CORRELATION INEQUALITIES OF GKS TYPE FOR THE POTTS MODEL

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Abstract. Correlation inequalities are presented for functionals of a ferromagnetic Potts model with external field, using the random-cluster representation. These results extend earlier inequalities of Ganikhodjaev–Razak and Schonmann, and yield also GKS-type inequalities when the spin-space is taken as the set of qth roots of unity.

1. Introduction

Our purpose in this brief note is to derive certain correlation inequalities for a ferromagnetic Potts model. The main technique is the random-cluster representation of this model, and particularly the FKG inequality. Some, at least, of the arguments given here are probably known to others. Our results generalize the work of Ganikhodjaev and Razak, who have shown in [5] how to formulate and prove GKS inequalities for the Potts model with a general number q of local states. Furthermore, our Theorems 2.5 and 2.7 extend the correlation inequalities of Schonmann to be found in [12].

2. The inequalities

Let $G = (V, E)$ be a finite graph, and let $J = (J_e : e \in E)$ and $h = (h_v : v \in V)$ be vectors of non-negative reals, and $q \in \{2, 3, \dots\}$. We take as local state space for the q-state Potts model the set $\mathcal{Q} = \{0, 1, \dots, q-1\}$. The Potts measure on G with parameters J has state space $\Sigma = \mathcal{Q}^V$ and probability measure

$$\pi(\sigma) = \frac{1}{Z} \exp\left\{ \sum_{e=\langle x,y\rangle \in E} J_e \delta_e(\sigma) + \sum_{v \in V} h_v \delta_v(\sigma) \right\},$$

Date: first posted 29 July 2007, revised 7 January 2009.

1991 *Mathematics Subject Classification.* 82B20, 60K35.

Key words and phrases. Griffiths inequality, GKS inequality, Ising model, Potts model, random-cluster model, angular spins.

for $\sigma = (\sigma_v : v \in V) \in \Sigma$, where $\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y}$ and $\delta_v(\sigma) = \delta_{\sigma_v,0}$ are Kronecker delta functions, and Z is the appropriate normalizing constant.

We shall make use of the random-cluster representation in this note, and we refer the reader to [9] for a recent account and bibliography. Consider a random-cluster model on the graph G^+ obtained by adding a 'ghost' vertex g, joined to each vertex $v \in V$ by a new edge $\langle g, v\rangle$. An edge $e \in E$ has parameter $p_e = 1 - e^{-J_e}$, and an edge $\langle g, v\rangle$ has parameter $p_v = 1 - e^{-h_v}$. With ϕ the corresponding random-cluster measure, we obtain the spin configuration as follows. The cluster C_g containing g has spin 0. To each open cluster of ω other than C_g, we allocate a uniformly chosen spin from $\mathcal{Q}$, such that every vertex in the cluster receives this spin, and the spins of different clusters are independent. The ensuing spin vector $\sigma = \sigma(\omega)$ has law π. See [9, Thm 1.3] for a proof of this standard fact, and for references to the original work of Fortuin and Kasteleyn.

Let $f : \mathcal{Q} \to \mathbb{C}$. For $\sigma \in \Sigma$, let

$$f(\sigma)^R = \prod_{v \in R} f(\sigma_v), \qquad R \subseteq V. \tag{2.1}$$

Thinking of σ as a random vector with law π, we write $\langle f(\sigma)^R\rangle$ for the mean value of $f(\sigma)^R$. Let $\mathcal{F}_q$ be the set of all functions $f : \mathcal{Q} \to \mathbb{C}$ such that, for all integers $m, n \geq 0$:

$$E(f(X)^m) \text{ is real and non-negative,} \tag{2.2}$$

$$E(f(X)^{m+n}) \geq E(f(X)^m)E(f(X)^n), \tag{2.3}$$

where X is a uniformly distributed random variable on $\mathcal{Q}$. That is, $f \in \mathcal{F}_q$ if each $S_m = \sum_{x \in \mathcal{Q}} f(x)^m$ is real and non-negative, and $qS_{m+n} \geq S_m S_n$. For $i \in \mathcal{Q}$, let $\mathcal{F}_q^i$ be the subset of $\mathcal{F}_q$ containing all f such that

$$f(i) = \max\{|f(x)| : x \in \mathcal{Q}\}. \tag{2.4}$$

This condition entails that $f(i)$ is real and non-negative.

Theorem 2.5. *Let $f \in \mathcal{F}_q^0$. For $R \subseteq V$, the mean $\langle f(\sigma)^R\rangle$ is real-valued and non-decreasing in the vectors J and h, and satisfies $\langle f(\sigma)^R\rangle \geq 0$. For $R, S \subseteq V$, we have that*

$$\langle f(\sigma)^R f(\sigma)^S\rangle \geq \langle f(\sigma)^R\rangle\langle f(\sigma)^S\rangle.$$

If there is no external field, in that $h \equiv 0$, it suffices for the above that $f \in \mathcal{F}_q$.

Theorem 2.6. *Let $q \geq 2$. The following functions belong to $\mathcal{F}_q^0$.*

(a) $f(x) = \frac{1}{2}(q-1) - x$.

(b) $f(x) = e^{2\pi ix/q}$, *a qth root of unity.*
(c) $f : \mathcal{Q} \to [0,\infty)$, *with* $f(x) \le f(0)$ *for all* x.

Case (a) gives us the inequalities of Ganikhodjaev and Razak, [5]. When $q = 2$, these reduce to the GKS inequalities for the Ising model, see [7, 8, 11]. We do not now if the implications of case (b) were known previously, or if they are useful. Perhaps they are elementary examples of the results of [6]. In case (c) with $f(x) = \delta_{x,0}$, we obtain the first correlation inequality of Schonmann, [12].

Theorem 2.7. *Let* $q \ge 2$, $f_0 \in \mathcal{F}_q^0$, *and let* $f_1 : \mathcal{Q} \to \mathbb{C}$ *satisfy* (2.2). *If* f_0 *and* f_1 *have disjoint support in that* $f_0 f_1 \equiv 0$ *then, for* $R, S \subseteq V$,

$$\langle f_0(\sigma)^R f_1(\sigma)^S \rangle \le \langle f_0(\sigma)^R \rangle \langle f_1(\sigma)^S \rangle.$$

If $h \equiv 0$, *it is enough to assume* $f_0 \in \mathcal{F}_q$.

Two correlation inequalities were proved in [12], a 'positive' inequality that is implied by Theorem 2.6(c), and a 'negative' inequality that is obtained as a special case of the last theorem, on setting $f_0(x) = \delta_{x,0}$ and $f_1(x) = \delta_{x,1}$. We note that Schonmann's inequalities were themselves (partial) generalizations of correlation inequalities proved in [4].

Amongst the feasible extensions of the above theorems that come to mind, we mention the classical space–time models used to study the quantum Ising/Potts models, see [1, 2, 3, 10].

3. Proof of Theorem 2.5

We use the coupling of the random-cluster and Potts model described in Section 2. Let E^+ be the edge-set of G^+, $\Omega^+ = \{0,1\}^{E^+}$, and $\omega \in \Omega^+$. Let $A_g, A_1, A_2, \dots, A_k$ be the vertex-sets of the open clusters of ω, where A_g is that of the cluster C_g containing g.

Let $R \subseteq V$, and let $f \in \mathcal{F}_q^0$. By (2.1),

$$f(\sigma)^R = f(0)^{|R\cap A_g|} \prod_{r=1}^{k} f(X_r)^{|R\cap A_r|},$$

where X_r is the random spin assigned to A_r. This has conditional expectation

$$g_R(\omega) := E(f(\sigma)^R \mid \omega) = f(0)^{|R\cap A_g|} \prod_{r=1}^{k} E(f(X)^{|R\cap A_r|} \mid \omega).$$

By (2.2) and (2.4), $g_R(\omega)$ is real and non-negative, whence so is its mean $\phi(g_R) = \langle f(\sigma)^R \rangle$.

We show next that g_R is a non-decreasing function on the partially ordered set Ω^+. It suffices to consider the case when the configuration

ω' is obtained from ω by adding an edge between two clusters of ω. In this case, by (2.3)–(2.4), $g_R(\omega') \geq g_R(\omega)$. That $\langle \sigma^R \rangle = \phi(g_R)$ is non-decreasing in J and h follows by the appropriate comparison inequality for the random-cluster measure ϕ, see [9, Thm 3.21].

Now,

$$E(f(\sigma)^R f(\sigma)^S \mid \omega) = f(0)^{|R\cap A_g|+|S\cap A_g|} \prod_{r=1}^{k} E\big(f(X)^{|R\cap A_r|+|S\cap A_r|} \bigm| \omega\big).$$

By (2.3),

$$E(f(\sigma)^R f(\sigma)^S \mid \omega) \geq g_R(\omega) g_S(\omega).$$

By the FKG property of ϕ, see [9, Thm 3.8],

$$\langle f(\sigma)^R f(\sigma)^S \rangle = \phi\big(E(f(\sigma)^R f(\sigma)^S \mid \omega)\big) \geq \langle f(\sigma)^R \rangle \langle f(\sigma)^S \rangle),$$

as required.

When $h \equiv 0$, the terms in $f(0)$ do not appear in the above, and it therefore suffices that $f \in \mathcal{F}_q$.

4. Proof of Theorem 2.6

We shall use the following elementary fact: if T is a non-negative random variable,

$$E(T^{m+n}) \geq E(T^m)E(T^n), \qquad m, n \geq 0. \tag{4.1}$$

This trivial inequality may be proved in several ways, of which one is the following. Let T_1, T_2 be independent copies of T. Clearly,

$$(T_1^m - T_2^m)(T_1^n - T_2^n) \geq 0, \tag{4.2}$$

since either $0 \leq T_1 \leq T_2$ or $0 \leq T_2 \leq T_1$. Inequality (4.1) follows by multiplying out (4.2) and averaging.

Case (a). Inequality (2.4) with $i = 0$ is a triviality. Since $f(X)$ is real-valued, with the same distribution as $-f(X)$, $E(f(X)^m) = 0$ when m is odd, and is positive when m is even. When $m+n$ is even, (2.3) follows from (4.1) with $T = f(X)^2$, and both sides of (2.3) are 0 otherwise.

Case (b). It is an easy calculation that

$$E(f(X)^m) = 1\{q \text{ divides } m\},$$

where $1\{F\}$ is the indicator function of the set F, and (2.2)–(2.3) follow.

Case (c). Inequality (2.3) follows by (4.1) with $T = f(X)$.

5. Proof of Theorem 2.7

We may as well assume that $f_0 \not\equiv 0$, so that $f_0(0) > 0$ and $f_1(0) = 0$. We use the notation of Section 3, and write

$$F_0(\omega) = f_0(0)^{|R\cap A_g|} \prod_{r=1}^{k} E(f_0(X)^{|R\cap A_r|} \mid \omega), \tag{5.1}$$

$$F_1(\omega) = \prod_{r=1}^{k} E(f_1(X)^{|S\cap A_r|} \mid \omega). \tag{5.2}$$

By (2.2), F_0 and F_1 are real-valued and non-negative. Since $f_0 \in \mathcal{F}_q^0$, F_0 is increasing.

Since $f_0 f_1 \equiv 0$,

$$E\big(f_0(\sigma)^R f_1(\sigma)^S \,\big|\, \omega\big) = 1_Z(\omega) F_0(\omega) F_1(\omega),$$

where 1_Z is the indicator function of the event $Z = \{S \not\leftrightarrow R \cup \{g\}\}$. Here, as usual, we write $U \leftrightarrow V$ if there exists an open path from some vertex of U to some vertex of V. Let T be the subset of V containing all vertices joined to S by open paths, and write ω_T for the configuration ω restricted to T. Using conditional expectation,

$$\begin{aligned} \langle f_0(\sigma)^R f_1(\sigma)^S \rangle &= \phi\big(1_Z F_0 F_1\big) \\ &= \phi\big(1_Z F_1 \phi(F_0 \mid T,\, \omega_T)\big), \end{aligned} \tag{5.3}$$

where we have used the fact that 1_Z and F_1 are functions of the pair T, ω_T only. On the event Z, F_0 is an increasing function of the configuration restricted to $V \setminus T$. Furthermore, given T, the conditional measure on $V \setminus T$ is the corresponding random-cluster measure. It follows that

$$\phi(F_0 \mid T,\, \omega_T) \le \phi(F_0) \quad \text{on} \quad Z,$$

by [9, Thm 3.21]. By (5.3),

$$\begin{aligned} \langle f_0(\sigma)^R f_1(\sigma)^S \rangle &\le \phi\big(1_Z F_1 \phi(F_0)\big) \\ &\le \phi(F_0)\phi(F_1) = \langle f_0(\sigma)^R \rangle \langle f_1(\sigma)^S \rangle, \end{aligned}$$

and the theorem is proved.

When $h \equiv 0$, $A_g = \varnothing$ in (5.1), and it suffices that $f_0 \in \mathcal{F}_q$.

Acknowledgements

The author is grateful to Jakob Björnberg, Chuck Newman, and Aernout van Enter for their comments and suggestions.

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