation here and later, and should write $\Lambda_{L,M} = ([-L, L]^2 \times [-M, M]) \cap \mathbb{Z}^3$.] We write $Q_L = \Lambda_{L,L}$, the cube of side-length $2L$, and $\Lambda_L = [-L, L]^2 \times \mathbb{Z}$, an infinite cylinder.

The configuration space of the random-cluster model on $\mathbb{L}$ is the set $\Omega = \{0, 1\}^{\mathbb{E}}$, which we endow with the σ-field $\mathcal{F}$ generated by the finite-dimensional cylinders. A configuration $\omega \in \Omega$ assigns to each edge e the value 0 or 1; we call the edge e *open* (in ω) if $\omega(e) = 1$, and *closed* otherwise. A set of edges (for example, a path) is called *open* if all the edges therein are open. For $\omega \in \Omega$, we write $x \leftrightarrow y$ if there exists an open path connecting the vertices x and y, and $x \leftrightarrow A$ if there exists $y \in A$ such that $x \leftrightarrow y$. Each $\omega \in \Omega$ is in one–one correspondence with its set $\eta(\omega) = \{e \in \mathbb{E} : \omega(e) = 1\}$ of open edges. We write $\eta_x(\omega)$ for the set of edges in the connected component of the graph $(\mathbb{Z}^3, \eta(\omega))$ containing the vertex x. The configuration which assigns 1 (respectively 0) to every edge is denoted 1 (respectively 0).

Let E be a finite subset of $\mathbb{E}$ and let $V = V(E)$, and suppose that $0 \leq p \leq 1$ and $q > 0$. The usual way (see [22]) of defining a random-cluster measure with parameters p, q on the graph $G = (V, E)$ with boundary condition ζ ($\in \Omega$) is via the formula

$$\phi^{\zeta}_{G,p,q}(\omega) = \frac{1}{Z^{\zeta}_{G,p,q}} \left\{ \prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k_G(\omega)} I\{\omega(f) = \zeta(f) \text{ if } f \notin E\},$$

defined for all $\omega \in \Omega$. Here, $k_G(\omega)$ is the number of connected components in the graph $(\mathbb{Z}^3, \eta(\omega))$ having at least one vertex belonging to V,

$$Z^{\zeta}_{G,p,q} = \sum_{\omega \in \Omega} \left\{ \prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k_G(\omega)} I\{\omega(f) = \zeta(f) \text{ if } f \notin E\} \tag{1}$$

is the normalizing partition function, and $I\{H\}$ is the indicator function of the event H. We shall write $k(\omega)$ for the total number of connected components of $(\mathbb{Z}^3, \eta(\omega))$.

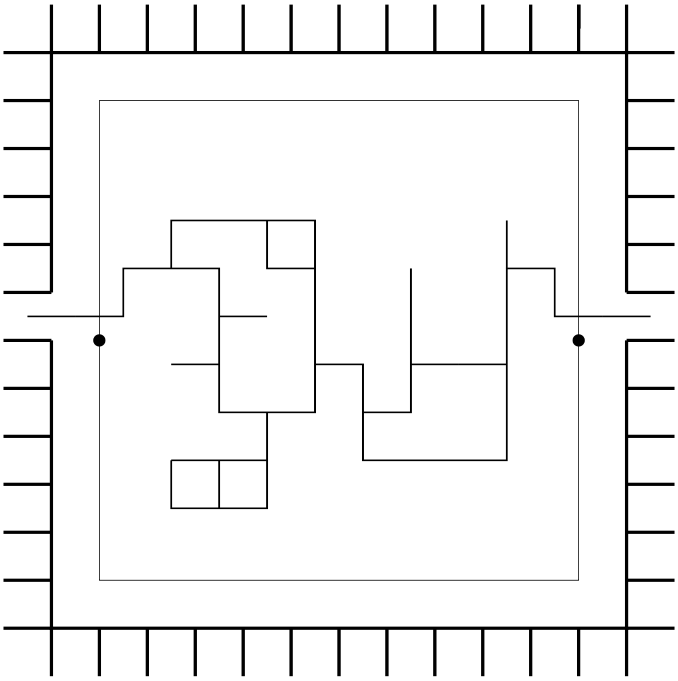

Figure 1: The box $\Lambda_{L,M}$. The heavy black edges are those given by the boundary condition μ, and there is a two-dimensional sketch of the interface Δ.

We shall be particularly concerned with the case $E = E_{L,M}$ and with a boundary condition μ corresponding to the mixed 'Dobrushin boundary' of [13]. To this end, we let μ be given by

$$\mu(e) = \begin{cases} 0 & \text{if } e = \langle x, y\rangle \text{ for some } x = (x_1, x_2, 0) \text{ and } y = (x_1, x_2, 1), \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

We let $\Omega^{\mu}_{L,M}$ be the set of all configurations $\omega \in \Omega$ such that $\omega(f) = \mu(f)$ if $f \notin E_{L,M}$. We define $\mathcal{I}_{L,M}$ to be the event that there exists no open path connecting a vertex of $\partial^+\Lambda_{L,M}$ to a vertex of $\partial^-\Lambda_{L,M}$. Let $\overline{\phi}^{\mu}_{\Lambda_{L,M},p,q}$ denote the measure $\phi^{\mu}_{G,p,q}$ conditioned on the event $\mathcal{I}_{L,M}$. See Figure 1.

The measure $\overline{\phi}^{\mu}_{\Lambda_{L,M},p,q}$ is only one of many such conditioned measures. Let E be a finite subset of $\mathbb{E}$, let $V = V(E)$, and write $G = (V, E)$ as usual. In a more general formulation, we take some boundary condition ζ, and we consider the set $\mathcal{C}(\zeta)$ of open components of ζ in the graph obtained from $\mathbb{L}$ by removing both E and all vertices adjacent to no edge in E^{c}. Let S be some set of labels, let $l : \mathcal{C}(\zeta) \to S$, and call $l(C)$ the *label* of $C \in \mathcal{C}$. We now consider the measure $\phi^{\zeta}_{G,p,q}$ conditioned on the event that no open path exists joining two vertices lying in components of $\mathcal{C}(\zeta)$ having different labels, and we denote this new measure by $\overline{\phi}^{\zeta,l}_{G,p,q}$. The case above arises when $E = E_{L,M}$ and $\zeta = \mu$, (note that $|\mathcal{C}(\mu)| = 2$), and the two members of $\mathcal{C}(\mu)$ have different labels.

3. Summary of main results

We summarise our main results as follows. The complete form of our main theorems appears with proofs in Section 9, using notation developed in the course of the work.

Many of our calculations concern the box $\Lambda_{L,M}$ and the measure $\overline{\phi}^{\mu}_{\Lambda_{L,M},p,q}$. We choose however to express our conclusions in terms of the infinite cylinder $\Lambda_L = \Lambda_{L,\infty}$ and the weak limit $\overline{\phi}_{L,p,q} = \lim_{M\to\infty} \overline{\phi}^{\mu}_{\Lambda_{L,M},p,q}$, which is shown in Lemma 8 to exist.

We show in Proposition 4 that, on the event $\mathcal{I}_{L,M} \cap \Omega^{\mu}_{L,M}$, there exists an 'interface which spans the equator' of $\Lambda_{L,M}$. (By the equator, we mean the circuit of $\Lambda_{L,M} \setminus \Lambda_{L-1,M}$ comprising all vertices x with $x_3 = \frac{1}{2}$.) Much of this paper is devoted to understanding the geometry of such an interface. We shall see in Theorem 2 that, in the limit as $M \to \infty$ and for sufficiently large p, this interface deviates, $\overline{\phi}_{L,p,q}$-almost surely, only locally from the flat plane through the equator of Λ_L. Indeed, the spatial density of such deviations approaches zero as p approaches 1. As a concrete application we present the following theorem, which we note to be a substantial weakening of Theorem 2 in Section 9.

Theorem 1 *Let $q \geq 1$. For all $\epsilon > 0$ there exists $\widehat{p} = \widehat{p}(\epsilon) < 1$ such that, if $p > \widehat{p}$,*

$$\overline{\phi}_{L,p,q}\Big(x \leftrightarrow \partial^-\Lambda_L\Big) > 1-\epsilon, \quad \overline{\phi}_{L,p,q}\Big(x + (0,0,1) \leftrightarrow \partial^+\Lambda_L\Big) > 1-\epsilon, \tag{3}$$

for all $L \geq 1$ and every $x = (x_1, x_2, 0) \in \Lambda_L$.

We have no proof that the sequence $\{\overline{\phi}_{L,p,q} : L \geq 1\}$ converges weakly as $L \to \infty$, but, by the usual compactness argument, there must exist weak limits of the sequence. It is a consequence of our main Theorem 2 that, for sufficiently large p, any such weak limit is non-translation-invariant. By making use of the relationship between random-cluster models and Potts models (see [2, 20] and the references therein), one obtains thereby a generalization of the theorem of Dobrushin [13] to include percolation and Potts models. We return to this point in Section 9, where it is shown in addition that there exists a geometric bound, uniform in L, on the tail of the displacement of the interface from the flat plane.

It would be interesting to know more of the random field defined by the locations where the interface coincides with the flat plane through the equator. It might be asked whether this field dominates (stochastically) a percolation process of some density $\rho(p)$, where $\rho(p) \to 1$ as $p \to 1$? Alan Stacey (personal communication) has pointed out that this does not hold, since the 'price' for a deviation from the flat plane over a region R depends on the length of the boundary of R rather than on its area.

Our strategy is to follow the milestones of the paper of Dobrushin [13], the methods of which are widely understood. Although Dobrushin's work is a helpful indicator of the overall route to the results, a considerable amount of extra work, involving new ideas, is necessary in the present context. For example, the geometry of interfaces is different for the random-cluster model from that for spin systems, and we shall furthermore require probabilistic estimates which are intrinsic to the present setting.

4. Properties of random-cluster measures

There follow some general lemmas concerning random-cluster measures. The first of these contains the comparison inequalities of Fortuin and Kasteleyn. There is a partial order on Ω given by $\omega_1 \le \omega_2$ if and only if $\omega_1(e) \le \omega_2(e)$ for all $e \in \mathbb{E}$. A function $h : \Omega \to \mathbb{R}$ is called *increasing* if it is increasing with respect to this partial order. Given two probability measures P_1, P_2 on $(\Omega, \mathcal{F})$, we write $P_1 \le_{\mathrm{st}} P_2$ if $\int h\, dP_1 \le \int h\, dP_2$ for all bounded measurable increasing functions h.

Lemma 1 *Let E be a finite subset of $\mathbb{E}$, and $G = (V, E)$ where $V = V(E)$. For any $\zeta \in \Omega$, we have that*

$$\begin{aligned} &\phi^{\zeta}_{G,p',q'} \le_{\mathrm{st}} \phi^{\zeta}_{G,p,q} \quad \text{if} \quad p' \le p,\ q' \ge q,\ q' \ge 1, \\ &\phi^{\zeta}_{G,p',q'} \ge_{\mathrm{st}} \phi^{\zeta}_{G,p,q} \quad \text{if} \quad \frac{p'}{q'(1-p')} \ge \frac{p}{q(1-p)},\ q' \ge q,\ q' \ge 1. \end{aligned} \tag{4}$$

See [2, 22] for a proof of these standard inequalities. Our second lemma is a formula for the partition function in terms of the edge densities. For $e \in \mathbb{E}$, we write J_e for the event that e is open.

Lemma 2 *Let E be a finite subset of $\mathbb{E}$, and $G = (V, E)$ where $V = V(E)$. For any $\zeta \in \Omega$, we have that*

$$\log Z^{\zeta}_{G,p,q} = k_G(\zeta^1) \log q + \sum_{e \in E} g^{\zeta}_{G,p,q}(e), \tag{5}$$

where ζ^1 is the configuration obtained from ζ by making every edge in E open, and

$$g^{\zeta}_{G,p,q}(e) = \int_p^1 \left[\frac{r - \phi^{\zeta}_{G,r,q}(J_e)}{r(1-r)} \right] dr. \tag{6}$$

Proof. We differentiate $\log Z^{\zeta}_{G,r,q}$ with respect to r, as in [22], p. 1479, to obtain that

$$\frac{d}{dr} \log Z^{\zeta}_{G,r,q} = \sum_{e \in E} \frac{\phi^{\zeta}_{G,r,q}(J_e) - r}{r(1-r)}.$$

This we integrate from p to 1, and note that $\log Z^{\zeta}_{G,1,q} = k_G(\zeta^1) \log q$. ∎

Let $q \ge 1$. We have by Lemma 1 that $\phi^{\zeta}_{G,r',1} \le_{\mathrm{st}} \phi^{\zeta}_{G,r,q} \le_{\mathrm{st}} \phi^{\zeta}_{G,r,1}$ where $r' = r/(r + (1-r)q)$, and hence

$$\frac{r}{r + (1-r)q} \le \phi^{\zeta}_{G,r,q}(J_e) \le r.$$

By substitution into (6),

$$0 \le g^{\zeta}_{G,p,q}(e) \le \int_p^1 (q-1)\,dr = (1-p)(q-1) \quad \text{for } e \in E, \tag{7}$$

uniformly in the choice of E and ζ. The above inequalities are reversed if $q < 1$.

We recall for the next lemma that $Q_n = \Lambda_{n,n}$, and, for $e \in \mathbb{E}$, we write $Q_n(e) = e + Q_n$, the set of translates of the endvertices of e by vectors in $\Lambda_{n,n}$.

Lemma 3 *Let $q \ge 1$. There exists $p^* = p^*(q) < 1$ and a constant $\alpha > 0$ such that the following holds. Let E_1 and E_2 be finite edge sets such that $e \in E_1 \cap E_2$, and let $n \ge 1$ be such that $E_1 \cap Q_n(e) = E_2 \cap Q_n(e)$. If $p > p^*$,*

$$|g^1_{G_1,p,q}(e) - g^1_{G_2,p,q}(e)| \le e^{-\alpha n},$$

where $G_i = (V(E_i), E_i)$.

Proof. Let L_e be the event that the endvertices of the edge e are joined by an open path which does not use e itself. It is an elementary argument, using equation (3.10) of [22], that

$$\frac{r - \phi^1_{G,r,q}(J_e)}{r(1-r)} = \frac{(q-1)(1-\phi^1_{G,r,q}(L_e))}{r + (1-r)q},$$

whence

$$|g^1_{G_1,p,q}(e) - g^1_{G_2,p,q}(e)| \le \int_p^1 \frac{(q-1)}{r+(1-r)q} |\phi^1_{G_1,r,q}(L_e) - \phi^1_{G_2,r,q}(L_e)|\,dr. \tag{8}$$

Fix $n \ge 1$. We shall now follow an argument of [22], pp. 1486–1487, and [29], pp. 138–152, of which we give some details next. Let $\mathcal{L}$ be derived from $\mathbb{L}$ by adding edges between any pair x, y of vertices with $\|x - y\| = 1$. For $\omega \in \Omega$, we call a vertex x *white* if $\omega(e) = 1$ for all e incident with x in $\mathbb{L}$, and *black* otherwise. Let V be the set of vertices which are incident in $\mathbb{L}$ to edges of both $Q_n(e)$ and its complement. We define B as the union of V together with all vertices $x_0 \in \mathbb{Z}^3$ for which there exists a path $x_0, x_1, \dots, x_m$ of $\mathcal{L}$ such that $x_0, x_1, \dots, x_{m-1} \notin V$, $x_m \in V$, and $x_0, x_1, \dots, x_{m-1}$ are black. Let K_n be the event that there exists no $x \in B$ such that $\|x - z\| \le 10$, say, where z is the centre of e. Using (5.17)–(5.18) of [22], together with estimates at the beginning of the proof of Lemma (2.24) of [29], we find that

$$\phi^0_{Q_n(e),r,q}(K_n) \ge 1 - c^n(1-\pi)^{en} \tag{9}$$

where c and e are absolute positive constants, and $\pi = r/(r + (1-r)q)$. Since K_n is an increasing event, we deduce that

$$\phi^1_{G_1,r,q}(K_n) \ge 1 - c^n(1-\pi)^{en}. \tag{10}$$

Let $H = E_1 \cap Q_n(e)$. It follows by the arguments of [22], p. 1487, and by coupling, that

$$0 \le \phi^1_{H,r,q}(L_e) - \phi^1_{G_1,r,q}(L_e) \le 1 - \phi^1_{G_1,r,q}(K_n).$$

The claim then follows by (8), (10), and the triangle inequality. ∎

5. Interfaces and geometry

We shall have much recourse to the dual of the random-cluster model, being a probability measure on the set of 'plaquettes' of the dual lattice $\mathbb{L}_{\mathrm{d}}$ obtained by shifting the vertices and edges of $\mathbb{L}$ through the vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (see [1, 23]). A *plaquette* of $\mathbb{L}_{\mathrm{d}}$ is a (topologically) closed unit square of $\mathbb{R}^3$ with corners lying in $\mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. We denote by $\mathbb{H}$ the set of all plaquettes of $\mathbb{L}_{\mathrm{d}}$. The straight line segment joining the vertices of an edge $\langle x, y\rangle$ passes through the middle of exactly one plaquette, denoted $h(\langle x, y\rangle)$, which we call the *dual* plaquette of $\langle x, y\rangle$. We declare this plaquette *open* (respectively *closed*) if $\langle x, y\rangle$ is closed (respectively open). The plaquette $h(\langle x, y\rangle)$ is called *horizontal* if $y = x + (0, 0, \pm 1)$, and *vertical* otherwise.

Two distinct plaquettes h_1 and h_2 are called 0-*connected*, written $h_1 \overset{0}{\sim} h_2$ if $h_1 \cap h_2 \neq \emptyset$. They are said to be 1-*connected*, written $h_1 \overset{1}{\sim} h_2$, if $h_1 \cap h_2$ is homeomorphic to the unit interval $[0, 1]$. A set of plaquettes is called 0-connected (respectively 1-connected) if they are connected when viewed as the vertex-set of a graph with adjacency relation $\overset{0}{\sim}$ (respectively $\overset{1}{\sim}$). The L^∞ distance between two plaquettes h_1, h_2 is defined to be the distance between their centres, and is denoted $\|h_1, h_2\|$. For any set H of plaquettes, we write $E(H)$ for the set of edges of $\mathbb{L}$ to which they are dual.

We define the *regular interface* as the set δ_0 given by

$$\delta_0 = \Big\{h \in \mathbb{H} : h = h(\langle x, y\rangle) \text{ for some } x = (x_1, x_2, 0) \text{ and } y = (x_1, x_2, 1)\Big\}.$$

The *interface* $\Delta(\omega)$ of a configuration $\omega \in \mathcal{I}_{L,M} \cap \Omega^\mu_{L,M}$ is defined to be the maximal 1-connected set of open plaquettes containing the plaquettes of $\delta_0 \setminus \{h(e) : e \in E_{L,M}\}$. The set of all interfaces is

$$\mathcal{D}_{L,M} = \{\Delta(\omega) : \omega \in \mathcal{I}_{L,M} \cap \Omega^\mu_{L,M}\}. \tag{11}$$

While it is tempting to think of an interface as part of a deformed plane, it may in fact have a much more complex geometry involving cavities and attachments. The following proposition, which will be proved later in this section, confirms that the interfaces in $\mathcal{D}_{L,M}$ separate the top of $\Lambda_{L,M}$ from its bottom.

Proposition 4 *The event $\mathcal{I}_{L,M} \cap \Omega^\mu_{L,M}$ is the set of all configurations $\omega \in \Omega^\mu_{L,M}$ for which there exists $\delta \in \mathcal{D}_{L,M}$ such that $\omega(e) = 0$ whenever $h(e) \in \delta$.*

For $\delta \in \mathcal{D}_{L,M}$, we define its *extended interface* $\overline{\delta}$ to be the set

$$\overline{\delta} = \delta \cup \{h \in \mathbb{H} : h \text{ is 1-connected to some member of } \delta\}. \tag{12}$$

It will be useful to introduce the 'maximal' ($\overline{\omega}_\delta$) and 'minimal' ($\underline{\omega}_\delta$) configurations in $\Omega^\mu_{L,M}$ which are compatible with δ:

$$\overline{\omega}_\delta(e) = \begin{cases} 0 & \text{if } e \in \delta, \\ 1 & \text{otherwise,} \end{cases} \qquad \underline{\omega}_\delta(e) = \begin{cases} \mu(e) & \text{if } e \notin E_{L,M}, \\ 1 & \text{if } e \in E_{L,M} \cap (\overline{\delta} \setminus \delta), \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

In Section 6, we shall consider interfaces spanning the equator of the infinite cylinder Λ_L.

We consider next some geometrical matters. The words 'connected' and 'component' should be interpreted for the moment in the topological sense. Let $T \subseteq \mathbb{R}^3$, and write $\overline{T}$ for the closure of T in $\mathbb{R}^3$. We define the *inside* $\mathrm{ins}(T)$ of T to be the union of all the bounded connected components of $\mathbb{R}^3 \setminus T$; the *outside* $\mathrm{out}(T)$ is the union of all the unbounded connected components of $\mathbb{R}^3 \setminus T$. The set T is said to *separate* $\mathbb{R}^3$ if $\mathbb{R}^3 \setminus T$ has more than one connected component. For a set $H \subseteq \mathbb{H}$ of plaquettes, we define the set $[H] \subseteq \mathbb{R}^3$ by $[H] = \{x \in \mathbb{R}^3 : x \in h \text{ for some } h \in H\}$. We call a finite set H of plaquettes a *splitting set* if $[H]$ is 1-connected in $\mathbb{R}^3$ and $\mathbb{R}^3 \setminus [H]$ contains at least one bounded connected component.

The following two propositions are in a sense dual to one another, and we believe they will find applications elsewhere. The first is an analogue in three dimensions of Proposition 2.1 of the Appendix of [28], where two-dimensional mosaics are considered. Related results may be found in [8, 12, 23].

Proposition 5 *Let $G = (V, E)$ be a finite connected subgraph of $\mathbb{L}$. There exists a splitting set Q of plaquettes such that*:

(i) $V \subseteq \mathrm{ins}([Q])$,

(ii) *every plaquette in Q is dual to some edge of $\mathbb{E}$ having exactly one endvertex in V,*

(iii) *if W is a connected set of vertices such that $V \cap W = \emptyset$, and there exists an infinite path on $\mathbb{L}$ starting in W which uses no vertices in V, then $W \subseteq \mathrm{out}([Q])$.*

Let $\delta = \{h(e) : e \in D\}$ be a 1-connected set of plaquettes, and let $\overline{\delta}$ be given as in (12). Consider the graph $(\mathbb{Z}^3, \mathbb{E} \setminus D)$, and let C be a connected component of this graph. Let $\Delta_{\mathrm{v}}C$ be the set of all vertices v in C for which there exists $w \in \mathbb{Z}^3$ with $h(\langle v, w\rangle) \in \overline{\delta}$, and let $\Delta_{\mathrm{e}}C$ be the set of edges f of C for which $h(f) \in \overline{\delta} \setminus \delta$. Note that edges in $\Delta_{\mathrm{e}}C$ have both endvertices belonging to $\Delta_{\mathrm{v}}C$.

Proposition 6 *For any finite connected component C of the graph $(\mathbb{Z}^d, \mathbb{E} \setminus D)$, the graph $(\Delta_{\mathrm{v}}C, \Delta_{\mathrm{e}}C)$ is connected.*

We shall apply this proposition in the following way. Let $\delta \in \mathcal{D}_{L,M}$. Consider the connected components of the graph $(\mathbb{Z}^3, \eta(\overline{\omega}_\delta))$, and denote these components as (S^i_δ, U^i_δ), $i = 1, 2, \ldots, K_\delta$, where $K_\delta = k(\overline{\omega}_\delta)$. Note that U^i_δ is empty whenever S^i_δ is a singleton. We define $W(\delta)$ as the set of edges in $E_{L,M} \setminus \{e \in \mathbb{E} : h(e) \in \overline{\delta}\}$.

Let $\omega \in \mathcal{I}_{L,M} \cap \Omega^\mu_{L,M}$ be such that $\Delta(\omega) = \delta$. It must be the case that

$$\omega(e) = \begin{cases} 0 & \text{if } h(e) \in \delta, \\ 1 & \text{if } h(e) \in \overline{\delta} \setminus \delta. \end{cases} \tag{14}$$

Let D be the set of edges having both endvertices in $\Lambda_{L+2,M+2}$ which either are dual to plaquettes in δ or join a vertex of $\Lambda_{L+1,M+1}$ to a vertex of $\partial\Lambda_{L+1,M+1}$. We apply Proposition 6 to the set D, and deduce that the number of components in the graph $(\mathbb{Z}^3, \eta(\omega))$ having a vertex in $V(\overline{\delta})$ is simply K_δ. We shall make use of this observation in the next section when we consider conditioning on events of the form $\{\Delta(\omega) = \delta\}$.

Proof of Proposition 5. This may be proved by extending the proof of Lemma 7.2 of [23]. Instead, we present a variant of that proof. Consider the set of edges with exactly one endvertex in V and let P be the corresponding set of plaquettes.

Let $x \in V$. We first show that $x \in \mathrm{ins}([P])$. Let $\mathcal{U}$ be the set of all closed unit cubes of $\mathbb{R}^3$ having centres in V. Since all relevant sets in this proof are simplicial, the notions of path-connectedness and arc-connectedness coincide. We recall that an unbounded path of $\mathbb{R}^3$ from x is defined to be a continuous mapping $\gamma : [0,\infty) \to \mathbb{R}^3$ with $\gamma(0) = x$ whose image is unbounded. Any such path γ satisfying $|\gamma(t)| \to \infty$ as $t \to \infty$ has a final point $z(\gamma)$ belonging to the (closed) union of all cubes in $\mathcal{U}$. Now $z(\gamma) \in [P]$ for all such γ, and therefore $x \in \mathrm{ins}([P])$.

Let $P_1, P_2, \dots, P_n$ be the partition of P such that the sets $[P_1], [P_2], \dots, [P_n]$ are the 1-connected components of $[P]$ in $\mathbb{R}^3$. Note that $[P_i] \cap [P_j]$ is a finite (or empty) set for $i \neq j$. We show next that there exists i such that $x \in \mathrm{ins}([P_i])$. Suppose for the sake of contradiction that this is false, which is to say that $x \notin \mathrm{ins}([P_i])$ for all i. Then $x \notin \overline{P}_i = [P_i] \cup \mathrm{ins}([P_i])$ for $i = 1, 2, \dots, n$. Note that each $\overline{P}_i$ is a closed set which does not separate $\mathbb{R}^3$.

Let $i \neq j$. We claim that: either $\overline{P}_i \cap \overline{P}_j$ is a finite set, or one of the sets $\overline{P}_i$, $\overline{P}_j$ is a subset of the other. To see this, suppose that $\overline{P}_i \cap \overline{P}_j$ is an infinite set. Suppose further that $\overline{P}_i \cap [P_j]$ is infinite. Since $[P_j]$ is a union of unit squares and $\overline{P}_i$ is a union of unit squares and cubes, all with corners in $\mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, there exists some edge f of $\mathbb{L}_\mathrm{d}$ such that $f \subseteq \overline{P}_i \cap [P_j]$. We cannot have $f \subseteq [P_i]$ since $[P_i] \cap [P_j]$ is finite, whence $f^o \subseteq \mathrm{ins}([P_i])$, where f^o denotes the open straight-line segment of $\mathbb{R}^3$ joining the endvertices of f. Now $[P_j]$ is 1-connected and $[P_i] \cap [P_j]$ is finite, so that $[P_j]$ is contained in the closure of $\mathrm{ins}([P_i])$, implying that $[P_j] \subseteq \overline{P}_i$ and therefore $\overline{P}_j \subseteq \overline{P}_i$.

Suppose next that $\overline{P}_i \cap [P_j]$ is finite but $\overline{P}_i \cap \mathrm{ins}([P_j])$ is infinite. Since $[P_i]$ is 1-connected, it has by definition no finite cutset. Since $[P_i] \cap [P_j]$ is finite, either $[P_i] \subseteq \overline{P}_j$ or $[P_i]$ is contained in the closure of the unbounded component of $\mathbb{R}^3 \setminus [P_j]$. The latter cannot hold since $\overline{P}_i \cap \mathrm{ins}([P_j])$ is infinite, whence $[P_i] \subseteq \overline{P}_j$ and therefore $\overline{P}_i \subseteq \overline{P}_j$.

It follows that we may write $R = \bigcup_{i=1}^n \overline{P}_i$ as the union of a collection of closed bounded sets $\widetilde{P}_i$, $i = 1, 2, \dots, k$ where $k \leq n$, that do not separate $\mathbb{R}^3$ and such that $\widetilde{P}_i \cap \widetilde{P}_j$ is finite for $i \neq j$. We shall now use Theorem 11 of [31] (§59, Section II) which, for clarity of exposition, we state in the language of the original: *If none of the closed sets F_0 and F_1 cuts $\mathcal{S}_n$ between the points p and q and if* $\dim(F_0 \cap F_1) \leq n - 3$*, their union $F_0 \cup F_1$ does it neither*. [Here, $\mathcal{S}_n$ is the n-sphere, and we shall apply this with

$n = 3$.] It follows by this theorem that R does not separate $\mathbb{R}^3$. Now $x \notin R$, whence x lies in the unique component of the complement $\mathbb{R}^3 \setminus R$, in contradiction of the assumption that $x \in \text{ins}([P])$. We deduce that there exists k such that $x \in \text{ins}([P_k])$, and we define $Q = P_k$.

Consider now a vertex $y \in V$. Since $G = (V, E)$ is connected, there exists a path in $\mathbb{L}$ that connects x with y using only vertices in V. Whenever u and v are two consecutive vertices on this path, $h(\langle u, v\rangle)$ does not belong to P. It follows that y lies in the inside of $[Q]$. Claims (i) and (ii) are now proved with Q as given, and it remains to prove (iii).

Let W be as in (iii), and let $w \in W$. There exists a path on $\mathbb{L}$ from w to infinity using no vertices of V. Whenever u and v are two consecutive vertices on such a path, the plaquette $h(\langle u, v\rangle)$ does not lie in P. It follows that $w \in \text{out}([P])$, and therefore $w \in \text{out}([Q])$. ∎

Proof of Proposition 6. Let $H = (\Delta_{\text{v}} C, \Delta_{\text{e}} C)$, and let H_x be the connected component of H containing the vertex x. We claim that there exists a plaquette $h_x = h(\langle y, z\rangle) \in \delta$ such that $y \in H_x$. This may be shown as follows.

The claim holds with $y = x$ and $h_x = h(\langle x, z\rangle)$ if x has a neighbour z with $h(\langle x, z\rangle) \in \delta$. Assume therefore that x has no such neighbour z. There exists a neighbour u of x with $h(\langle x, u\rangle) \in \overline{\delta} \setminus \delta$. By a consideration of the various possibilities, there exists $\widetilde{h} \in \delta$ such that $\widetilde{h} \overset{1}{\sim} h(\langle x, u\rangle)$, and

$$\begin{aligned} \text{either (i) } & \widetilde{h} = h(\langle u, z\rangle) \text{ for some } z, \\ \text{or (ii) } & \widetilde{h} = h(\langle v, z\rangle) \text{ for some } v \sim x,\ z \sim u. \end{aligned}$$

If (i) holds we take $y = u$, $h_x = \widetilde{h}$, and if (ii) holds we take $y = v$ $(\in H_x)$, $h_x = \widetilde{h}$.

We apply Proposition 5 with $G = H_x$ to obtain a splitting set Q_x, and we claim that

$$Q_x \cap \delta \neq \emptyset. \tag{15}$$

This we prove as follows. If $h_x \in Q_x$, the claim is immediate. Suppose then that $h_x \notin Q_x$, so that $[h_x] \cap \text{ins}([Q_x]) \neq \emptyset$, implying that δ intersects both $\text{ins}([Q_x])$ and $\text{out}([Q_x])$. Since δ and Q_x are 1-connected sets of plaquettes, it follows that $\delta \cup Q_x$ is 1-connected. Therefore there exist $h' \in \delta$, $h'' \in Q_x$ such that $h' \overset{1}{\sim} h''$. If $h'' \in \delta$, then (15) holds, so we may assume that $h'' \notin \delta$, and hence $h'' \in \overline{\delta} \setminus \delta$. Then $h'' = h(\langle u, v\rangle)$ where $u \in H_x$, and therefore $v \in H_x$, a contradiction. We conclude that (15) holds.

We claim that (15) implies $Q_x \subseteq \delta$. Suppose on the contrary that $Q_x \nsubseteq \delta$, so that there exist $h' \in \delta$, $h'' \in Q_x \setminus \delta$ such that $h' \overset{1}{\sim} h''$. This leads to a contradiction by the argument just given, whence $Q_x \subseteq \delta$.

Suppose now that x and y are vertices of H such that H_x and H_y are distinct connected components. Then either H_x lies in $\text{out}([Q_y])$, or H_y lies in $\text{out}([Q_x])$. Since $Q_x, Q_y \subseteq \delta$, either possibility contradicts the assumption that x and y are connected in C. Therefore $H_x = H_y$ as claimed. ∎

Proof of Proposition 4. If $\omega \in \mathcal{I}_{L,M} \cap \Omega^\mu_{L,M}$, then by definition $\omega(e) = 0$ whenever $h(e) \in \Delta(\omega)$. Suppose conversely that $\delta \in \mathcal{D}_{L,M}$, and let $\omega \in \Omega^\mu_{L,M}$ satisfy $\omega(e) = 0$ whenever $h(e) \in \delta$. Since $\omega \le \overline{\omega}_\delta$, it suffices to show that $\overline{\omega}_\delta \in \mathcal{I}_{L,M}$. Since $\delta \in \mathcal{D}_{L,M}$, there exists $\xi \in \mathcal{I}_{L,M} \cap \Omega^\mu_{L,M}$ such that $\delta = \Delta(\xi)$. Note that $\xi \le \overline{\omega}_\delta$. Suppose for the sake of a contradiction that $\overline{\omega}_\delta \notin \mathcal{I}_{L,M}$, and think of $\overline{\omega}_\delta$ as being obtained from ξ by declaring a certain sequence $e_1, e_2, \dots, e_r$ with $\xi(e_i) = 0$ for $1 \le i \le r$, in turn, to be open. Let ξ^k be obtained from ξ by $\eta(\xi^k) = \eta(\xi) \cup \{e_1, e_2, \dots, e_k\}$. By assumption, there exists K such that $\xi^K \in \mathcal{I}_{L,M}$ but $\xi^{K+1} \notin \mathcal{I}_{L,M}$. For $\psi \in \Omega^\mu_{L,M}$, let $J(\psi)$ denote the set of all edges e having endvertices in $\Lambda_{L,M}$, with $\psi(e) = 1$, and both of whose endvertices are attainable from $\partial^+\Lambda_{L,M}$ by open paths of ψ. We apply Proposition 5 to the finite connected graph induced by $J(\xi^K)$ to find that there exists a splitting set Q of plaquettes such that: $\partial^+\Lambda_{L,M} \subseteq \mathrm{ins}([Q])$, $\partial^-\Lambda_{L,M} \subseteq \mathrm{out}([Q])$, and $\xi^K(e) = 0$ whenever $e \in E_{L,M}$ and $h(e) \in Q$. It must be the case that $h(e_{K+1}) \in Q$, since $\xi^{K+1} \notin \mathcal{I}_{L,M}$. By the 1-connectedness of Q, there exists a sequence $f_1 = e_{K+1}, f_2, f_3, \dots, f_t$ of edges such that:

(i) $h(f_i) \in Q$ for all i,

(ii) $f_i \in E_{L,M}$ for $1 \le i < t$, $f_t = h(\langle x, x - (0,0,1)\rangle)$ for some $x = (x_1, x_2, 1) \in \partial^+\Lambda_{L,M}$,

(iii) $h(f_i) \stackrel{1}{\sim} h(f_{i+1})$ for $1 \le i < t$.

It follows that $h(f_i) \in \delta$ for $1 \le i \le t$. In particular, $h(e_{K+1}) \in \delta$ and so $\overline{\omega}_\delta(e_{K+1}) = 0$, a contradiction. Therefore $\overline{\omega}_\delta \in \mathcal{I}_{L,M}$ as claimed. ∎

6. Probability distribution of the interface

For conciseness of notation, we shall henceforth abbreviate $\phi^\mu_{\Lambda_{L,M},p,q}$ to $\phi_{L,M}$, and $\overline{\phi}^\mu_{\Lambda_{L,M},p,q}$ to $\overline{\phi}_{L,M}$. Let $\delta \in \mathcal{D}_{L,M}$. We derive next an expression for the probability $\phi_{L,M}(\Delta = \delta)$, which we abbreviate to $\phi_{L,M}(\delta)$.

Let K_δ be the number of components of the graph $(\mathbb{Z}^3, \eta(\overline{\omega}_\delta))$, and recall from the discussion after Proposition 6 that, if $\omega \in \mathcal{I}_{L,M} \cap \Omega^\mu_{L,M}$ and $\Delta(\omega) = \delta$, then ω has exactly K_δ open components intersecting $V(\overline{\delta})$. We have that

$$\begin{aligned}\phi_{L,M}(\delta) &= \frac{1}{Z(E_{L,M})} p^{|\overline{\delta}\setminus\delta|}(1-p)^{|\delta|} \sum_{\substack{\omega \in \Omega^\mu_{L,M}:\\ \Delta(\omega)=\delta}} \left\{ \prod_{e \in W(\delta)} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega)} \\ &= \frac{Z^1(\delta)}{Z(E_{L,M})} p^{|\overline{\delta}\setminus\delta|}(1-p)^{|\delta|} q^{K_\delta - 1}, \qquad (16)\end{aligned}$$

where $Z(E_{L,M}) = Z^\mu_{\Lambda_{L,M},p,q}$ and $Z^1(\delta) = Z^1_{W(\delta),p,q}$ as in (1). [As before, $W(\delta) = E_{L,M} \setminus \{e \in \mathbb{E} : h(e) \in \overline{\delta}\}$.] In this expression and later, for $H \subseteq \mathbb{H}$, $|H|$ denotes the number of plaquettes in the set $H \cap \{h(e) : e \in E_{L,M}\}$. The term $q^{K_\delta - 1}$ arises since the application of '1' boundary conditions to δ has the effect of uniting the boundaries of the cavities of δ, whereby the number of clusters diminishes by $K_\delta - 1$.

We next exploit properties of the partition functions $Z(\cdot)$ in order to rewrite (16). For $i = 1, 2$, let $L_i > 0$, $M_i > 0$, $\delta_i \in \mathcal{D}_{L_i,M_i}$, and $e_i \in E(\delta_i) \cap E_{L_i,M_i}$, and let

$$G(e_1, \delta_1, E_{L_1,M_1}; e_2, \delta_2, E_{L_2,M_2}) = \sup \Big\{ L : \begin{array}{l} Q_L(e_1) \cap E_{L_1,M_1} \simeq Q_L(e_2) \cap E_{L_2,M_2} \\ \text{and } Q_L(e_1) \cap E(\delta_1) \simeq Q_L(e_2) \cap E(\delta_2) \end{array} \Big\},$$

where $Q_L(e) = e + Q_L$ as before. We write $Z^1(E_{L,M}) = Z^1_{\Lambda_{L,M},p,q}$.

Proposition 7 *Let $L, M \geq 1$ and $\delta \in \mathcal{D}_{L,M}$. We may write $\phi_{L,M}(\delta)$ in the form*

$$\phi_{L,M}(\delta) = \frac{Z^1(E_{L,M})}{Z(E_{L,M})} p^{|\overline{\delta} \setminus \delta|} (1-p)^{|\delta|} q^{K_\delta - 1} \exp\left(\sum_{e \in E(\delta) \cap E_{L,M}} f_p(e, \delta, L, M) \right), \quad (17)$$

for functions $f_p(e, \delta, L, M)$ with the following properties. For $q \geq 1$ there exist $p^* < 1$ and constants C_1, C_2, $\gamma > 0$ such that, if $p > p^*$,

$$|f_p(e, \delta, L, M)| < C_1, \quad (18)$$

$$|f_p(e_1, \delta_1, L_1, M_1) - f_p(e_2, \delta_2, L_2, M_2)| \leq C_2 e^{-\gamma G},\ e_1 \in \delta_1,\ e_2 \in \delta_2,\ e_1 \simeq e_2, \quad (19)$$

where $G = G(e_1, \delta_1, E_{L_1,M_1}; e_2, \delta_2, E_{L_2,M_2})$. Inequalities (18) and (19) are valid for all relevant values of their arguments.

Proof. We have by Lemma 2 that

$$\log\left(\frac{Z^1(\delta)}{Z^1(E_{L,M})} \right) = \sum_{f \in W(\delta)} \Big[g(f, W(\delta)) - g(f, E_{L,M}) \Big] - \sum_{f \in E(\overline{\delta})} g(f, E_{L,M}), \quad (20)$$

where $g(f, D) = g^1_{D,p,q}(f)$. The summations may be expressed as sums over edges e lying in $E(\delta)$ in the following way. The edges in $\mathbb{E}$ may be ordered according to the lexicographic ordering of their centres. Let $f \in E_{L,M}$ and $\delta \in \mathcal{D}_{L,M}$. Amongst all edges in $E(\delta) \cap E_{L,M}$ which are closest to f (in the sense that their centres are closest in L^∞ norm), we write $\nu(f, \delta)$ for the earliest edge in this ordering. We have by (20) that

$$\log\left(\frac{Z^1(\delta)}{Z^1(E_{L,M})} \right) = \sum_{e \in E(\delta) \cap E_{L,M}} f_p(e, \delta, L, M) \quad (21)$$

where

$$f_p(e, \delta, L, M) = \sum_{\substack{f \in W(\delta): \\ \nu(f,\delta) = e}} \Big[g(f, W(\delta)) - g(f, E_{L,M}) \Big] - \sum_{\substack{f \in E(\overline{\delta}): \\ \nu(f,\delta) = e}} g(f, E_{L,M}). \quad (22)$$

This establishes (17) via (16).

It remains to show the required properties of the f_p. Suppose $e = \nu(f, \delta)$ and set $r = \|e, f\|$. Then $Q_{r-2,r-2}(f)$ does not intersect $\overline{\delta}$, implying by Lemma 3 that

$$|g(f, W(\delta)) - g(f, E_{L,M})| \le e^{-\alpha\|e,f\|+2\alpha} \quad \text{if } p > p^*, \tag{23}$$

where p^* and α are given as in that lemma. Secondly, there exists an absolute constant K such that, for all e and δ, the number of edges $f \in E(\overline{\delta})$ with $e = \nu(f, \delta)$ is no greater than K. Therefore, by (7),

$$|f_p(e, \delta, L, M)| \le \sum_{f \in \mathbb{E}} e^{-\alpha\|e,f\|+2\alpha} + K(1-p)(q-1)$$

as required for (18).

Finally we show (19) for $p > p^*$ and appropriate C_2, γ. Let $e \in \delta_1$, $e_2 \in \delta_2$, and let G be given as in the proposition; we may suppose that $G > 9$. By assumption, $e_1 \simeq e_2$, whence there exists a translate τ of $\mathbb{L}$ such that $\tau e_1 = e_2$. We have for $f \in W(\delta_1) \cap Q_{G/3}(e_1)$ that

$$\tau\Big[Q_{G/3}(f) \cap E_{L_1,M_1}\Big] = Q_{G/3}(\tau f) \cap E_{L_2,M_2}, \tag{24}$$

$$\tau\Big[Q_{G/3}(f) \cap \delta_1\Big] = Q_{G/3}(\tau f) \cap \delta_2, \tag{25}$$

and that

$$\text{for } \|f, e_1\| \le \tfrac{1}{3}G,\ \nu(f, \delta_1) = e_1 \text{ if and only if } \nu(\tau f, \delta_2) = e_2. \tag{26}$$

It follows from the definition (22) of the functions f_p that

$$\begin{aligned}
&|f_p(e_1, \delta_1, L_1, M_1) - f_p(e_2, \delta_2, L_2, M_2)| \\
&\quad\le \sum_{\substack{f \in W(\delta_1) \cap Q_{G/3}(e_1): \\ \nu(f,\delta_1) = e_1}} \Big\{|g(f, W(\delta_1)) - g(\tau f, W(\delta_2))| + |g(f, E_{L_1,M_1}) - g(\tau f, E_{L_2,M_2})|\Big\} \\
&\qquad + \sum_{\substack{f \in W(\delta_1) \setminus Q_{G/3}(e_1): \\ \nu(f,\delta_1) = e_1}} |g(f, W(\delta_1)) - g(f, E_{L_1,M_1})| \\
&\qquad + \sum_{\substack{f \in W(\delta_2) \setminus Q_{G/3}(e_2): \\ \nu(f,\delta_2) = e_2}} |g(f, W(\delta_2)) - g(f, E_{L_2,M_2})| + S,
\end{aligned} \tag{27}$$

where

$$S = \left| \sum_{\substack{f \in E(\overline{\delta}_1): \\ \nu(f,\delta_1) = e_1}} g(f, E_{L_1,M_1}) - \sum_{\substack{f \in E(\overline{\delta}_2): \\ \nu(f,\delta_2) = e_2}} g(f, E_{L_2,M_2}) \right|.$$

By (24), (25), and Lemma 3, the first summation in (27) is bounded above by $2G^3 e^{-\frac{1}{3}\alpha G}$. Using the definition of the $\nu(f, \delta_i)$, the second and third summations of (27) are bounded above, respectively, by

$$\sum_{f \notin Q_{G/3}(e_i)} e^{-\alpha\|f,e_i\|+2\alpha} \le C' e^{-\frac{1}{3}\alpha G + 2\alpha},$$

for some $C' < \infty$, as in (23). We have by (26) that

$$S = \left| \sum_{\substack{f \in E(\overline{\delta}_1): \\ \nu(f,\delta_1)=e_1}} g(f, E_{L_1,M_1}) - g(\tau f, E_{L_2,M_2}) \right| \leq K e^{-\frac{1}{3}\alpha G},$$

and inequality (19) is proved for an appropriate choice of γ. ■

In the next part of this section, we consider measures and interfaces for the infinite cylinder $\Lambda_L = \Lambda_{L,\infty} = [-L, L]^2 \times \mathbb{Z}$. We note first that, if $q \geq 1$, then $\phi_{L,M+1} \leq_{\mathrm{st}} \phi_{L,M}$, as in [21], Theorem 3.1(a), whence the (decreasing) weak limit

$$\phi_L = \lim_{M\to\infty} \phi_{L,M} \tag{28}$$

exists. We write Ω_L^μ for the set of all configurations ω such that $\omega(e) = \mu(e)$ for $e \notin E_L = \lim_{M\to\infty} E_{L,M}$, and $\mathcal{I}_L$ for the event that no vertex of $\partial\Lambda_L^+$ is joined by an open path to a vertex of $\partial\Lambda_L^-$. The set of interfaces on which we concentrate is $\mathcal{D}_L = \bigcup_M \mathcal{D}_{L,M} = \lim_{M\to\infty} \mathcal{D}_{L,M}$. Thus $\mathcal{D}_L$ is the set of interfaces which span Λ_L, and every member of $\mathcal{D}_L$ is bounded in the direction of the third coordinate. It is easy to see that $\mathcal{I}_L \supseteq \lim_{M\to\infty} \mathcal{I}_{L,M}$, and it is a consequence of the next lemma that the difference between these two events has ϕ_L-probability zero.

Lemma 8 *We have, if $q \geq 1$, that $\phi_{L,M}(\cdot \mid \mathcal{I}_{L,M}) \Rightarrow \phi_L(\cdot \mid \mathcal{I}_L)$ as $M \to \infty$, and that*

$$\phi_L\Big(\mathcal{I}_L \setminus \lim_{M\to\infty} \mathcal{I}_{L,M}\Big) = 0.$$

For $L_i > 0$, $\delta_i \in \mathcal{D}_{L_i}$, and $e_i \in E(\delta_i) \cap E_{L_i}$, let

$$\begin{aligned} G(e_1, \delta_1, E_{L_1}; e_2, \delta_2, E_{L_2}) = \sup\Big\{L :\ & Q_L(e_1) \cap E_{L_1} \simeq Q_L(e_2) \cap E_{L_2} \\ & \text{and } Q_L(e_1) \cap E(\delta_1) \simeq Q_L(e_2) \cap E(\delta_2)\Big\}. \end{aligned}$$

On the event $\mathcal{I}_L$, Δ is defined as before to be the maximal 1-connected set of open plaquettes which intersects $\delta_0 \setminus E_L$.

Lemma 9 (a) *Suppose $L > 0$, $\delta \in \mathcal{D}_L$, and $e \in E(\delta) \cap E_L$. The functions f_p given in (22) are such that the limit*

$$f_p(e, \delta, L) = \lim_{M\to\infty} f_p(e, \delta, L, M) \tag{29}$$

exists. Furthermore, if $p > p^*$,

$$|f_p(e, \delta, L)| < C_1, \tag{30}$$

and, for $L_i > 0$, $\delta_i \in \mathcal{D}_{L_i}$, and $e_i \in E(\delta_i) \cap E_{L_i}$ satisfying $e_1 \simeq e_2$,

$$|f_p(e_1, \delta_1, L_1) - f_p(e_2, \delta_2, L_2)| \leq C_2 e^{-\gamma G},$$

where p^*, C_1, C_2, γ are given as in Proposition 7 and $G = G(e_1, \delta_1, E_{L_1}; e_2, \delta_2, E_{L_2})$.
(b) *For $q \geq 1$ and $\delta \in \mathcal{D}_L$, the probability $\phi_L(\delta \mid \mathcal{I}_L) = \phi_L(\Delta = \delta \mid \mathcal{I}_L)$ is given by*

$$\phi_L(\delta \mid \mathcal{I}_L) = \frac{1}{Z_L} p^{|\overline{\delta}\setminus\delta|}(1-p)^{|\delta|} q^{K_\delta} \exp\left(\sum_{e \in E(\delta)\cap E_L} f_p(e, \delta, L) \right), \tag{31}$$

where Z_L is the appropriate normalizing constant.

Proof of Lemma 8. It suffices for the claim of weak convergence that

$$\phi_{L,M}(F \cap \mathcal{I}_{L,M}) \to \phi_L(F \cap \mathcal{I}_L) \quad \text{for all cylinder events } F. \tag{32}$$

Let $A_{L,M} = [-L, L]^2 \times \{-M\}$ and $B_{L,M} = [-L, L]^2 \times \{M\}$, and let $T_{L,M}$ be the event that no open path exists between a vertex of $\partial\Lambda^+_{L,M} \setminus B_{L,M}$ and a vertex of $\partial\Lambda^-_{L,M} \setminus A_{L,M}$. Note that $T_{L,M} \to \mathcal{I}_L$ as $M \to \infty$. Let F be a cylinder event. Then

$$\begin{aligned} \phi_{L,M}(F \cap \mathcal{I}_{L,M}) &\leq \phi_{L,M}(F \cap T_{L,M'}) && \text{for } M' \leq M \\ &\to \phi_L(F \cap T_{L,M'}) && \text{as } M \to \infty \\ &\to \phi_L(F \cap \mathcal{I}_L) && \text{as } M' \to \infty. \end{aligned} \tag{33}$$

In order to obtain a corresponding lower bound, we introduce the event K_r that all edges of E_L, both of whose endvertices have third coordinate equal to $\pm r$, are open. We may suppose without loss of generality that $p > 0$. We have by Lemma 1 that $\phi_{L,M}$ dominates product measure with density $\pi = p/\{p + (1-p)q\}$, whence there exists $\beta = \beta_L < 1$ such that

$$\phi_{L,M}(K_r \text{ for some } r \leq R) \geq 1 - \beta^R \quad \text{for } R < M.$$

Now $\mathcal{I}_{L,M} \subseteq T_{L,M}$, and $T_{L,M} \setminus \mathcal{I}_{L,M} \subseteq \bigcap_{r=1}^{M-1} K_r^{\mathrm{c}}$, whence

$$\begin{aligned} \phi_{L,M}(F \cap \mathcal{I}_{L,M}) &\geq \phi_{L,M}(F \cap T_{L,M}) - \beta^{M-1} \\ &\geq \phi_{L,M}(F \cap \mathcal{I}_L) - \beta^{M-1} \\ &\to \phi_L(F \cap \mathcal{I}_L) && \text{as } M \to \infty. \end{aligned} \tag{34}$$

Equation (32) follows from (33) and (34). The second claim of the lemma follows by taking $F = \Omega$, the entire sample space. ■

Proof of Lemma 9. (a) The existence of the limit follows from the monotonicity of $g(f, D_i)$ for an increasing sequence $\{D_i\}$, and the proof of (18). The inequalities are implied by (18) and (19).
(b) Let $\delta \in \mathcal{D}_L$, so that $\delta \in \mathcal{I}_{L,M}$ for all large M. By Lemma 8, $\phi_L(\delta \mid \mathcal{I}_L) = \lim_{M\to\infty} \phi_{L,M}(\delta \mid \mathcal{I}_{L,M})$. We take the limit as $M \to \infty$ in (17), and use part (a) to obtain the claim. ■

7. Geometry of interfaces

Next, we describe in more detail the interfaces in $\mathcal{D}_L = \lim_{M\to\infty} \mathcal{D}_{L,M}$. While it was natural in Section 5 to introduce the extended interface $\overline{\delta}$ of a member δ of $\mathcal{D}_L$, it turns out to be useful when studying its geometry to introduce its *semi-extended interface*

$$\delta^* = \delta \cup \Big\{h \in \mathbb{H} : h \text{ is a horizontal plaquette that is 1-connected to } \delta\Big\}.$$

Let $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$. The *projection* $\pi(h)$ of a horizontal plaquette $h = h(\langle x, x+(0,0,1)\rangle)$ onto the regular interface δ_0 is defined to be the plaquette $\pi(h) = h(\langle (x_1,x_2,0),(x_1,x_2,1)\rangle) \in \delta_0$. The projection of the vertical plaquette $h = h(\langle x, x+(1,0,0)\rangle)$ is the interval $\pi(h) = [(x_1+\frac12, x_2-\frac12, \frac12), (x_1+\frac12, x_2+\frac12, \frac12)]$. Similarly, $h = h(\langle x, x+(0,1,0)\rangle)$ has projection $\pi(h) = [(x_1-\frac12, x_2+\frac12, \frac12), (x_1+\frac12, x_2+\frac12, \frac12)]$.

Let $\delta \in \mathcal{D}_L$. A horizontal plaquette h of the semi-extended interface δ^* is called a *c-plaquette* if h is the unique member of δ^* having projection $\pi(h)$. All other plaquettes of δ^* are called *w-plaquettes.* A *ceiling* of δ is a maximal 0-connected set of c-plaquettes. The *projection* of a ceiling C is the set $\pi(C) = \{\pi(h) : h \in C\}$. Similarly, we define a *wall* W of δ as a maximal 0-connected set of w-plaquettes, and its *projection* as

$$\pi(W) = \{\pi(h) : h \text{ is a horizontal plaquette of } W\}.$$

We collect together some properties of interfaces thus.

Lemma 10 *Let $\delta \in \mathcal{D}_L$.*

(i) *The set $\delta^* \setminus \delta$ contains no c-plaquette.*

(ii) *All plaquettes of δ^* that are 1-connected to some c-plaquette are horizontal plaquettes of δ. All horizontal plaquettes that are 0-connected to some c-plaquette belong to δ^*.*

(iii) *Let C be a ceiling. There is a unique plane parallel to the regular interface which contains all the c-plaquettes of C.*

(iv) *Let C be a ceiling. We have that $\{h \in \delta^* : \pi(h) \subseteq [\pi(C)]\} = C$.*

(v) *Let W be a wall. We have that $\{h \in \delta^* : \pi(h) \subseteq [\pi(W)]\} = W$.*

(vi) *For each wall W, $\delta_0 \setminus \pi(W)$ has exactly one maximal infinite 0-connected component (respectively, 1-connected component).*

(vii) *Let W be a wall, and suppose that $\delta_0 \setminus \pi(W)$ comprises n maximal 0-connected sets $H_1, H_2, \dots, H_n$. The set of all plaquettes $h \in \delta^* \setminus W$ which are 0-connected to W comprises only c-plaquettes, which belong to the union of exactly n distinct ceilings $C_1, C_2 \dots, C_n$ such that $\{\pi(h) : \ h \text{ is a c-plaquette of } C_i\} \subseteq H_i$.*

(viii) *The projections $\pi(W_1)$ and $\pi(W_2)$ of two different walls W_1 and W_2 of δ^* are not 0-connected.*

(ix) *The projection* $\pi(W)$ *of any wall* W *contains at least one plaquette of* δ_0.

The displacement of the plane in (iii) from the regular interface, counted positive or negative, is called the *height* of the ceiling C.

Proof. (i) Let h be a c-plaquette of δ^* with $\pi(h) = h_0$. Since $\delta \in \mathcal{D}_L$, it contains at least one plaquette with projection h_0. Yet, according to the definition of a c-plaquette, there is no such a plaquette besides h. Therefore $h \in \delta$.

(ii) Suppose h is a c-plaquette. Necessarily, h belongs to δ and any horizontal plaquette which is 1-connected to h belongs to δ^*. It may be seen in addition that any vertical plaquette which is 1-connected to h lies in $\overline{\delta} \setminus \delta$. Suppose, on the contrary, that one such vertical plaquette h' lies in δ. Then the horizontal plaquettes which are 1-connected to h' lie in δ^*. One of these latter plaquettes has projection $\pi(h)$, in contradiction of the assumption that h is a c-plaquette.

We may now see as follows that any horizontal plaquette h'' which is 1-connected to h must lie in δ. Suppose, on the contrary, that one such plaquette h'' lies in $\overline{\delta} \setminus \delta$. We may construct a path of open edges on $(\mathbb{Z}^3, \eta(\underline{\omega_\delta}))$ that connects the vertex x just above h with the vertex $x - (0, 0, 1)$ just below h, using the open edges of $\underline{\omega_\delta}$ corresponding to the three relevant plaquettes of $\overline{\delta} \setminus \delta$. This contradicts the assumption that h is a c-plaquette of the interface δ.

The second claim of (ii) follows immediately, by the definition of δ^*.

(iii) The first part follows by the definition of ceiling, since the only horizontal plaquettes that are 0-connected with a given c-plaquette h lie in the plane containing h.

(iv) Assume that $h \in \delta^*$ and $\pi(h) \subseteq [\pi(C)]$. If h is horizontal, the conclusion holds by the definition of c-plaquette. If h is vertical, then $h \in \delta$, and all 1-connected horizontal plaquettes lie in δ^*. At least two such horizontal plaquettes project onto the same plaquette in $\pi(C)$, in contradiction of the assumption that C is a ceiling.

(v) Let C be a ceiling and let $\gamma_1, \gamma_2, \dots, \gamma_n$ be the maximal 0-connected sets of plaquettes of $\delta_0 \setminus \pi(C)$. Let $\delta_i^* = \{h \in \delta^* : \pi(h) \subseteq [\gamma_i]\}$, and let $\beta_i^* = \{h \in \delta_i^* : h \text{ horizontal}, h \overset{0}{\sim} h' \text{ for some } h' \in C\}$. We note that, by the geometry of $\mathbb{Z}^2$, β_i^* is a 0-connected subset of δ_i^*. [This is a consequence of statement (5.3) of [35], see also footnote 2 on page 40 of [34].]

We have by part (iv) that $\delta^* = C \cup (\bigcup_{i=1}^n \delta_i^*)$. We claim that each δ_i^* is 0-connected, and we prove this as follows. Let $h_1, h_2 \in \delta_i^*$. Since δ^* is 0-connected, it contains a sequence $h_1 = f_0, f_1, \dots, f_m = h_2$ of plaquettes such that $f_{i-1} \overset{0}{\sim} f_i$ for $1 \le i \le m$. We need to show that such a sequence exists containing no plaquettes in C. Suppose on the contrary that the sequence (f_i) has a non-empty intersection with C. Let $k = \min\{i : f_i \in C\}$ and $l = \max\{i : f_i \in C\}$, and note that $0 < k \le l < n$.

If f_{k-1} and f_{l+1} are horizontal, then $f_{k-1}, f_{l+1} \in \beta_i^*$, whence they are 0-connected by a path of horizontal plaquettes of β_i^*, and the claim follows. A similar argument is valid if either or both of f_{k-1} and f_{l+1} is vertical. For example, if f_{k-1} is vertical, by (ii) it cannot be 1-connected to a plaquette of C. Hence it is 1-connected to some horizontal plaquette in $\delta^* \setminus C$ which is itself 1-connected to a plaquette of C. The same conclusion is valid for f_{l+1} if vertical. In any such case, as above there exists a 0-connected sequence of w-plaquettes connecting f_{k-1} with f_{l+1}, and the claim follows.

To prove (v), we note by the above that the wall W is a subset of one of the sets δ_i^*, say δ_1^*. Next we let C_1 be a ceiling contained in δ_1^*, if this exists, and we repeat the above procedure. We consider the 0-connected components of $\gamma_1 \setminus \pi(C_1)$, and we use the fact that δ_1^* is 0-connected to deduce that the set of plaquettes which project onto one of these components is itself 0-connected.

This procedure is repeated until all ceilings have been removed, the result being a 0-connected set of w-plaquettes of which, by definition of a wall, all members belong to W.

Finally, (vi) is a simple observation since walls are finite. Claim (vii) is immediate from claim (ii) and the definitions of wall and ceiling. Claim (viii) follows from (v) and (vii), and (ix) is a consequence of the definition of the semi-extended interface δ^*. ■

The properties described in Lemma 10 allow us to describe a wall W in more detail. By (vi) and (vii), there exists a unique ceiling that is 0-connected to W and with projection in the infinite 0-connected component of $\delta_0 \setminus \pi(W)$. We call this ceiling the *base* of W. The *altitude* of W is the height of the base of W; see (iii). The *height* $D(W)$ of W is the maximum absolute value of the displacement in the third coordinate direction of $[W]$ from the horizontal plane $\{(x_1, x_2, s + \frac{1}{2}) : x_1, x_2 \in \mathbb{Z}\}$, where s is the altitude of W. The *interior* $\mathrm{int}(W)$ (of the projection $\pi(W)$) of W is the complement in δ_0 of the unique maximal infinite 0-connected component of $\delta_0 \setminus \pi(W)$ (cf. (vi)).

We next define the concept of a standard wall. Let $S = (A, B)$ where A, B are sets of plaquettes. We call S a *standard wall* if there exists $\delta \in \mathcal{D}_L$ such that $A \subseteq \delta$, $B \subseteq \delta^* \setminus \delta$, and $A \cup B$ is the unique wall of δ. If $S = (A, B)$ is a standard wall, we shall refer to plaquettes of either A or B as plaquettes of S, and we write $\pi(S) = \pi(A \cup B)$.

Lemma 11 *Let $S = (A, B)$ be a standard wall. There exists a unique $\delta \in \mathcal{D}_L$ such that: $A \subseteq \delta$, $B \subseteq \delta^* \setminus \delta$, and $A \cup B$ is the unique wall of δ.*

This will be proved soon. We denote by δ_S the unique such $\delta \in \mathcal{D}_L$ corresponding to the standard wall S. We shall see that standard walls are the basic building blocks for a general interface. Notice that the base of a standard wall is a subset of the regular interface. We introduce an ordering on the plaquettes of δ_0, and we define the *origin* of the standard wall S to be the earliest plaquette in $\pi(S)$ which is 1-connected to some plaquette of $\delta_0 \setminus \pi(S)$. Such an origin exists by Lemma 10(ix), and the origin belongs to S by (ii). For $h \in \delta_0$, we denote by $\mathcal{S}_h$ the set of all standard walls with origin h. We attach to $\mathcal{S}_h$ the *empty wall* $\mathcal{E}_h$ interpreted as a wall with origin h but containing no plaquettes.

A family $\{S_i = (A_i, B_i) : 1 \le i \le m\}$ of standard walls is called *admissible* if:

(i) for $i \neq j$, there exists no pair $h_1 \in \pi(S_i)$ and $h_2 \in \pi(S_j)$ such that $h_1 \stackrel{0}{\sim} h_2$,

(ii) if, for some i, $h(e) \in S_i$ where $e \notin E_L$, then $h(e) \in A_i$ if and only if $\mu(e) = 0$.

The members of any such family have distinct origins. For our future convenience we label each S_i according to its origin $h(i)$, and write $\{S_h : h \in \delta_0\}$ for the family, where S_h is to be interpreted as $\mathcal{E}_h$ when h is the origin of none of the S_i. We adopt the convention that, when a standard wall is denoted as S_h for some $h \in \delta_0$, then $S_h \in \mathcal{S}_h$.

We introduce next the concept of a group of walls. Let $h \in \delta_0$, $\delta \in \mathcal{D}_L$, and denote by $\rho(h, \delta)$ the number of (vertical or horizontal) plaquettes in δ whose projection is a subset of h. Two standard walls S_1, S_2 are called *close* if there exist $h_1 \in \pi(S_1)$ and $h_2 \in \pi(S_2)$ such that

$$\|h_1, h_2\| < \sqrt{\rho(h_1, \delta_{S_1})} + \sqrt{\rho(h_2, \delta_{S_2})}.$$

A family G of non-empty standard walls is called a *group of (standard) walls* if it is admissible and if, for any pair $S_1, S_2 \in G$, there exists a sequence $T_0 = S_1, T_1, T_2, \ldots, T_n = S_2$ of members of G such that T_i and T_{i+1} are close for $0 \leq i < n$.

The *origin* of a group of walls is defined as the earliest of the origins of the standard walls therein. We write $\mathcal{G}_h$ for the set of all possible groups of walls with origin $h \in \delta_0$. As before, we attach to $\mathcal{G}_h$ the *empty group* with origin h but containing no standard wall which we denote also as $\mathcal{E}_h$. A family $\{G_i : 1 \leq i \leq m\}$ of groups of walls is called *admissible* if, for $i \neq j$, there exists no pair $S_1 \in G_i$, $S_2 \in G_j$ such that S_1 and S_2 are close.

We adopt the convention that, when a group of walls is denoted as G_h for some $h \in \delta_0$, then $G_h \in \mathcal{G}_h$. Thus a family of groups of walls may be written as a collection $\mathbf{G} = \{G_h : h \in \delta_0\}$ where $G_h \in \mathcal{G}_h$.

Lemma 12 *The set $\mathcal{D}_L$ is in one–one correspondence with both the collection of admissible families of standard walls, and with the collection of admissible families of groups of walls.*

Equally important to the existence of these one–one correspondences is their nature, as described in the proof of the lemma. We write δ_G (respectively $\delta_{\mathbf{G}}$) for the interface corresponding thus to an admissible family G of standard walls (respectively an admissible family $\mathbf{G}$ of groups of walls).

Proof of Lemma 11. Let $\delta \in \mathcal{D}_L$ have unique wall $S = (A, B)$. By definition, every plaquette of δ^* other than those in $A \cup B$ is a c-plaquette, so that $\Sigma = \delta^* \setminus (A \cup B)$ is a union of ceilings $C_1, C_2, \ldots, C_n$. Each C_i contains some plaquette h_i which is 1-connected to some $h_i' \in A$, whence, by Lemma 10(iii), the height of C_i is determined uniquely by knowledge of S. Hence δ is unique. ■

Proof of Lemma 12. Let $\delta \in \mathcal{D}_L$. Let $W_1, W_2, \ldots, W_n$ be the non-empty walls of δ^*, and write $W_i = (A_i, B_i)$ where $A_i = W_i \cap \delta$, $B_i = W_i \cap (\delta^* \setminus \delta)$. Let s_i be the altitude of W_i. We claim that $\tau_{(0,0,-s_i)}W_i$ is a standard wall, and we prove this as follows. Let C_{i_j}, $j = 1, 2, \dots, k$, be the ceilings that are 0-connected to W_i, and let H_{i_j} be the maximal 0-connected set of plaquettes in $\delta_0 \setminus \pi(W_i)$ onto which C_{i_j} projects. (See Lemma 10(vii).) It suffices to construct an interface $\delta(W_i)$ having $\tau_{(0,0,-s_i)}W_i$ as its unique wall. To this end we add to $\tau_{(0,0,-s_i)}A_i$ the plaquettes in $\tau_{(0,0,-s_i)}C_{i_j}$, $j = 1, 2, \dots, k$, together with, for each j, the horizontal plaquettes in the maximal 0-connected set of horizontal plaquettes that contains $\tau_{(0,0,-s_i)}C_{i_j}$ and elements of which project onto H_{i_j}.

We now define the family $\{S_h : h \in \delta_0\}$ of standard walls by

$$S_h = \begin{cases} \tau_{(0,0,-s_i)}W_i & \text{if } h \text{ is the origin of } \tau_{(0,0,-s_i)}W_i, \\ \mathcal{E}_h & \text{if } h \text{ is the origin of no } \tau_{(0,0,-s_i)}W_i. \end{cases}$$

More precisely, in the first case, $S_h = (A_h, B_h)$ where $A_h = \tau_{(0,0,-s_i)}A_i$ and $B_h = \tau_{(0,0,-s_i)}B_i$. That this is an admissible family of standard walls follows from Lemma 10(viii) and from the observation that $s_i = 0$ when $E(W_i) \cap E_L^{\mathrm{c}} \neq \emptyset$.

Conversely, let $\{S_h = (A_h, B_h) : h \in \delta_0\}$ be an admissible family of standard walls. We shall show that there is a unique interface δ corresponding in a certain way to this family. Let $S_1, S_2 \dots, S_n$ be the non-empty walls of the family, and let δ_i be the unique interface in $\mathcal{D}_L$ having S_i as its only wall.

We introduce the partial ordering on the walls given by $S_i < S_j$ if $\mathrm{int}(S_i) \subseteq \mathrm{int}(S_j)$, and we re-order the non-empty walls in such a way that $S_i < S_j$ implies $i < j$.

When it exists, we take the first index $k > 1$ such that $S_1 < S_k$ and we modify δ_k as follows. First we remove the c-plaquettes that project onto $\mathrm{int}(S_1)$, and then we add translates of the plaquettes of A_1. This is done by translating these plaquettes so that the base of S_1 is raised (or lowered) to the plane containing the ceiling that is 0-connected to S_k and that projects on the maximal 0-connected set of plaquettes in $\delta_0 \setminus \pi(S_k)$ that contains $\pi(S_1)$. (See Lemma 10(viii).) We write δ'_k for the ensuing interface. We now repeat this procedure starting from the set of standard walls $S_2, S_3, \dots, S_n$ and interfaces $\delta_2, \delta_3, \dots, \delta_{k-1}, \delta'_k, \delta_{k+1}, \dots, \delta_n$. If no such k exists, we continue the procedure with the interfaces $\delta_2, \delta_3, \dots, \delta_{k-1}, \delta_k, \delta_{k+1}, \dots, \delta_n$.

We continue this process until we are left with interfaces δ''_{i_k}, $k = 1, 2, \dots, r$, having indices which refer to standard walls that are smaller than no other wall. The final interface δ is now constructed as follows. For each k, we remove from the regular interface δ_0 all horizontal plaquettes contained in $\mathrm{int}(S_{i_k})$, and we replace them by the plaquettes of δ''_{i_k} that project onto $\mathrm{int}(S_{i_k})$.

The final assertion concerning admissible families of groups of walls is straightforward. ■

Next we derive certain combinatorial properties of walls. For $S = (A, B)$ a standard wall, we write $N(S) = |A|$ and we set $\Pi(S) = N(S) - |\pi(S)|$. For an admissible set $F = \{S_1, S_2, \dots, S_m\}$ of standard walls, we write $\Pi(F) = \sum_{i=1}^m \Pi(S_i)$, $N(F) = \sum_{i=1}^m N(S_i)$, and $\pi(F) = \bigcup_{i=1}^m \pi(S_i)$.

Lemma 13 *Let $S=(A,B)$ be a standard wall, and $D(S)$ its height.*

(i) $N(S) \ge \frac{14}{13}|\pi(S)|$. *Consequently,* $\Pi(S) \ge \frac{1}{13}|\pi(S)|$ *and* $\Pi(S) \ge \frac{1}{14}N(S)$.

(ii) $N(S) \ge \frac{1}{5}|S|$.

(iii) $\Pi(S) \ge D(S)$.

Proof. (i) Define for each $h_0 \in \delta_0$ the set $U(h_0) = \{h \in \delta_0 : h = h_0 \text{ or } h \stackrel{1}{\sim} h_0\}$. We call two plaquettes $h_1, h_2 \in \delta_0$ *separated* if $U(h_1) \cap U(h_2) = \emptyset$. Denote by $H_{\mathrm{sep}} = H_{\mathrm{sep}}(S) \subseteq \pi(S)$ a set of pairwise-separated plaquettes in $\pi(S)$ having maximum cardinality, and let $H = \bigcup_{h_1 \in H_{\mathrm{sep}}} [U(h_1) \cap \pi(S)]$. Note that

$$|H_{\mathrm{sep}}| \ge \tfrac{1}{13}|\pi(S)|. \tag{35}$$

For every $h_0 \in \pi(S)$, there exists a horizontal plaquette $h_1 \in \delta_S$ such that $\pi(h_1) = h_0$. Since $A \cup B$ contains no c-plaquette of δ_S, it is the case that h_1 is a w-plaquette, whence $h_1 \in A$. In particular, $N(S) \ge |\pi(S)|$.

For $h_0 = \pi(h_1) \in H_{\mathrm{sep}}$ where $h_1 \in A$, we claim that

$$\Big|\Big\{h \in A : \text{either } \pi(h) \subseteq [h_0] \text{ or } \pi(h) \in U(h_0)\Big\}\Big| \ge |U(h_0) \cap \pi(S)| + 1. \tag{36}$$

It follows from (35) and (36) that

$$\begin{aligned} N(S) &\ge \sum_{h_0 \in H_{\mathrm{sep}}} \Big\{|U(h_0) \cap \pi(S)| + 1\Big\} + |\pi(S) \setminus H| \\ &= |H| + |H_{\mathrm{sep}}| + |\pi(S)| - |H| \ge \tfrac{14}{13}|\pi(S)| \end{aligned}$$

as required.

In order to prove (36), we argue first that $U(h_0) \cap \pi(S)$ contains at least one (horizontal) plaquette besides h_0. Suppose that this is not true. Then $U(h_0) \setminus h_0$ contains the projections of c-plaquettes of δ_S^* only. By Lemma 10(ii, iii), these c-plaquettes belong to the same ceiling C and therefore lie in the same plane. Since h_1 is by assumption a w-plaquette, there must be at least one other horizontal plaquette of δ_S^* projecting onto h_0. Only one such plaquette, however, is 1-connected with the c-plaquettes. Since δ_S^* is 1-connected, the other plaquettes projecting onto h_0 must be 1-connected with at least one other plaquette of δ_S^*. Each of these further plaquettes projects into $\pi(C)$, in contradiction of Lemma 10(iv).

We may now verify (36) as follows. Since h_1 is a w-plaquette, there exists $h_2 \in A \cup B$, $h_2 \ne h_1$, such that $\pi(h_2) = h_0$. If there exists such h_2 belonging to A, then (36) holds. We assume the contrary, and let h_2 be such a plaquette with $h_2 \in B$. Since $h_1 \in A$, for every $\eta \in U(h_0) \cap \pi(S)$, $\eta \ne h_0$, there exists $\eta' \in A$ such that $\pi(\eta') \subseteq [\eta]$ and $\eta' \stackrel{1}{\sim} h_1$. [If this fails for some η, then, as in the proof of Lemma 10(ii), in any configuration with interface δ_S, there exists a path of open edges joining the vertex just above h_1 to the vertex just beneath h_1. Since, by

assumption, all plaquettes of $A \cup B$ other than h_1, having projection h_0, lie in B, this contradicts the fact that δ_S is an interface.] If any such η' is vertical, then (36) follows. Assume that all such η' are horizontal. Since $h_2 \in B$, there exists $h_3 \in A$ such that $h_3 \stackrel{1}{\sim} h_2$, and (36) holds in this case also.
(ii) The second part of the lemma follows from the observation that each of the plaquettes in A is 1-connected to no more than four horizontal plaquettes of B.
(iii) Recall from the remark after (35) that A contains at least $|\pi(S)|$ horizontal plaquettes. Furthermore, A must contain at least $D(S)$ vertical plaquettes, and the claim follows. ■

Finally in this section, we derive an exponential bound for the number of groups of walls satisfying certain constraints.

Lemma 14 *Let $h \in \delta_0$. There exists a constant K such that: the number of groups of walls $G \in \mathcal{G}_h$ satisfying $\Pi(G) = k$ is no greater than K^k.*

Proof. Let $G = \{S_1, S_2, \dots, S_n\} \in \mathcal{G}_h$ where the $S_i = (A_i, B_i)$ are non-empty standard walls and $S_1 \in \mathcal{S}_h$. For $j \in \delta_0$, define

$$R_j = \left\{h' \in \delta_0 : \|j, h'\| \le \sqrt{\rho(j, \delta_G)}\right\} \setminus \pi(G)$$

and

$$\widetilde{G} = \Big(\bigcup_{i=1}^{n}[A_i \cup B_i]\Big) \cup \Big(\bigcup_{j \in \pi(G)} R_j\Big).$$

There exist constants C' and C'' such that, by Lemma 13,

$$|\widetilde{G}| \le |G| + C' \sum_{j \in \pi(G)} \rho(j, \delta_G) \le C''|G| \le 5 \cdot 14 C'' \Pi(G),$$

where $|G| = |\bigcup_i (A_i \cup B_i)|$.

It may be seen that $\widetilde{G}$ is a 0-connected set of plaquettes containing h. Moreover, the 0-connected sets obtained by removing all the horizontal plaquettes $h' \in \widetilde{G}$, for which there exists no other plaquette $h'' \in \widetilde{G}$ with $\pi(h'') = \pi(h')$, are the standard walls of G. Hence, the number of such groups of walls with $\Pi(G) = k$ is no greater than the number of 0-connected sets of plaquettes containing no more than $70C''k$ elements including h. It is proved in [13], Lemma 2, that there exists $\nu < \infty$ such that the number of 0-connected sets of size n containing h is no larger than ν^n. Corresponding to each such set there are at most 2^n ways of partitioning the plaquettes between the A_i and the B_i. The claim of the lemma follows. ■

8. Exponential bounds for probabilities

Let $\mathbf{G} = \{G_h : h \in \delta_0\}$ be a family of groups of walls. If $\mathbf{G}$ is admissible, there exists by Lemma 12 a unique corresponding interface $\delta_{\mathbf{G}}$. We may pick a random family $\zeta = \{\zeta_h : h \in \delta_0\}$ of groups of walls according to the probability measure $\mathbb{P}_L$ induced by ϕ_L thus:

$$\mathbb{P}_L(\zeta = \mathbf{G}) = \begin{cases} \overline{\phi}_L(\Delta = \delta_{\mathbf{G}}) & \text{if } \mathbf{G} \text{ is admissible,} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 15 *Let $q \geq 1$, and let p^* be as in Proposition 7. There exist constants C_3, C_4 such that*

$$\mathbb{P}_L\Big(\zeta_{h'} = G_{h'} \,\Big|\, \zeta_h = G_h \text{ for } h \in \delta_0,\ h \neq h'\Big) \leq C_3[C_4(1-p)]^{\Pi(G_{h'})},$$

for $p > p^*$, and for all $h' \in \delta_0$, $G_{h'} \in \mathcal{G}_{h'}$, $L > 0$, and for any admissible family $\{G_h : h \in \delta_0,\ h \neq h'\}$ of groups of walls.

Proof. The claim is trivial if $\mathbf{G} = \{G_h : h \in \delta_0\}$ is not admissible, and therefore we may assume it admissible. Let $h' \in \delta_0$, and let $\mathbf{G}'$ agree with $\mathbf{G}$ except at h', where $G_{h'}$ is replaced by the empty group $\mathcal{E}_{h'}$. We write $\delta = \delta_{\mathbf{G}}$ and $\delta' = \delta_{\mathbf{G}'}$. Then

$$\mathbb{P}_L\Big(\zeta_{h'} = G_{h'} \,\Big|\, \zeta_h = G_h \text{ for } h \in \delta_0,\ h \neq h'\Big) \leq \frac{\overline{\phi}_L(\delta)}{\overline{\phi}_L(\delta')}. \tag{37}$$

We will use (31) to bound the right-hand side of this expression. In doing so, we shall require bounds for $|\delta| - |\delta'|$, $|\overline{\delta} \setminus \delta| - |\overline{\delta}' \setminus \delta'|$, $K_\delta - K_{\delta'}$, and

$$\sum_{e \in E(\delta) \cap E_L} f_p(e, \delta, L) - \sum_{e \in E(\delta') \cap E_L} f_p(e, \delta', L). \tag{38}$$

It is easy to see from the definition of δ that

$$|\delta| = |\delta_0| + \sum_{h \in \delta_0} \Big[N(G_h) - |\pi(G_h)|\Big],$$

and it follows that

$$|\delta| - |\delta'| = N(G_{h'}) - |\pi(G_{h'})| = \Pi(G_{h'}). \tag{39}$$

A little thought leads to the inequality

$$|\overline{\delta} \setminus \delta| - |\overline{\delta}' \setminus \delta'| \geq 0, \tag{40}$$

and the reader may wish to omit the explanation which follows. We claim that (40) follows from the inequality

$$|P(\overline{\delta})| - |P(\overline{\delta}')| \geq 0, \tag{41}$$

where $P(\overline{\delta})$ (respectively $P(\overline{\delta}'))$ is the set of plaquettes in $\overline{\delta} \setminus \delta$ (respectively $\overline{\delta}' \setminus \delta'$) which project into $[\pi(G_{h'})]$. In order to see that (41) implies (40), we argue as follows. We may construct the extended interface $\overline{\delta}$ from $\overline{\delta}'$ in the following manner. First we remove all the plaquettes from $\overline{\delta}'$ that project into $[\pi(G_{h'})]$, and we fill the gaps by introducing the walls of $G_{h'}$ one by one along the lines of the proof of Lemma 12. Then we add the plaquettes of $\overline{\delta} \setminus \delta$ that project into $[\pi(G_{h'})]$. During this operation on interfaces, we remove $P(\overline{\delta}')$ and add $P(\overline{\delta})$; the claim follows.

By Lemma 10(viii), there exists no vertical plaquette of $\overline{\delta}' \setminus \delta'$ that projects into $[\pi(G_{h'})]$ and is in addition 1-connected to some wall not belonging to $G_{h'}$. Moreover, since all the horizontal plaquettes of $\overline{\delta}'$ belong to the semi-extended interface δ'^*, those that project onto $[\pi(G_{h'})]$ are c-plaquettes of δ'^*; hence, such plaquettes lie in δ'. It follows that $P(\overline{\delta}')$ comprises the vertical plaquettes that are 1-connected with $\pi(G_{h'})$.

It is therefore sufficient to construct an injective map T that maps each vertical plaquette 1-connected with $\pi(G_{h'})$ to a different vertical plaquette in $P(\overline{\delta})$. We noted in the proof of Lemma 13(i) that, for every $h_0 \in \pi(G'_h)$, there exists a horizontal plaquette $h_1 \in \delta$ with $\pi(h_1) = h_0$. For every vertical plaquette $h^{\mathrm{v}} \stackrel{1}{\sim} h_0$, there exists a translate $h_1^{\mathrm{v}} \stackrel{1}{\sim} h_1$. Suppose h^{v} lies above δ_0. If $h_1^{\mathrm{v}} \in \overline{\delta} \setminus \delta$, we set $T(h^{\mathrm{v}}) = h_1^{\mathrm{v}}$. If $h_1^{\mathrm{v}} \in \delta$, we consider the (unique) vertical plaquette 'above' it, which we denote h_2^{v}. We repeat this procedure up to the first n that we meet a plaquette $h_n^{\mathrm{v}} \in \overline{\delta} \setminus \delta$, and we set $T(h^{\mathrm{v}}) = h_n^{\mathrm{v}}$. When h^{v} lies below δ_0, we act similarly to find a plaquette $T(h^{\mathrm{v}})$ of $\overline{\delta} \setminus \delta$ beneath h^{v}. The resulting T is as required.

Turning to $K_\delta - K_{\delta'}$, we recall the notation after Proposition 6. Notice that exactly two of the components (S_δ^i, U_δ^i) are infinite, and we suppose that these are assigned indices 1 and 2. For $i = 3, 4, \ldots, K_\delta$, let $H(S_\delta^i)$ be the set of plaquettes that are the dual to an edge having one vertex in S_δ^i and one vertex in ∂S_δ^i. The finite component (S_δ^i, U_δ^i) is in a natural way surrounded by a particular wall, namely that to which all the plaquettes of $H(S_\delta^i)$ belong. This follows from Lemma 10(v, viii) and the facts that

$$P_i = \Big\{\pi\Big(h(\langle x, x + (0,0,1)\rangle)\Big) : x \in S_\delta^i\Big\}$$

is a 1-connected subset of δ_0, and that $[\pi(H(S_\delta^i))] = [P_i]$.

Therefore,

$$K_\delta - K_{\delta'} = K_{\delta''} - 2, \tag{42}$$

where $\delta'' = \delta_{G_{h'}}$. It is elementary by Lemma 13(i) that

$$K_{\delta''} \le 2N(G_{h'}) \le 28\Pi(G_{h'}). \tag{43}$$

Finally, we estimate (38). Let $H_1, H_2, \ldots, H_r$ be the maximal 0-connected sets of plaquettes in $\delta_0 \setminus \pi(G_{h'})$, and let δ_i (respectively δ'_i) be the set of plaquettes of δ (respectively δ') that project into $[H_i]$. Recalling the construction of an interface from its standard walls in the proof of Lemma 12, there is a natural one–one correspondence between the plaquettes of δ_i and those of δ'_i, and hence between the plaquettes in $U = \bigcup_{i=1}^r \delta_i$ and those in $U' = \bigcup_{i=1}^r \delta'_i$. We denote by T the corresponding bijection that maps an edge e with $h(e) \in \bigcup_{i=1}^r \delta_i$ to the edge $T(e)$ with corresponding dual plaquette in $\bigcup_{i=1}^r \delta'_i$. Note that $T(e)$ is a vertical translate of e.

If e is such that $h(e) \in U$,

$$G(e, \delta, E_L; T(e), \delta', E_L) \geq \|\pi'(h(e)), \pi(G_{h'})\| - 1,$$

where $\pi'(h)$ is the earliest plaquette h'' of δ_0 such that $\pi(h) \subseteq [h'']$, and

$$\|h_1, H\| = \min\Big\{\|h_1, h_2\| : h_2 \in H\Big\}.$$

Let $p > p^*$. Using the notation of Proposition 7 and Lemma 9,

$$\begin{aligned}
&\left|\sum_{e\in E(\delta)\cap E_L} f_p(e, \delta, L) - \sum_{e\in E(\delta')\cap E_L} f_p(e, \delta', L)\right| \\
&\quad \leq \sum_{e\in E(U)\cap E_L} |f_p(e, \delta, L) - f_p(T(e), \delta', L)| \\
&\qquad\qquad + \sum_{e\in E(\delta\setminus U)\cap E_L} f_p(e, \delta, L) + \sum_{e\in E(\delta'\setminus U')\cap E_L} f_p(e, \delta', L) \\
&\quad \leq C_2 e^\gamma \sum_{e\in E(U)\cap E_L} \exp\Big(-\gamma\|\pi'(h(e)), \pi(G_{h'})\|\Big) + C_1\Big[N(G_{h'}) + |\pi(G_{h'})|\Big]. \qquad (44)
\end{aligned}$$

By Lemma 13, the second term of the last line is no greater than $C_5\Pi(G_{h'})$ for some constant C_5. Using the same lemma and the definition of a group of walls, we see that the first term is no larger than

$$\begin{aligned}
&C_2 e^\gamma \sum_{h\in\delta_0\setminus\pi(G_{h'})} \rho(h, \delta) \exp\Big(-\gamma\|h, \pi(G_{h'})\|\Big) \\
&\qquad \leq C_2 e^\gamma \sum_{h\in\delta_0\setminus\pi(G_{h'})} \|h, \pi(G_{h'})\|^2 \exp\Big(-\gamma\|h, \pi(G_{h'})\|\Big) \\
&\qquad \leq C_2 e^\gamma \sum_{h''\in\pi(G_{h'})} \sum_{h\in\delta_0\setminus\pi(G_{h'})} \|h, h''\|^2 \exp\Big(-\gamma\|h, h''\|\Big) \\
&\qquad \leq C_6|\pi(G_{h'})| \leq 13C_6\Pi(G_{h'}), \qquad (45)
\end{aligned}$$

for some constant C_6.

The required conditional probability is, by (31) and (37),

$$p^{|\overline{\delta}\setminus\delta| - |\overline{\delta}'\setminus\delta'|}(1-p)^{|\delta|-|\delta'|} q^{K_\delta - K_{\delta'}} \exp\left(\sum_{e\in E(\delta)\cap E_L} f_p(e, \delta, L) - \sum_{e\in E(\delta')\cap E_L} f_p(e, \delta', L)\right),$$

which, by (39)–(45), is bounded above as in the statement of the lemma. ∎

9. Main theorem

Let $h \in \delta_0$. For $\omega \in \Omega_L^\mu$, we write $h \leftrightarrow \infty$ if there exists a sequence $h = h_0, h_1, \dots, h_r$ of plaquettes in δ_0 such that: $h_i \stackrel{1}{\sim} h_{i+1}$ for $0 \le i < r$; each h_i is a c-plaquette of $\Delta(\omega)$; $h_r = h(e)$ for some $e \notin E_L$.

Theorem 2 *Let $q \ge 1$. For all $\epsilon > 0$, there exists $\widehat{p} = \widehat{p}(\epsilon) < 1$ such that, if $p > \widehat{p}$,*

$$\overline{\phi}_L(h \leftrightarrow \infty) > 1 - \epsilon \tag{46}$$

for all $h \in \delta_0$ and all $L \ge 1$.

Since, following Theorem 2, h is a c-plaquette with high probability, it follows by Proposition 6 and the discussion immediately thereafter that the vertex of $\mathbb{Z}^3$ immediately beneath (respectively above) the centre of h is joined to $\partial^-\Lambda_L$ (respectively $\partial^+\Lambda_L$) with high probability. Thus Theorem 1 holds. Furthermore, since $h \leftrightarrow \infty$ with high probability, such connections may be found within the plane of $\mathbb{Z}^3$ comprising vertices x with $x_3 = 0$ (respectively $x_3 = 1$).

The existence of non-translation-invariant (conditioned) random-cluster measures follows from Theorem 2, as in the following sketch argument. For $e \in \mathbb{E}$, we write $e^\pm = e \pm (0, 0, 1)$. Let $\omega \in \Omega$. If $h = h(e) \in \delta_0$ is a c-plaquette of $\Delta(\omega)$, then e is closed, and $h(e^\pm) \notin \overline{\Delta(\omega)}$. The configurations in the two regions above and below $\Delta(\omega)$ are governed by wired random-cluster measures. [We have used Lemma 8 here.] Hence, under (46),

$$\overline{\phi}_L(\omega(e) = 1) \le \epsilon, \quad \overline{\phi}_L(\omega(e^\pm) = 1) \ge \frac{(1-\epsilon)p}{p + (1-p)q},$$

by Lemma 1. Note that these inequalities concern the probabilities of cylinder events.

Our second main result concerns the vertical displacement of the interface, and states roughly that there exists a geometric bound on the tail of the displacement, uniformly in L. Let $\delta \in \mathcal{D}_L$, $(x_1, x_2) \in \mathbb{Z}^2$, and write $x = (x_1, x_2, \frac{1}{2})$. We define the *displacement* of δ at x by

$$D(x, \delta) = \sup\Big\{|d - \tfrac{1}{2}| : (x_1, x_2, d) \in [\delta]\Big\}.$$

Theorem 3 *Let $q \ge 1$. There exists $\widehat{p} < 1$ and $\alpha(p)$ satisfying $\alpha(p) > 0$ when $p > \widehat{p}$ such that*

$$\overline{\phi}_L\Big(D(x, \Delta) \ge d\Big) \le e^{-d\alpha(p)} \quad \textit{for } d \ge 1,$$

for all $(x_1, x_2) \in \mathbb{Z}^2$ and $L \ge 1$.

Proof of Theorem 2. Let $h \in \delta_0$. We have not so far specified the ordering of plaquettes in δ_0 used to identify the origin of a standard wall or of a group of walls.

We assume henceforth that this ordering is such that: for all $h_1, h_2 \in \delta_0$, $h_1 > h_2$ implies $\|h, h_1\| \geq \|h, h_2\|$.

For any standard wall S there exists, by Lemma 10(vi), a unique maximal infinite 1-connected component $I(S)$ of $\delta_0 \setminus \pi(S)$. Let $\omega \in \Omega_L^\mu$. The interface $\Delta(\omega)$ gives rise to a family of standard walls, and $h \leftrightarrow \infty$ if and only if, for each such wall S, h belongs to $I(S)$. (This is a consequence of a standard property of $\mathbb{Z}^2$; see the appendix of [28].) Suppose on the contrary that $h \notin I(S_j)$ for some such standard wall S_j, for some $j \in \delta_0$, belonging in turn to some maximal admissible group $G_{h'} \in \mathcal{G}_{h'}$ of walls of Δ, for some $h' \in \delta_0$. We have by Lemma 13 and the above ordering on members of δ_0 that

$$13\Pi(G_{h'}) \geq |\pi(G_{h'})| \geq |\pi(S_j)| \geq \|h, j\| + 1 \geq \|h, h'\| + 1.$$

Let K be as in Lemma 14, and p^*, C_4 as in Lemma 15. We let $\widetilde{p}$ be sufficiently large that $\widetilde{p} > p^*$ and

$$\lambda = \lambda(p) = -\tfrac{1}{13} \log[KC_4(1-p)]$$

satisfies $\lambda(\widetilde{p}) > 0$. By the latter lemma, when $p > \widetilde{p}$,

$$\begin{aligned}
1 - \overline{\phi}_L(h \leftrightarrow \infty) &\leq \sum_{h' \in \delta_0} \mathbb{P}_L\Big(\Pi(\zeta_{h'}) \geq \tfrac{1}{13}(\|h, h'\| + 1)\Big) \\
&\leq \sum_{h' \in \delta_0} \sum_{n \geq (\|h,h'\|+1)/13} \sum_{\substack{G \in \mathcal{G}_{h'}: \\ \Pi(G) = n}} \mathbb{P}_L(\zeta_{h'} = G) \\
&\leq \sum_{h' \in \delta_0} \sum_{n \geq (\|h,h'\|+1)/13} K^n C_3 [C_4(1-p)]^n \\
&\leq C_3 \sum_{h' \in \delta_0} \exp(-\lambda(\|h, h'\| + 1)) \leq C_7 e^{-\lambda},
\end{aligned}$$

for appropriate constants C_i. The claim follows on choosing p sufficiently close to 1. ■

Proof of Theorem 3. This is related to the proof of Proposition 2.4 of [6]. If $D(x, \Delta) \geq d$, there exists r satisfying $1 \leq r \leq d$ such that the following statement holds. There exist distinct plaquettes $h_1, h_2, \dots, h_r \in \delta_0$, and maximal admissible groups G_{h_i}, $1 \leq i \leq r$, of walls of Δ such that: $x = (x_1, x_2, \frac{1}{2})$ lies in the interior of one or more standard wall of each G_{h_i}, and $\sum_{i=1}^r \Pi(G_{h_i}) \geq d$ (recall Lemma 13(iii)). Let $m_i = \lfloor \frac{1}{13}(\|x, h_i\| + 1) \rfloor$ where $\|x, h\| = \|x - y\|$ and y is the centre of h. By Lemma 15, and as in the previous proof,

$$\begin{aligned}
\overline{\phi}_L(D(x, \Delta) \geq d) &\leq \sum_{\substack{h_1, h_2, \dots, h_r \\ 1 \leq r \leq d}} \mathbb{P}_L\left(\sum_i \Pi(\zeta_{h_i}) \geq d,\ \Pi(\zeta_{h_i}) \geq m_i \vee 1\right) \\
&= \sum_{\substack{h_1, h_2, \dots, h_r \\ 1 \leq r \leq d}} \sum_{s=d}^{\infty} \sum_{\substack{z_1, z_2, \dots, z_r: \\ z_1 + z_2 + \cdots + z_r = s \\ z_i \geq m_i \vee 1}} \mathbb{P}_L\Big(\Pi(\zeta_{h_i}) = z_i \text{ for } 1 \leq i \leq r\Big)
\end{aligned}$$

$$\leq \sum_{h_i} \sum_{s \geq d} C_8 [KC_4(1-p)]^s \sum_{\substack{z_1, z_2, \ldots, z_r: \\ z_1+z_2+\cdots+z_r=s \\ z_i \geq m_i \vee 1}} 1,$$

for some constant C_8. The last summation is the number of ordered partitions of the integer s into r parts, the ith of which is at least $m_i \vee 1$. By adapting the classical solution to this enumeration valid for the case $m_i \equiv 1$ (see, for example, [24]), we see that

$$\sum_{\substack{z_1, z_2, \ldots, z_r: \\ z_1+z_2+\cdots+z_r=s \\ z_i \geq m_i \vee 1}} 1 \leq \binom{s-1-\sum_i m_i \vee 1}{r-1} \leq 2^{s-1-\sum_i m_i \vee 1} \leq 2^{s-1-\sum_i m_i},$$

whence, for some C_9,

$$\overline{\phi}_L(D(x,\Delta) \geq d) \leq C_9 \sum_{s \geq d} [2KC_4(1-p)]^s \left[\sum_{h \in \delta_0} 2^{-\lfloor \|x,h\|/13 \rfloor} \right]^d,$$

which decays exponentially as $d \to \infty$ when $2KC_4(1-p)$ is sufficiently small. ∎

Acknowledgement

We thank Roman Kotecký for his comments on an aspect of this work, and an anonymous referee for indicating a proof in an earlier version.

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