RANDOM EVEN GRAPHS

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Abstract. We study a random even subgraph of a finite graph G with a general edge-weight $p \in (0,1)$. We demonstrate how it may be obtained from a certain random-cluster measure on G, and we propose a sampling algorithm based on coupling from the past. A random even subgraph of a planar lattice undergoes a phase transition at the parameter-value $\frac{1}{2}p_c$, where p_c is the critical point of the $q = 2$ random-cluster model on the dual lattice. The properties of such a graph are discussed, and are related to Schramm–Löwner evolutions (SLE).

1. Introduction

Our purpose in this paper is to study a random even subgraph of a finite graph $G = (V, E)$, and to show how to sample such a subgraph. A subset F of E is called *even* if, for all $x \in V$, x is incident to an even number of elements of F. We call the subgraph (V, F) *even* if F is even, and we write $\mathcal{E}$ for the set of all even subsets F of E. It is standard that every even set F may be decomposed as an edge-disjoint union of cycles. Let $p \in [0, 1)$. The *random even subgraph* of G with parameter p is that with law

$$\rho_p(F) = \frac{1}{Z^{\mathrm{E}}} p^{|F|}(1-p)^{|E\setminus F|}, \qquad F \in \mathcal{E}, \tag{1.1}$$

where $Z^{\mathrm{E}} = Z^{\mathrm{E}}_G(p)$ is the appropriate normalising constant.

We may express ρ_p as follows in terms of product measure on E. Let ϕ_p be product measure with density p on the configuration space $\Omega = \{0,1\}^E$. For $\omega \in \Omega$ and $e \in E$, we call e *ω-open* if $\omega(e) = 1$, and *ω-closed* otherwise. Let $\partial\omega$ denote the set of vertices $x \in V$ that are incident to an odd number of ω-open edges. Then

$$\rho_p(F) = \frac{\phi_p(\omega_F)}{\phi_p(\partial\omega = \varnothing)}, \qquad F \subseteq E, \tag{1.2}$$

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where ω_F is the edge-configuration whose open edge-set is F. In other words, ϕ_p describes the random subgraph of G obtained by randomly and independently deleting each edge with probability $1-p$, and ρ_p is the law of this random subgraph conditioned on being even.

Random even graphs are closely related to the Ising model and the random-cluster model on G, and we review these models briefly. Let $\beta \in (0,\infty)$ and

$$p = 1 - e^{-2\beta} = \frac{2\tanh\beta}{1+\tanh\beta}. \tag{1.3}$$

The Ising model on G has configuration space $\Sigma = \{-1,+1\}^V$, and probability measure

$$\pi_\beta(\sigma) = \frac{1}{Z^{\mathrm{I}}}\exp\left\{\beta\sum_{e\in E}\sigma_x\sigma_y\right\}, \qquad \sigma\in\Sigma. \tag{1.4}$$

where $Z^{\mathrm{I}} = Z^{\mathrm{I}}_G(\beta)$ is the partition function that makes π_β a probability measure. A *spin-cluster* of a configuration $\sigma \in \Sigma$ is a maximal connected subgraph of G each of whose vertices v has the same spin-value σ_v. A spin-cluster is termed a k cluster if $\sigma_v = k$ for all v belonging to the cluster. An important quantity associated with the Ising model is the 'two-point correlation function'

$$\tau_\beta(x,y) = \pi_\beta(\sigma_x=\sigma_y) - \tfrac{1}{2} = \tfrac{1}{2}\pi_\beta(\sigma_x\sigma_y), \qquad x,y\in V, \tag{1.5}$$

where $P(f)$ denotes the expectation of a random variable f under the probability measure P.

The random-cluster measure on G with parameters $p\in(0,1)$ and $q=2$ is given as follows [it may be defined for general $q>0$ but we are concerned here only with the case $q=2$]. Let

$$\begin{aligned}\phi_{p,2}(\omega) &= \frac{1}{Z^{\mathrm{RC}}}\left\{\prod_{e\in E}p^{\omega(e)}(1-p)^{1-\omega(e)}\right\}2^{k(\omega)}\\ &= \frac{1}{Z^{\mathrm{RC}}}p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}2^{k(\omega)}, \qquad \omega\in\Omega,\end{aligned} \tag{1.6}$$

where $k(\omega)$ denotes the number of ω-open components on the vertex-set V, $\eta(\omega) = \{e\in E : \omega(e)=1\}$ is the set of open edges, and $Z^{\mathrm{RC}} = Z^{\mathrm{RC}}_G(p)$ is the appropriate normalizing factor.

The relationship between the Ising and random-cluster models on G is well established, and hinges on the fact that, in the notation introduced above,

$$\tau_\beta(x,y) = \tfrac{1}{2}\phi_{p,2}(x\leftrightarrow y),$$

where $\{x\leftrightarrow y\}$ is the event that x and y are connected by an open path. See [11] for an account of the random-cluster model. There is a relationship between the Ising model and the random even graph also, known misleadingly as the 'high-temperature expansion'. This may be stated as follows (see [3] for a proof). Consider the Ising model on G with parameter β, and let $p = 1 - e^{-2\beta}$. Then (cf. (1.2))

$$2\tau_{\beta,2}(x,y) = \frac{\phi_{p/2}(\partial\omega = \{x,y\})}{\phi_{p/2}(\partial\omega = \varnothing)}, \qquad x,y \in V,\ x \neq y. \tag{1.7}$$

This note is laid out in the following way. In Section 2 we define a random even subgraph of a finite or infinite graph, and we explain how to sample a *uniform* even subgraph. In Section 3 we explain how to sample a non-uniform random even graph, starting with a sample from a random-cluster measure. An algorithm for exact sampling is presented in Section 4 based on the method of coupling from the past. The structure of random even subgraphs of the square and hexagonal lattices is summarized in Section 5.

In a second paper [13], we study the asymptotic properties of a random even subgraph of the complete graph K_n. Whereas the special relationship with the random-cluster and Ising models is the main feature of the current work, the analysis of [13] is more analytic, and extends to random graphs whose vertex degrees are constrained to lie in any given subsequence of the non-negative integers.

Remark 1.8. The definition (1.1) may be generalized by replacing the single parameter p by a family $\mathbf{p} = (p_e : e \in E)$, just as sometimes is done for the random-cluster measure (1.6), see for example [24]; we let

$$\rho_{\mathbf{p}}(F) = \frac{1}{Z} \prod_{e\in F} p_e \prod_{e\notin F} (1 - p_e). \tag{1.9}$$

For simplicity we will mostly consider the case of a single p.

2. Uniform random even subgraphs

In the case $p = \frac{1}{2}$ in (1.1), every even subgraph has the same probability, so $\rho_{\frac{1}{2}}$ describes a uniform random even subgraph of G. Such a random subgraph can be obtained as follows.

We identify the family of all spanning subgraphs of $G = (V, E)$ with the family of all subsets of E. This family can further be identified with $\{0,1\}^E = \mathbb{Z}_2^E$, and is thus a vector space over $\mathbb{Z}_2$; the addition is componentwise addition modulo 2 in $\{0,1\}^E$, which translates into taking the symmetric difference of edge-sets: $F_1 + F_2 = F_1 \triangle F_2$ for $F_1, F_2 \subseteq E$.

The family of even subgraphs of G forms a subspace $\mathcal{E}$ of this vector space $\{0,1\}^E$, since $F_1 + F_2 = F_1 \triangle F_2$ is even if F_1 and F_2 are even. (In fact, $\mathcal{E}$ is the cycle space Z_1 in the $\mathbb{Z}_2$-homology of G as a simplical complex.) In particular, the number of even subgraphs of G equals $2^{c(G)}$ where $c(G) = \dim(\mathcal{E})$; $c(G)$ is thus the number of independent cycles in G, and, as is well known,

$$c(G) = |E| - |V| + k(G). \tag{2.1}$$

Proposition 2.2. *Let $C_1, \dots, C_c$ be a maximal set of independent cycles in G. Let $\xi_1, \dots, \xi_c$ be independent* $\mathrm{Be}(\frac{1}{2})$ *random variables (i.e., the results of fair coin tosses). Then $\sum_i \xi_i C_i$ is a uniform random even subgraph of G.*

Proof. $C_1, \dots, C_c$ is a basis of the vector space $\mathcal{E}$ over $\mathbb{Z}_2$. □

One standard way of choosing $C_1, \dots, C_c$ is exploited in the next proposition. Another, for planar graphs, is given by the boundaries of the finite faces; this will be used in Section 5. In the following proposition, we use the term *spanning subforest* of G to mean a maximal forest of G, that is, the union of a spanning tree from each component of G.

Proposition 2.3. *Let (V, F) be a spanning subforest of G. Each subset X of $E \setminus F$ can be completed by a unique $Y \subseteq F$ to an even edge-set $E_X = X \cup Y \in \mathcal{E}$. Choosing a uniform random subset $X \subseteq E \setminus F$ thus gives a uniform random even subgraph E_X of G.*

Proof. It is easy to see, and well known, that each edge $e_i \in E \setminus F$ can be completed by edges in F to a unique cycle C_i; these cycles form a basis of $\mathcal{E}$ and the result follows by Proposition 2.2. (It is also easy to give a direct proof.) □

There is a natural definition of a uniform random even subgraph of an *infinite*, locally finite graph $G = (V, E)$. In fact, we can define the subspace $\mathcal{E}$ of $\{0,1\}^E$ as before. (Note that we need G to be locally finite in order to do so.) Further, let E_1 be a finite subset of E. The natural projection $\pi_{E_1} : \{0,1\}^E \to \{0,1\}^{E_1}$ given by $\pi_{E_1}(\omega) = (\omega_e)_{e \in E_1}$ maps $\mathcal{E}$ onto a subspace $\mathcal{E}_{E_1} = \pi_{E_1}(\mathcal{E})$ of $\{0,1\}^{E_1}$.

Theorem 2.4. *Let G be a locally finite, countably infinite graph. There exists a unique probability measure ρ on $\Omega = \{0,1\}^E$ such that, for every finite set $E_1 \subset E$, $(\omega_e)_{e \in E_1}$ is uniformly distributed on $\mathcal{E}_{E_1}$, i.e.,*

$$\rho(\pi_{E_1}^{-1}(A)) = |A \cap \mathcal{E}_{E_1}| / |\mathcal{E}_{E_1}|, \qquad A \subseteq \{0,1\}^{E_1}. \tag{2.5}$$

Proof. We have defined ρ on all cylinder events by (2.5). By the Kolmogorov extension theorem, it suffices to show that this definition is consistent as E_1 varies, which amounts to showing that if $E_1 \subseteq E_2 \subset E$ with E_1, E_2 finite, then the projection $\pi_{E_2 E_1} : \{0,1\}^{E_2} \to \{0,1\}^{E_1}$ maps the uniform distribution on $\mathcal{E}_{E_2}$ to the uniform distribution on $\mathcal{E}_{E_1}$. This is an immediate consequence of the fact that $\pi_{E_2 E_1}$ is a linear map of $\mathcal{E}_{E_2}$ onto $\mathcal{E}_{E_1}$. □

3. Random even subgraphs via coupling

We return to the random even subgraph with parameter $p \in [0, 1)$ defined by (1.1) for a finite graph $G = (V, E)$. We show next how to couple the $q = 2$ random-cluster model and the random even subgraph of G. Let $p \in [0, \frac{1}{2}]$, and let ω be a realization of the random-cluster model on G with parameters $2p$ and $q = 2$. Let $R = (V, \gamma)$ be a uniform random even subgraph of $(V, \eta(\omega))$.

Theorem 3.1. *Let $p \in [0, \frac{1}{2}]$. The graph $R = (V, \gamma)$ is a random even subgraph of G with parameter p.*

This recipe for random even subgraphs provides a neat method for their simulation, provided $p \leq \frac{1}{2}$. One may sample from the random-cluster measure by the method of coupling from the past (see [19]), and then sample a uniform random even subgraph by either Proposition 2.2 or Proposition 2.3.

Proof. Let $g \subseteq E$ be even. By the observations in Section 2, with $c(\omega) = c(V, \eta(\omega))$ denoting the number of independent cycles in the open subgraph,

$$\mathbb{P}(\gamma = g \mid \omega) = \begin{cases} 2^{-c(\omega)} & \text{if } g \subseteq \eta(\omega), \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\mathbb{P}(\gamma = g) = \sum_{\omega : g \subseteq \eta(\omega)} 2^{-c(\omega)} \phi_{2p,2}(\omega).$$

Now $c(\omega) = |\eta(\omega)| - |V| + k(\omega)$, so that, by (1.6),

$$\begin{aligned} \mathbb{P}(\gamma = g) &\propto \sum_{\omega : g \subseteq \eta(\omega)} (2p)^{|\eta(\omega)|}(1-2p)^{|E \setminus \eta(\omega)|} 2^{k(\omega)} \frac{1}{2^{|\eta(\omega)| - |V| + k(\omega)}} \\ &\propto \sum_{\omega : g \subseteq \eta(\omega)} p^{|\eta(\omega)|}(1-2p)^{|E \setminus \eta(\omega)|} \\ &= [p + (1-2p)]^{|E \setminus g|} p^{|g|} \\ &= p^{|g|}(1-p)^{|E \setminus g|}, \qquad g \subseteq E. \end{aligned}$$

The claim follows. □

Let $p \in (\frac{1}{2}, 1)$. If G is even, we can sample from ρ_p by first sampling a subgraph $(V, \widetilde{F})$ from ρ_{1-p} and then taking the complement $(V, E \setminus \widetilde{F})$, which has the distribution ρ_p. If G is not even, we adapt this recipe as follows. For $W \subseteq V$ and $H \subseteq E$, we say that H is W*-even* if each component of (V, H) contains an even number of members of W. Let $W \neq \varnothing$ be the set of vertices of G with odd degree, so that, in particular, E is W-even. Let $\Omega^W = \{\omega \in \Omega : \eta(\omega) \text{ is } W\text{-even}\}$. For $\omega \in \Omega^W$, we pick disjoint subsets $P^i = P^i_\omega$, $i = 1, 2, \dots, \frac{1}{2}|W|$, of $\eta(\omega)$, each of which constitutes an open non-self-intersecting path with distinct endpoints lying in W, and such that every member of W is the endpoint of exactly one such path. Write $P_\omega = \bigcup_i P^i_\omega$.

Let $r = 2(1-p)$, and let $\phi^W_{r,2}$ be the random-cluster measure on Ω with parameters r and $q = 2$ conditional on the event Ω^W. We sample from $\phi^W_{r,2}$

to obtain a subgraph $(V, \eta(\omega))$, from which we select a uniform random even subgraph (V, γ) by the procedure of the previous section.

Theorem 3.2. *Let $p \in (\frac{1}{2}, 1)$. The graph $S = (V, E \setminus (\gamma \bigtriangleup P_\omega))$ is a random even subgraph of G with parameter p.*

The recipes in Theorems 3.1 and 3.2 can be combined as follows, if we consider the generalized model mentioned in Remark 1.8 with one parameter $p_e \in (0, 1)$ for each edge $e \in E$. Let $A = \{e \in E : p_e > \frac{1}{2}\}$. Define $r_e = 2p_e$ when $e \notin A$ and $r_e = 2(1 - p_e)$ when $e \in A$. (Thus $0 < r_e \le 1$.) Let $W = W_A$ be the set of vertices that are *A-odd*, i.e., endpoints of an odd number of edges in A. Sample ω from the random-cluster measure with parameters $\mathbf{r} = (r_e : e \in E)$ and $q = 2$, conditioned on $\eta(\omega)$ being W-even, let P_ω be as above (for $W = W_A$), and sample a uniform random even subgraph (V, γ) of $(V, \eta(\omega))$.

Theorem 3.3. *The graph $S = (V, \gamma \bigtriangleup P_\omega \bigtriangleup A)$ is a random even subgraph of G with the distribution $\rho_{\mathbf{p}}$ given in* (1.9)*.*

Note that Theorems 3.1 and 3.2 are special cases of Theorem 3.3, with $A = \varnothing$ and $A = E$ respectively. We find it more illuminating to present the proof of Theorem 3.2 in this more general setup.

Proof of Theorem 3.3, and thus of Theorem 3.2. Let $F = \gamma \bigtriangleup P_\omega \bigtriangleup A$ be the resulting edge-set, and note that

$$\eta(\omega) \supseteq \gamma \bigtriangleup P_\omega = F \bigtriangleup A. \tag{3.4}$$

Furthermore, if F is even, then $F \bigtriangleup A$ has odd degree exactly at vertices in $W = W_A$; hence (3.4) implies that necessarily $\omega \in \Omega^W$.

Given an even edge set $f \subseteq E$, we thus obtain $F = f$ if we first choose $\omega \in \Omega^W$ with $\eta(\omega) \supseteq f \bigtriangleup A$ and then (having chosen P_ω) select γ as the even subgraph $f \bigtriangleup A \bigtriangleup P_\omega$. Hence, for every $\omega \in \Omega^W$ with $\eta(\omega) \supseteq f \bigtriangleup A$, we have $\mathbb{P}(F = f \mid \omega) = 2^{-c(\omega)}$, and summing over such ω we find

$$\begin{aligned}
\mathbb{P}(F = f) &\propto \sum_{\omega:\eta(\omega)\supseteq f \triangle A} 2^{-c(\omega)} \phi_{\mathbf{r},2}(\omega) \\
&\propto \sum_{\omega:\eta(\omega)\supseteq f \triangle A} 2^{-c(\omega)} 2^{k(\omega)} \prod_{e\in E} r_e^{\omega(e)} (1 - r_e)^{1-\omega(e)} \\
&\propto \sum_{\omega:\eta(\omega)\supseteq f \triangle A} 2^{-|\eta(\omega)|} \prod_{e\in E} r_e^{\omega(e)} (1 - r_e)^{1-\omega(e)} \\
&= \sum_{\omega:\eta(\omega)\supseteq f \triangle A} \prod_{e\in E} \left(\frac{r_e}{2}\right)^{\omega(e)} (1 - r_e)^{1-\omega(e)} \\
&= \prod_{e\in f \triangle A} \left(\frac{r_e}{2}\right) \prod_{e\notin f \triangle A} \left(1 - \frac{r_e}{2}\right).
\end{aligned}$$

With 1_e denoting the indicator function of the event $\{e \in F\}$, this can be rewritten as

$$\begin{aligned}\mathbb{P}(F = f) &\propto \prod_{e \notin A} (r_e/2)^{1_e}(1 - r_e/2)^{1-1_e} \prod_{e \in A} (r_e/2)^{1-1_e}(1 - r_e/2)^{1_e} \\ &= \prod_{e \notin A} p_e^{1_e}(1 - p_e)^{1-1_e} \prod_{e \in A} (1 - p_e)^{1-1_e} p_e^{1_e} \\ &= \prod_{e \in E} p_e^{1_e}(1 - p_e)^{1-1_e} \\ &\propto \rho_{\mathbf{p}}(f).\end{aligned}$$

The claim follows. □

There is a converse to Theorem 3.1. Take a random even subgraph (V, F) of $G = (V, E)$ with parameter $p \le \frac{1}{2}$. To each $e \notin F$, we assign an independent random colour, *blue* with probability $p/(1-p)$ and *red* otherwise. Let H be obtained from F by adding in all blue edges.

Theorem 3.5. *The graph (V, H) has law $\phi_{2p,2}$.*

Proof. For $h \subseteq E$,

$$\begin{aligned}\mathbb{P}(H = h) &\propto \sum_{J \subseteq h,\ J \text{ even}} \left(\frac{p}{1-p}\right)^{|J|} \left(\frac{p}{1-p}\right)^{|h \setminus J|} \left(\frac{1-2p}{1-p}\right)^{|E \setminus h|} \\ &\propto p^{|h|}(1-2p)^{|E \setminus h|} N(h),\end{aligned}$$

where $N(h)$ is the number of even subgraphs of (V, h). As in the above proof, $N(h) = 2^{|h|-|V|+k(h)}$ where $k(h)$ is the number of components of (V, h), and the proof is complete. □

An edge e of a graph is called *cyclic* if it belongs to some cycle of the graph.

Corollary 3.6. *For $p \in [0, \frac{1}{2}]$ and $e \in E$,*

$$\rho_p(e \text{ is open}) = \tfrac{1}{2}\phi_{2p,2}(e \text{ is a cyclic edge of the open graph}).$$

By summing over $e \in E$, we deduce that the mean number of open edges under ρ_p is one half of the mean number of cyclic edges under $\phi_{2p,2}$.

Proof. Let $\omega \in \Omega$ and let $\mathcal{C}$ be a maximal family of independent cycles of ω. Let $R = (V, \gamma)$ be a uniform random even subgraph of $(V, \eta(\omega))$, constructed using Proposition 2.2 and $\mathcal{C}$. For $e \in E$, let M_e be the number of elements of $\mathcal{C}$ that include e. If $M_e \ge 1$, the number of these M_e cycles of γ that are selected in the construction of γ is equally likely to be even as odd. Therefore,

$$\mathbb{P}(e \in \gamma \mid \omega) = \begin{cases} \frac{1}{2} & \text{if } M_e \ge 1, \\ 0 & \text{if } M_e = 0. \end{cases}$$

The claim follows by Theorem 3.1. □

4. Sampling an even subgraph

It was remarked earlier that Theorem 3.1 gives a neat way of sampling an even subgraph of G according to the probability measure η_p with $p \le \frac{1}{2}$. Simply use coupling-from-the-past (cftp) to sample from the random-cluster measure $\phi_{2p,2}$, and then flip a fair coin once for each member of some maximal independent set of cycles of G. The situation is slightly more complicated when $p > \frac{1}{2}$ and G is not itself even, since the conditioned random-cluster measure used in Theorems 3.2 and 3.3 is neither monotone nor anti-monotone. We indicate briefly in this section how to adapt the technique of cftp to such a situation.

Let E be a non-empty finite set, and let μ be a probability measure on the product space $\Omega = \{0,1\}^E$. We call μ *monotone* (respectively, *anti-monotone*) if $\mu(1_e \mid \xi_e)$ is non-decreasing (respectively, non-increasing) in $\xi \in \Omega$. Here, 1_e is the indicator function that e is open, and ξ_e is the configuration obtained from ξ on $E \setminus \{e\}$. For $e \in E$, $\psi \in \Omega$, and $b = 0,1$, we write ψ_e^b for the configuration that agrees with ψ off e and takes the value b on e.

It is standard that cftp may be used to sample from μ if μ is monotone (see [26, 19]), and it is explained in [14] how to adapt this when μ is anti-monotone. We propose below a mechanism that results in an exact sample from μ without any assumption of (anti-)monotonicity. This mechanism may be cast in the more general framework of the 'bounding chain' of [17], but, unlike in that work, it makes use of the fact that Ω is partially ordered. Write $S_\mu = \{\omega \in \Omega : \mu(\omega) > 0\}$, the subset of Ω on which μ is strictly positive, and *assume for simplicity that S_μ is increasing, and that* $1 \in S_\mu$, where 1 (respectively, 0) denotes the configuration of 'all 1s' (respectively, 'all 0s'). This assumption is valid in the current setting, but is not necessary for all that follows.

We start with the usual Gibbs sampler for μ. This is a discrete-time Markov chain $G = (G_n : n \ge 0)$ on the state space Ω that updates as follows. Suppose $G_n = \xi$. A uniformly distributed member of E is chosen, e say, and also a random variable U with the uniform distribution on $[0,1]$. Then $G_{n+1} = \xi'$ where $\xi'(f) = \xi(f)$ for $f \ne e$, and

$$\xi'(e) = \begin{cases} 0 & \text{if } U > \mu(1_e \mid \xi_e), \\ 1 & \text{if } U \le \mu(1_e \mid \xi_e). \end{cases}$$

The transition rule is well defined whenever $\xi_e^1 \in S_\mu$. It is convenient to use the device of [14] to extend this definition to configurations not in S_μ, and to this end we set

$$\mu(1_e \mid \xi_e) = \max\bigl\{\mu(1_e \mid \psi_e) : \psi_e \ge \xi_e,\ \psi_e^1 \in S_\mu\bigr\} \tag{4.1}$$

when $\xi_e^1 \notin S_\mu$. There is a degree of arbitrariness about this definition, which we follow for consistency with [14].

Let (e_n, U_n) be an independent sequence as above. Let $(A_n, B_n : n \geq 0)$ be a Markov chain with state space Ω^2, and $(A_0, B_0) = (0, 1)$. Suppose $(A_n, B_n) = (\xi, \eta)$ where $\xi \leq \eta$. We set $(A_{n+1}, B_{n+1}) = (\xi', \eta')$ where $\xi'(f) = \xi(f)$, $\eta'(f) = \eta(f)$ for $f \neq e_{n+1}$. At $e = e_{n+1}$ we set

$$\xi'(e) = 1 \quad \text{if and only if} \quad U_{n+1} \leq \alpha,$$
$$\eta'(e) = 1 \quad \text{if and only if} \quad U_{n+1} \leq \beta,$$

where

$$\begin{aligned} \alpha &= \alpha(\xi, \eta) = \min\big\{\mu(1_e \mid \psi_e) : \xi_e \leq \psi_e \leq \eta_e\big\}, \\ \beta &= \beta(\xi, \eta) = \max\big\{\mu(1_e \mid \psi_e) : \xi_e \leq \psi_e \leq \eta_e\big\}. \end{aligned} \tag{4.2}$$

Since $\alpha \leq \beta$, we have that $\xi' \leq \eta'$.

We run the chain (A, B) starting at negative times, in the manner prescribed by cftp, and let T be the coalescence time. More precisely, for $m \geq 0$, let $(A_k(m), B_k(m) : -m \leq k \leq 0)$ denote the chain beginning with $A_{-m}(m) = 0$, $B_{-m}(m) = 1$, using a fixed random sequence $(e_n, U_n)_{-\infty}^0$ for all m, and set

$$T = \min\{m \geq 0 : A_0(m) = B_0(m)\},$$

so that $A_0(T) = B_0(T)$.

Theorem 4.3. *If S_μ is increasing and $1 \in S_\mu$, then $P(T < \infty) = 1$, and $A_0(T)$ has law μ.*

Proof. We prove only that $P(T < \infty) = 1$. The second part is a standard exercise in cftp, and is easily derived as in [14], Theorem 2.2. By the definition of S_μ and (4.1), there exists $\eta = \eta(E, \mu) > 0$ such that $\mu(1_e \mid \xi_e) \geq \eta$ for all $e \in E$ and $\xi \in \Omega$. In any given time-interval of length $|E|$, there is a strictly positive probability that the corresponding sequence (e_i, U_i) satisfies $E = \{e_i\}$ and $U_i < \eta$ for all i. On this event, the lower process A takes the value 1 after the interval is past, so that coalescence has taken place. The corresponding events for distinct time-intervals are independent, whence the tail of T is no greater than geometric. □

The above recipe is exactly that of [19] when μ is monotone, and that of [14] when μ is anti-monotone.

Let $G = (V, E)$ be a finite graph, and $W \subseteq V$ a non-empty set of vertices with $|W|$ even. Let $\mathbf{r} = (r_e : e \in E)$ be a vector of numbers from $(0, 1]$, and let $\phi_{\mathbf{r},q}$ be the random-cluster measure on G with edge-parameters $\mathbf{r}$ and $q \geq 1$. We write $\phi_{\mathbf{r},q}^W$ for $\phi_{\mathbf{r},q}$ conditioned on the event that the open graph is W-even, and note that $\phi_{\mathbf{r},q}^W$ is neither monotone nor anti-monotone. The event S_μ is easily seen to be increasing, and $1 \in S_\mu$. We may therefore apply Theorem 4.3 to the measure $\mu = \phi_{\mathbf{r},q}^W$.

When implementing the above, it is convenient to have a quick way to calculate α and β in (4.2). We shall not go into this here, but note that there is no serious difficulty in the case in question.

5. Random even subgraphs of planar lattices

In this section, we consider random even subgraphs of the square and hexagonal lattices. We show that properties of the Ising models on these lattices imply properties of the random even graphs. In so doing, we shall review certain known properties of the Ising model, and we include a 'modern' proof of the established fact that the Ising model on the square lattice has a unique Gibbs state at the critical point.

Let $G = (V, E)$ be a planar graph embedded in $\mathbb{R}^2$, with dual graph $G_{\mathrm{d}} = (V_{\mathrm{d}}, E_{\mathrm{d}})$, and write e_{d} for the dual edge corresponding to the primal edge $e \in E$. [See [11] for an account of planar duality in the context of the random-cluster model.] Let $p \in (0, \frac{1}{2}]$ and let $\omega \in \Omega = \{0,1\}^E$ have law $\phi_{2p,2}$. There is a one–one correspondence between Ω and $\Omega_{\mathrm{d}} = \{0,1\}^{E_{\mathrm{d}}}$ given by $\omega(e)+\omega_{\mathrm{d}}(e_{\mathrm{d}}) = 1$. It is well known that ω_{d} has the law of the random-cluster model on G_{d} with parameters $(1-2p)/(1-p)$ and 2, see [11] for example. For $A \subseteq V$, the *boundary* of A is given by $\partial A = \{v \in A : v \sim w \text{ for some } w \notin A\}$. For $\omega \in \Omega$, let $f_0, f_1, \dots, f_c$ be the faces of $(V, \eta(\omega))$, with f_0 the infinite face. These faces are in one–one correspondence with the clusters of $(V_{\mathrm{d}}, \eta(\omega_{\mathrm{d}}))$, which we thus denote by $K_0, K_1, \dots, K_c$, and the boundaries of the finite faces form a basis of $\mathcal{E} = \mathcal{E}(V, \eta(\omega))$. More precisely, the boundary of each finite face f_i consists of an 'outer boundary' and zero, one or several 'inner boundaries'; each of these parts is a cycle (and two parts may have up to one vertex in common). If we orient the outer boundary cycle counter-clockwise (positive) and the inner boundary cycles clockwise (negative), the face will always be on the left side along the boundaries, and the winding numbers of the boundary cycles sum up to 1 at every point inside the face and to 0 outside the face. It is easy to see that the outer boundary cycles form a maximal family of independent cycles of $(V, \eta(\omega))$, and thus a basis of $\mathcal{E}$; another basis is obtained by the complete boundaries C_i of the finite faces. We use the latter basis, and select a random subset of the basis by randomly assigning (by fair coin tosses) $+$ and $-$ to each cluster in the dual graph $(V_{\mathrm{d}}, \eta(\omega_{\mathrm{d}}))$, or equivalently to each face f_i of $(V, \eta(\omega))$. We then select the boundaries C_i of the finite faces f_i that have been given a sign different from the sign of the infinite face f_0. The union (modulo 2) of the selected boundaries is by Proposition 2.2 and Theorem 3.1 a random even subgraph of G with parameter p. On the other hand, this union is exactly the dual boundary of the $+$ clusters of G_{d}, that is, the set of open edges $e \in E$ with the property that one endpoint of the corresponding dual edge e_{d} is labelled $+$ and the other is labelled $-$. [Such an edge e_{d} is called a $+/-$ edge.]

It is standard that the $+/-$ configuration on G_{d} is distributed as the Ising model on G_{d} with parameter β satisfying

$$\frac{1-2p}{1-p} = 1 - e^{-2\beta} = \frac{2\tanh\beta}{1+\tanh\beta}. \tag{5.1}$$

In summary, we have the following.

Theorem 5.2. *Let G be a finite planar graph with dual G_{d}. A random even subgraph of G with parameter $p \in (0, \frac{1}{2}]$ is dual to the $+/-$ edges of the Ising model on G_{d} with β satisfying* (5.1)*.*

Much is known about the Ising model on subsets of two-dimensional lattices, and the above fact permits an analysis of random even subgraphs of their dual lattices.

Consider the case when G is a box in the square lattice $\mathbb{Z}^2$. That is, $G = G_{m,n}$ is the subgraph of $\mathbb{Z}^2$ induced by the vertex-set $[-m,m] \times [-n,n]$, where $m, n \in \mathbb{Z}_+$ and $[a,b]$ is to be interpreted as $[a,b] \cap \mathbb{Z}$. It is a mild inconvenience that $G_{m,n}$ is not an even graph, and we adjust the 'boundary' to rectify this. For definiteness, we consider the so-called 'wired boundary condition' on $G_{m,n}$, which is to say that we consider the random-cluster measure on the graph $G^{\mathrm{w}}_{m,n}$ obtained from $G_{m,n}$ by identifying as one the set of vertices lying in its boundary $\partial G_{m,n}$.

The Ising model on $\mathbb{Z}^2$ with parameter β is *critical* when $e^{2\beta} = 1 + \sqrt{2}$, or equivalently when the above random-cluster model on the dual lattice has parameter satisfying

$$\frac{1-2p}{1-p} = \frac{\sqrt{2}}{1+\sqrt{2}} = 2 - \sqrt{2},$$

i.e., $p = p_{\mathrm{c}}$ where

$$p_{\mathrm{c}} = \frac{1}{2+\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}. \tag{5.3}$$

The Ising model has been studied extensively in the physics literature, and physicists have a detailed knowledge of the two-dimensional case particularly. There is a host of 'exact calculations', rigorous proofs of which can present challenges to mathematicians, see [3, 18].

We shall use the established facts stated in the following theorem. The continuity of the magnetization at the critical point contributes to the proof that the boundary of a large spin-cluster of the critical Ising model converges weakly to the Schramm–Löwner curve SLE_3, see [22, 23].

Theorem 5.4. *The critical value of the Ising model on the square lattice is $\beta_{\mathrm{c}} = \beta_{\mathrm{sd}}$ where $\beta_{\mathrm{sd}} = \frac{1}{2}\log(1+\sqrt{2})$ is the 'self-dual point'. The magnetization (and therefore the corresponding random-cluster percolation-probability also) is a continuous function of β on $[0,\infty)$.*

We note the corollary that the wired and free random-cluster measures on $\mathbb{Z}^2$ are identical for $p \in [0, 1]$; see [11], Thms 5.33 and 6.17.

Proof. These facts are 'classical' and have received much attention, see [18] for example; they may be proved as follows using 'modern' arguments. Recall first that the magnetization equals the percolation probability of the corresponding wired random-cluster model, and the two-point correlation function of the Ising model equals the two-point connectivity function of the random-cluster model (see [11]). We have that $\beta_{\mathrm{sd}} \le \beta_{\mathrm{c}}$, by Theorem 6.17(a) of [11] or otherwise, and similarly the random-cluster model with free boundary conditions has percolation-probability 0 whenever either $\beta \le \beta_{\mathrm{sd}}$ or $\beta < \beta_{\mathrm{c}}$. One may deduce by a coupling argument that the Ising model with $\beta < \beta_{\mathrm{c}}$ has a unique Gibbs state which we denote by π_β.

By the results of [2, 20], the two-point correlation function $\pi_\beta(\sigma_x\sigma_y)$ of the spins at x and y decays exponentially as $|x - y| \to \infty$ when $\beta < \beta_{\mathrm{c}}$, and it follows by the final statement of [12] or otherwise that $\beta_{\mathrm{c}} = \beta_{\mathrm{sd}}$. The continuity of the magnetization at $\beta \neq \beta_{\mathrm{c}}$ is standard, see for example [11], Theorems 5.16 and 6.17(b). When $\beta = \beta_{\mathrm{c}}$, it suffices to show that the $\pm$ boundary-condition Gibbs states $\pi^{\pm}_{\beta_{\mathrm{c}}}$ and the free boundary-condition Gibbs state $\pi^0_{\beta_{\mathrm{c}}}$ satisfy $\pi^+_{\beta_{\mathrm{c}}} = \pi^-_{\beta_{\mathrm{c}}} = \pi^0_{\beta_{\mathrm{c}}}$. Suppose this does not hold, so that $\pi^+_{\beta_{\mathrm{c}}} \neq \pi^-_{\beta_{\mathrm{c}}} \neq \pi^0_{\beta_{\mathrm{c}}}$. By the random-cluster representation or otherwise, the two-point correlation functions $\pi^{\pm}_{\beta_{\mathrm{c}}}(\sigma_x\sigma_y)$ are bounded away from 0 for all pairs x, y of vertices. By the main result of [1, 15] (see also [9]) and the symmetry of $\pi^0_{\beta_{\mathrm{c}}}$, we have that $\pi^0_{\beta_{\mathrm{c}}} = \frac{1}{2}\pi^+_{\beta_{\mathrm{c}}} + \frac{1}{2}\pi^-_{\beta_{\mathrm{c}}}$, whence $\pi^0_{\beta_{\mathrm{c}}}(\sigma_x\sigma_y)$ is bounded away from 0. By [11], Theorem 5.17, this contradicts the above remark that the percolation-probability of the free-boundary condition random-cluster measure is 0 at $\beta = \beta_{\mathrm{sd}} = \beta_{\mathrm{c}}$. □

We consider now the so-called thermodynamic limit of the random even graph on $G^{\mathrm{w}}_{m,n}$ as $m, n \to \infty$. It is long established that the (free boundary-condition) Ising measure on $G_{m,n}$ converges weakly (in the product topology) to an infinite-volume limit measure denoted π_β. This may be seen as follows using the theory of the corresponding random-cluster model on $\mathbb{Z}^2$ (see [11]). When $\beta \le \beta_{\mathrm{c}}$, the existence of the limit follows more or less as discussed above, using the coupling with the random-cluster measure, and the fact that the percolation probability of the latter measure is 0 whenever $\beta \le \beta_{\mathrm{c}}$. We write π_β for the limit Ising measure as $m, n \to \infty$.

The thermodynamic limit is slightly more subtle when $\beta > \beta_{\mathrm{c}}$, since the infinite-volume Ising model has a multiplicity of Gibbs states in this case. The (wired) random-cluster measure on $G^{\mathrm{w}}_{m,n}$ converges to the wired limit measure. By the uniqueness of infinite-volume random-cluster measures, the limit Ising measure is obtained by allocating random spins to the clusters of the infinite-volume random-cluster model (see Section 4.6 of [11]). Once again, we write π_β for the ensuing measure on $\{-1,+1\}^{\mathbb{Z}^2}$, and we note that $\pi_\beta = \frac{1}{2}\pi_\beta^+ + \frac{1}{2}\pi_\beta^-$ where $\pi_\beta^\pm$ denotes the infinite-volume Ising measure with $\pm$ boundary conditions.

It has been shown in [6] (see also [8], Corollary 8.4) that there exists (with strictly positive π_β-probability) an infinite spin-cluster in the Ising model if and only if $\beta > \beta_c$. More precisely:

(a) if $\beta \le \beta_c$, there is π_β-probability 1 that all spin-clusters are finite,
(b) if $\beta > \beta_c$, there is π_β-probability 1 that there exists a unique infinite spin-cluster, which is equally likely to be a $+$ cluster as a $-$ cluster. Furthermore, by the main theorem of [7] or otherwise, for any given finite set S of vertices, the infinite spin-cluster contains, π_β-a.s., a cycle containing S in its interior.

On passing to the dual graph, one finds that the random even subgraph of $G^{\mathrm{w}}_{m,n}$ with parameter $p \in (0, \frac{1}{2}]$ converges weakly as $m, n \to \infty$ to a probability measure ρ_p that is concentrated on even subgraphs of $\mathbb{Z}^2$ and satisfies:

(a′) if $p \ge p_c$, there is ρ_p-probability 1 that all faces of the graph are bounded,
(b′) if $p < p_c$, there is ρ_p-probability 1 that the graph is the vertex-disjoint union of finite clusters.

(Note that (5.1) defines β as a decreasing function of p, so the order relations are reversed.)

We have thus obtained a description of the weak-limit measure ρ_p when $p \le \frac{1}{2}$, and we note the phase transition at the parameter-value $p = p_c$. When $p > \frac{1}{2}$, a random even subgraph of $G^{\mathrm{w}}_{m,n}$ is the complement of a random even subgraph with parameter $1-p$. [It is a convenience at this point that $G^{\mathrm{w}}_{m,n}$ is an even graph.] Hence the weak-limit measure ρ_p exists for all $p \in [0,1]$ and gives meaning to the expression "a random even subgraph on $\mathbb{Z}^2$ with parameter p". [It is easily verified that $\rho_{\frac{1}{2}}$ equals the measure defined in Theorem 2.4 for $\mathbb{Z}^2$, and thus describes a uniform random even subgraph of $\mathbb{Z}^2$.] There is a sense in which the random even subgraph on $\mathbb{Z}^2$ has two points of phase transition, corresponding to the values p_c and $1-p_c$.

We consider finally the question of the size of a typical face of the random even graph on $\mathbb{Z}^2$ when $p_c \le p \le \frac{1}{2}$. When $p > p_c$, this amounts to asking about the size of a (sub)critical Ising spin-cluster. Higuchi [16] has proved an exponential upper bound for the radius of such a cluster, and this has been extended by van den Berg [4] to the cluster-volume. Thus, the law of the area of a typical face has an exponential tail.

The picture is quite different when the square lattice is replaced by the hexagonal lattice $\mathbb{H}$. Any even subgraph of $\mathbb{H}$ has vertex degrees 0 and/or 2, and thus comprises a vertex-disjoint union of cycles, doubly infinite paths, and isolated vertices. The (dual) Ising model inhabits the (Whitney) dual lattice of $\mathbb{H}$, namely the triangular lattice $\mathbb{T}$. Once again there exists a critical point $p_c = p_c(\mathbb{T}) < \frac{1}{2}$ such that the random even subgraph of $\mathbb{H}$ satisfies (a′) and (b′) above. In particular, the random even subgraph has a.s. only cycles and isolated vertices but no infinite paths. Recall that site percolation on $\mathbb{T}$ has critical value $\frac{1}{2}$. Therefore, for $p = \frac{1}{2}$, the face F_x of the random even subgraph containing the dual vertex x corresponds to a *critical* percolation cluster. It follows that its volume and radius have polynomially decaying tails, and that the boundary of F_x, when conditioned to be increasingly large, approaches SLE_6. See [21, 23] and [5, 25]. The spin-clusters of the Ising model on $\mathbb{T}$ are 'critical' (in a certain sense described below) for all $p \in (p_c(\mathbb{T}), \frac{1}{2}]$, and this suggests the possibility that the boundary of F_x, when conditioned to be increasingly large, approaches SLE_6 for any such p. This is supported by the belief in the physics community that the so-called universality class of the spin-clusters of the subcritical Ising model on $\mathbb{T}$ is the same as that of critical percolation.

The 'criticality' of such Ising spin-clusters (mentioned above) may be obtained as follows. Note first that, since $\beta < \beta_c$, there is a unique Gibbs state π_β for the Ising model. Therefore, π_β is invariant under the interchange of spin-values $-1 \leftrightarrow +1$. Let R_n be a rhombus of the lattice with side-lengths n and axes parallel to the horizontal and one of the diagonal lattice directions, and consider the event A_n that R_n is traversed from left to right by a $+$ path (i.e., a path ν satisfying $\sigma_y = +1$ for all $y \in \nu$). It is easily seen that the complement of A_n is the event that R_n is crossed from top to bottom by a $-$ path (see Lemma 11.21 of [10] for the analogous case of bond percolation on the square lattice). Therefore,

$$\pi_\beta(A_n) = \tfrac{1}{2}, \qquad 0 \le \beta < \beta_c. \tag{5.5}$$

For $x \in \mathbb{Z}^2$, let S_x denote the spin-cluster containing x, and define

$$\mathrm{rad}(S_x) = \max\{|z - x| : z \in S\},$$

where $|y|$ is the graph-theoretic distance from 0 to y. By (5.5), there exists a vertex x such that $\pi_\beta(\mathrm{rad}(S_x) \ge n) \ge (2n)^{-1}$. By the translation-invariance of π_β,

$$\pi_\beta\big(\mathrm{rad}(S_0) \ge n\big) \ge \frac{1}{2n}, \qquad 0 \le \beta < \beta_c, \tag{5.6}$$

where 0 denotes the origin of the lattice. The left side of (5.6) tends to 0 as $n \to \infty$, and the polynomial lower bound is an indicator of the criticality of the model.

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