RANDOM ELECTRICAL NETWORKS ON COMPLETE GRAPHS II: PROOFS

Geoffrey Grimmett and Harry Kesten

Abstract. This paper contains the proofs of Theorems 2 and 3 of the article entitled *Random electrical networks on complete graphs*, written by the same authors and published in the Journal of the London Mathematical Society, vol. 30 (1984), pp. 171–192. The current paper was written in 1983 but was not published in a journal, although its existence was announced in the LMS paper. This TEX version was created on 9 July 2001. It incorporates minor improvements to formatting and punctuation, but no change has been made to the mathematics.

We study the effective electrical resistance of the complete graph K_{n+2} when each edge is allocated a random resistance. These resistances are assumed independent with distribution $\mathbb{P}(R = \infty) = 1 - n^{-1}\gamma(n)$, $\mathbb{P}(R \le x) = n^{-1}\gamma(n)F(x)$ for $0 \le x < \infty$, where F is a fixed distribution function and $\gamma(n) \to \gamma \ge 0$ as $n \to \infty$. The asymptotic effective resistance between two chosen vertices is identified in the two cases $\gamma \le 1$ and $\gamma > 1$, and the case $\gamma = \infty$ is considered. The analysis proceeds via detailed estimates based on the theory of branching processes.

1. Introduction

In these notes we give complete proofs of Theorems 2 and 3 and a further indication of the proof of Theorem 1 in Grimmett and Kesten (1983). We use the same notation as in that paper and we therefore repeat only the barest necessities. K_{n+2} denotes the complete graph with $n+2$ vertices, which we label as $\{0, 1, \dots, n, \infty\}$. (See Bollobás (1979) for definition). Each edge e is given a random resistance $R(e)$ with distribution

$$\begin{aligned} \mathbb{P}(R(e) \le x) &= \frac{\gamma(n)}{n} F(x) \quad \text{for } 0 \le x < \infty \\ \mathbb{P}(R(e) = \infty) &= 1 - \frac{\gamma(n)}{n}, \end{aligned} \tag{1.1}$$

where F is a fixed distribution function concentrated on $[0, \infty)$ and $\gamma(n)$ a sequence of numbers such that $0 \le \gamma(n) \le n$. All the resistances $R(e)$, $e \in K_{n+2}$, are assumed independent. R_n denotes the resulting (random) effective resistance in K_{n+2} between the vertices 0 and ∞. We shall prove the following result (the numbering is taken from Grimmett and Kesten (1983)):

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Theorem 2. *If*

$$\lim_{n\to\infty} \gamma(n) = \gamma \le 1 \tag{1.2}$$

then

$$\lim_{n\to\infty} \mathbb{P}\big(R_n = \infty\big) = 1. \tag{1.3}$$

To describe the limit distribution of R_n when $\gamma(n) \to \gamma > 1$ we need a (one-type) Bienaymé–Galton–Watson process $\{Z_n\}_{n\ge 0}$ in which the offspring distribution is a Poisson distribution with mean γ and $Z_0 = 1$. (See Harris (1963) Ch. I; this book uses the more traditional name Galton–Watson process for the branching process). We denote the random family tree of such a process by T and label its root by $\langle 0\rangle$, and the children in the nth generation of the individual $\langle i_1, \dots, i_{n-1}\rangle$ (or $\langle 0\rangle$ if $n = 1$) in the $(n-1)$th generation by $\langle i_1, \dots, i_n\rangle$ with $1 \le i_n \le N = N(i_1, \dots, i_{n-1}) :=$ number of children of $\langle i_1, \dots, i_{n-1}\rangle$. Thus, not all $\langle i_1, \dots, i_n\rangle$ with $i_1, \dots, i_n \ge 1$ occur as vertices of T, but only those which correspond to individuals which are actually born or "realized". For more details, see Harris (1963), Ch. VI.2 or Jagers (1975), Ch. 1.2 or Grimmett and Kesten (1983). $T_0 := \{\langle 0\rangle\}$ is called the *0th generation* of T, and for $n \ge 1$ the collection of vertices $\langle i_1, \dots, i_n\rangle$ of T is called the *nth generation* of T, and denoted by T_n. $|T_n|$, the cardinality of T_n, is just Z_n. The subtree of T consisting of all vertices in $T_0 \cup T_1 \cup \dots \cup T_n$ together with all the edges between these vertices is denoted by $T_{[n]}$. We write $R(T_{[n]})$ for the resistance between $\langle 0\rangle$ and T_n in $T_{[n]}$ (formally this is defined by first identifying all vertices in T_n — or shortcircuiting them — and finding the resistance between $\langle 0\rangle$ and the single vertex obtained by this identification). Note that $R(T_{[n]}) = \infty$ if and only if $T_n = \emptyset$, i.e., if and only if the branching process is extinct by the nth generation. The limit

$$R(T) := \lim_{n\to\infty} R(T_{[n]})$$

always exists by the monotonicity property (2.6) below.

Theorem 3. *If*

$$\lim \gamma(n) = \gamma > 1, \tag{1.4}$$

then, as $n \to \infty$, the distribution of R_n converges to that of $R'(\gamma) + R''(\gamma)$, where $R'(\gamma)$ and $R''(\gamma)$ are independent random variables, each with the distribution of $R(T)$, defined above, with the mean of the Poisson offspring distribution equal to the value γ given by (1.4). In particular the atom at ∞ of the limit distribution of R_n equals $2q(\gamma) - q^2(\gamma)$ (< 1), where $q(\gamma)$ is the extinction probability of the branching process with Poisson offspring distribution with mean γ ($q(\gamma)$ is the smaller solution of the equation $q = \exp(-\gamma(1-q))$).

Remark. Theorems 2 and 3 show that the value one is a critical value for γ. If $\gamma \le 1$, then all the mass in the distribution of R_n escapes to ∞ as $n \to \infty$, while for

$\gamma > 1$, R_n has a limit distribution which puts some mass on $[0, \infty)$ (but also has an atom at ∞). This is closely related to the threshold phenomenon for the spread of epidemics discussed by von Bahr and Martin-Löf (1980, especially Section 5). Both the von Bahr and Martin-Löf paper and ours rest on the existence of imbedded random trees which behave like the family trees of branching processes (see next paragraph).

The idea of the proofs was already explained in Grimmett and Kesten (1983). It consists in looking at the graphs of vertices of K_{n+2} which are connected to 0 and ∞, respectively, by paths of finite resistance (such paths will be called *conducting paths* in the sequel). It will be shown that these graphs resemble two independent trees, each with the same distribution as T, given above. In addition, it will be shown that there are with high probability a great many edges with finite resistance joining pairs of vertices, one from each tree, which are far away from 0 and ∞, respectively. There are enough of these interconnections to make R_n nearly equal to the sum of the resistances of these two trees (one connected to 0 and one to ∞). All this will be done first under the assumption that $R(e) \geq \varepsilon > 0$ for all e with probability one. The next section is largely devoted to proving the continuity property in Proposition 1 which allows us to let ε go to 0 afterwards. This continuity property needs proof because T may be infinite. In finite networks continuity of the effective resistance between two vertices, as a function of the resistances of the individual edges, is comparatively easy (see Kesten (1982) Ch. 11).

2. Preliminaries

Some standard ways to combine resistances were already discussed in Section 2 of Grimmett and Kesten (1983). We need some (known) extensions of these rules, especially for the case where individual edges may have zero resistance.

Let G be a finite connected graph and A_0 and A_1 two disjoint sets of vertices of G. Assume that each edge e has been assigned a resistance, to be denoted by $R(e)$. To find the resistance between A_0 and A_1 in the network of edges of G one first identifies all vertices in A_0 (A_1) as a single vertex, $\widehat{A}_0$ ($\widehat{A}_1$) say. This is equivalent to setting $R(e)$ equal to zero, whenever both endpoints of e lie in the same A_i. We shall also identify as one vertex any maximal class of vertices of G which is already shortcircuited, i.e., any maximal class $\widehat{A} = \{v_1, \dots, v_m\}$ such that for any v_i, v_j in $\widehat{A}$ there exist $v_{i_1}, \dots, v_{i_r}$ and edges e_l between v_{i_l} and $v_{i_{l+1}}$, $l = 0, \dots, r$, with $v_{i_0} = v_i$, $v_{i_{r+1}} = v_j$, and $R(e_l) = 0$, $l = 0, \dots, r$. Let $\widehat{G}$ be the network resulting from these identifications. The resistance between A_0 and A_1 in G is defined as the resistance between $\widehat{A}_0$ and $\widehat{A}_1$ in $\widehat{G}$. To compute this resistance one introduces the potential function $V(\widehat{v})$ (with $\widehat{v}$ running through the vertices of $\widehat{G}$) with the boundary values 0 on $\widehat{A}_0$ and 1 on $\widehat{A}_1$ (to produce these boundary values physically, one has to connect $\widehat{A}_0$ and $\widehat{A}_1$ to a voltage source external to the network). $V(\cdot)$ is determined by Kirchhoff's laws:

$$V(\widehat{v}) = \left\{ \sum \frac{1}{R(e)} \right\}^{-1} \sum \frac{V(\widehat{w}(e))}{R(e)}, \quad \widehat{v} \neq \widehat{A}_0, \ \widehat{A}_1. \tag{2.1}$$

The sums in (2.1) run over all edges e of G with one endpoint in the class corresponding to $\widehat{v}$ and the other endpoint outside this class; the class of the endpoint of e outside $\widehat{v}$ is denoted by $\widehat{w}(e)$. Note that any $R(e)$ appearing in (2.1) is strictly positive by our choice of the classes $\widehat{v}$ and $\widehat{w}$ (see Kesten (1982) Ch. 11 for more details).

We shall frequently appeal to the following probabilistic interpretation of $V(\cdot)$ (see Doyle and Snell (1982), Griffeath and Liggett (1982)). Consider a Markov chain $\{X_\nu\}$ on $\widehat{G}$ with transition probability

$$P(\widehat{v}, \widehat{w}) = \left\{ \sum_{v} \frac{1}{R(e)} \right\}^{-1} \sum_{v,w} \frac{1}{R(e)}, \quad \widehat{v} \neq \widehat{w}, \tag{2.2}$$

where $\sum_v$ runs over all edges e of G with one endpoint inside and one endpoint outside $\widehat{v}$, while $\sum_{v,w}$ runs only over those edges with one endpoint in $\widehat{v}$ and the other in $\widehat{w}$. Then

$$V(\widehat{v}) = \mathbb{P}\{X. \text{ visits } \widehat{A}_1 \text{ before } \widehat{A}_0 \mid X_0 = \widehat{v}\}. \tag{2.3}$$

The resistance between $\widehat{A}_0$ and $\widehat{A}_1$ is given by

$$\left\{ \sum \frac{V(\widehat{w}(e))}{R(e)} \right\}^{-1}, \tag{2.4}$$

with the sum in (2.4) running over all edges e of G with one endpoint in A_0, and the other endpoint in any class disjoint from A_0. (In (2.4) the class of this other endpoint is denoted by $\widehat{w}(e)$; $\widehat{w}(e)$ varies with e.) See Kesten (1982) Ch. 11. The probability interpretation (2.3) together with (2.4) gives a probabilistic meaning to resistance as well. The reader should note that the above simplifies if $R(e) > 0$ for all e and if A_0 and A_1 consist of single vertices. In this case $\widehat{G} = G$.

Very intuitive is the following monotonicity property. Let G, A_0 and A_1 be as above, and let $\{R'(e)\}$, $\{R''(e)\}$ be two assignments of resistances to the edges of G. Denote the corresponding resistances between A_0 and A_1 in G by $R'(A_0, A_1)$ and $R''(A_0, A_1)$, respectively. Then

$$R'(e) \leq R''(e) \quad \text{for all } e \tag{2.5}$$

implies

$$R'(A_0, A_1) \leq R''(A_0, A_1). \tag{2.6}$$

Unfortunately the proof is not all that simple (see Griffeath and Liggett (1982), Doyle and Snell (1982), and for the case when $R'(e)$ and $R''(e)$ may take the values 0, ∞ see Kesten (1982)). Note that this monotonicity property states in particular that shortcircuiting some vertices, or insertion of additional edges (no matter what their resistance is) can only decrease the resistance between A_0 and A_1; also removal of any edges can only increase the latter resistance.

For the remainder of this section $Z_0 = 1, Z_1, Z_2, \dots$ is any Bienaymé–Galton–Watson branching process with the mean number γ of offspring per individual strictly greater than 1, but finite. That is to say,

$$1 < \gamma := \mathbb{E}Z_1 < \infty. \tag{2.7}$$

It is not assumed that the offspring distribution is a Poisson distribution. q denotes the extinction probability:

$$q = \mathbb{P}\{Z_n = 0 \text{ eventually}\}. \tag{2.8}$$

It is well known (see Harris (1963) Theorem I.6.1) that under (2.7)

$$q < 1. \tag{2.9}$$

We write $f(z)$ for the generating function of the offspring distributions:

$$f(z) = \sum_{n=0}^{\infty} \mathbb{P}\{Z_1 = n\} z^n, \quad |z| \le 1. \tag{2.10}$$

The convexity of f on $[0,1]$ and (2.9) imply that $f'(q) < 1$ (see Harris (1963) Fig. 1 and proof of Theorem I.8.4). We can therefore find an $\varepsilon_0 > 0$ such that

$$0 < q + 2\varepsilon_0 < 1, \ f'(q + 2\varepsilon_0) + 2\varepsilon_0 < 1. \tag{2.11}$$

T will be the family tree of the branching process as in Section 1. Also T_n and $T_{[n]}$ are as in Section 1 and each edge of T is given a resistance such that the $\{R(e) : e \in T\}$ are independent, and all have the same distribution F. As in Section 1 we define

$$R(T) = \lim_{n\to\infty} R(T_{[n]}) = \lim_{n\to\infty} \{\text{resistance between 0 and } T_n \text{ in } T_{[n]}\}.$$

Since we can think of $R(T_{[n]})$ as the resistance between 0 and T_{n+1} in $T_{[n+1]}$ when all vertices in $T_n \cup T_{n+1}$ are shortcircuited, it follows from the monotonicity property (2.6) that $R(T_{[n]}) \le R(T_{[n+1]})$ so that the limit $R(T)$ is well defined so long as we allow it to take the value ∞. We set

$$R^\varepsilon(e) = R(e) + \varepsilon,$$

and in general, for any resistance $R(*)$ calculated as a function of the $R(e)$, we denote by $R^\varepsilon(*)$ the corresponding resistance when $R(e)$ is everywhere replaced by $R^\varepsilon(e)$. In particular $R^\varepsilon(T)$ is the resistance of the family tree when each edge resistance is increased by ε.

Proposition 1.

$$\lim_{\varepsilon\downarrow 0} R^{\varepsilon}(T) = R(T) \quad \textit{w.p.1.} \tag{2.12}$$

The proof will be broken down into several lemmas. If $\langle i_1, \dots, i_n\rangle$ is a vertex of T then we write $T(i_1, \dots, i_n)$ for the subtree of T whose vertices are $\langle i_1, \dots, i_n\rangle$ and all its descendants. I.e., the vertex set of $T(i_1, \dots, i_n)$ is

$$\langle i_1, \dots, i_n\rangle \bigcup \Big\{\langle i_1, \dots, i_n,\ j_1, \dots, j_l\rangle \in T : l \ge 1, j_1, \dots, j_l \in \{1, 2, \dots\}\Big\}.$$

Two vertices of $T(i_1, \dots, i_n)$ have an edge of $T(i_1, \dots, i_n)$ between them if and only if they are connected by an edge in T. From the branching property, it follows that, conditionally on $\langle i_1, \dots, i_n\rangle \in T$, the distribution of $T(i_1, \dots, i_n)$ (as a random graph) is the same as the original distribution of T. More generally, given that $\langle i_1, \dots, i_m\rangle \in T$ and $1 \le n_1 < n_2 < \dots < n_l \le m-1$, $j_r \ne i_{n_r+1}$ and $\langle i_1, \dots, i_{n_r},\ j_r\rangle \in T$,

$$\begin{aligned}&T(i_1, \dots, i_{n_r}, j_r),\ r = 1, \dots, l, \text{ are conditionally independent, each}\\ &\text{with the same distribution as } T\text{, and the edges of these}\\ &\text{trees have independent resistances, all with distribution } F.\end{aligned} \tag{2.13}$$

We also introduce the notation $T^j(i_1, \dots, i_n)$ for the subtree of T whose vertices are $\langle i_1, \dots, i_n\rangle$, $\langle i_1, \dots, i_n, j\rangle$ and all descendants of $\langle i_1, \dots, i_n, j\rangle$. Note that $\langle i_1, \dots, i_n\rangle$ only has the one descendant $\langle i_1, \dots, i_n, j\rangle$ in this tree. Similarly $T^j = T^j(0)$ has as vertices $\langle 0\rangle, \langle j\rangle$ and the descendants of $\langle j\rangle$. We only use this notation if $\langle i_1, \dots, i_n, j\rangle \in T$ (respectively $\langle j\rangle \in T$ if $n = 0$). Given that $\langle i_1, \dots, i_n, j\rangle \in T$ the distribution of $T^j(i_1, \dots, i_n)$ is the same as the conditional distribution of T, given $|T_1| = 1$.[1]) There is also an independence property for several $T^{j_r}(i_1, \dots, i_{n_r})$ analogous to (2.13), given that $\langle i_1, \dots, i_{n_r}, j_r\rangle \in T$.

Lemma 1. *Let $\mathcal{P}$ be a property of rooted labeled trees with resistances assigned to their edges. If ε_0 satisfies (2.11), and if*

$$\mathbb{P}\{T^j \text{ does not have property } \mathcal{P} \mid \langle j\rangle \in T_1\} \le q + 2\varepsilon_0, \tag{2.14}$$

then

$$\begin{aligned}\mathbb{P}\{&\text{for each infinite path } \langle i_1, i_2, \dots\rangle \text{ in } T \text{ there exist infinitely}\\ &\text{many } n \text{ and integers } j_{n+1} \ne i_{n+1} \text{ such that } \langle i_1, \dots, i_n, j_{n+1}\rangle \in T\\ &\text{and such that } T^{j_{n+1}}(i_1, \dots, i_n) \text{ has property } \mathcal{P}\} = 1.\end{aligned} \tag{2.15}$$

[1] $|A|$ denotes the number of vertices in A.

Moreover, if $\Gamma(i_1, i_2, \dots, i_n)$ *denotes the number of* k, $2 \le k \le n$ *for which there exists a* $j_k \neq i_k$ *such that* $T^{j_k}(i_1, \dots, i_{k-1})$ *has property* $\mathcal{P}$, *then there exist constants* $0 < C_1, C_2 < \infty$ *such that for* $n \ge 2$

$$\mathbb{P}\Big\{T_n \neq \emptyset \text{ and } \min_{\langle i_1,\dots,i_n\rangle \in T_n} \Gamma(i_1, \dots, i_n) \le C_1 n\Big\} \le e^{-C_2 n}. \tag{2.16}$$

Proof. Assume (2.14). By virtue of (2.13) and its analogue for several $T^{j_r}(i_1, \dots, i_{n_r})$ it suffices for (2.15) to prove that

$$\begin{aligned}\lim_{n\to\infty} \sum_{\substack{i_1\ge 1,\dots,\\ i_n\ge 1}} \mathbb{P}\Big\{&\langle i_1, \dots, i_n\rangle \in T \text{ but there does not exist a } k \le n \\ &\text{and } j_k \neq i_k, \text{ such that } \langle i_1, \dots, i_{k-1}, j_k\rangle \in T \\ &\text{and such that } T^{j_k}(i_1, \dots, i_{k-1}) \text{ has property } \mathcal{P}\Big\} = 0.\end{aligned} \tag{2.17}$$

To see this, note that (2.17) says that if $\langle i_1, i_2, \dots\rangle$ is an infinite path in T, then there is with probability one at least one n and $j_{n+1} \neq i_{n+1}$ for which $T^{j_{n+1}}(i_1, \dots, i_n)$ has property $\mathcal{P}$. But $T^{j_{n+1}}(i_1, \dots, i_n)$ and $T^{i_{n+1}}(i_1, \dots, i_n)$ are independent, and (2.17) again applies to $T^{i_{n+1}}(i_1, \dots, i_n)$, so that with probability one there exists a further $m > n$ and $j_{m+1} \neq i_{m+1}$ such that $T^{i_{m+1}}(i_1, \dots, i_m)$ also has property $\mathcal{P}$ etc.

To prove (2.17) note that $\langle i_1, \dots, i_n\rangle \in T$ if and only if $\langle 0\rangle$ has $l_1 \ge i_1$ children, $\dots$, $\langle i_1, \dots, i_r\rangle$ has $l_{r+1} \ge i_{r+1}$ children, $r = 0, \dots, n-1$, for some integers l_{r+1}. The probability of this event for given $l_{r+1} \ge i_{r+1}$ is

$$p_{l_1} \cdot p_{l_2} \cdots p_{l_n},$$

where $p_l = \mathbb{P}\{Z_1 = l\}$. Given that the above event occurs with prescribed $l_1, \dots, l_n$, the trees $T^{j_{r+1}}(i_1, \dots, i_r)$, $r = 0, \dots, n-1$, $j_{r+1} = 1, \dots, l_{r+1}$, $j_{r+1} \neq i_{r+1}$ with their resistances are all independent, and each has the conditional distribution of T^j and its resistances, given $\langle j\rangle \in T_1$. Therefore the conditional probability that none of these trees has property $\mathcal{P}$ is at most

$$(q + 2\varepsilon_0)^{\sum_{r=1}^n (l_r - 1)} \tag{2.18}$$

(by virtue of (2.14)). The summand of (2.17) is therefore at most

$$\sum_{l_1\ge i_1,\dots,l_n\ge i_n} p_{l_1} \cdots p_{l_n} (q + 2\varepsilon_0)^{\sum_{r=1}^n (l_r - 1)}.$$

Finally, the sum in (2.17) is at most

(2.19)
$$\sum_{l_1\geq 1,\dots,l_n\geq 1}\ \sum_{\substack{1\leq i_1\leq l_1\\ \vdots\\ 1\leq i_n\leq l_n}} p_{l_1}\cdots p_{l_n}(q+2\varepsilon_0)^{\sum_{r=1}^n(l_r-1)} = \Big\{\sum_{l\geq 1} lp_l(q+2\varepsilon_0)^{l-1}\Big\}^n$$
$$= \{f'(q+2\varepsilon_0)\}^n.$$

By (2.11) the last member of (2.19) tends to zero as $n\to\infty$, so that (2.15) holds.

A slight strengthening of the above argument leads to (2.16). For every $\theta\geq 0$ and $l_{r+1}\geq i_{r+1}$,

(2.20)
$$\begin{aligned}&\mathbb{P}\{\langle i_1,\dots,i_n\rangle\in T_n \text{ and } \langle i_1,\dots,i_r\rangle \text{ has } l_{r+1} \text{ children in } T,\\ &\qquad r=0,\dots,n-1,\ \text{ but } \Gamma(i_1,\dots,i_n)\leq C_1n\}\\ &\leq p_{l_1}\cdots p_{l_n}e^{\theta C_1 n}\mathbb{E}\{e^{-\theta\Gamma(i_1,\dots,i_n)}\mid \langle i_1,\dots,i_r\rangle \text{ has } l_{r+1} \text{ children in } T,\ r=0,\dots,n-1\}.\end{aligned}$$

Denote the left hand side of (2.14) temporarily by ρ. Analogously to (2.18) we then obtain

$$\begin{aligned}(2.21)\qquad &\mathbb{E}\{e^{-\theta\Gamma(i_1,\dots,i_n)}\mid\langle i_1,\dots,i_r\rangle \text{ has } l_{r+1} \text{ children, } 0\leq r\leq n-1\}\\ &=\prod_{r=1}^{n-1}\{\rho^{l_{r+1}-1}+(1-\rho^{l_{r+1}-1})e^{-\theta}\}\\ &\leq\prod_{r=1}^{n-1}\{(q+2\varepsilon_0)^{l_{r+1}-1}+(1-(q+2\varepsilon_0^{l_{r+1}-1})e^{-\theta}\}.\end{aligned}$$

Moreover

$$\begin{aligned}(2.22)\qquad &\sum_{i=1}^{\infty}\sum_{l\geq i}p_l\{(q+2\varepsilon_0)^{l-1}+(1-(q+2\varepsilon_0)^{l-1})e^{-\theta}\}\\ &=\sum_{l=1}^{\infty}lp_l\{(q+2\epsilon_0)^{l-1}(1-e^{-\theta})+e^{-\theta}\}\\ &=(1-e^{-\theta})f'(q+2\varepsilon_0)+e^{-\theta}\gamma.\end{aligned}$$

Now choose $\theta>0$ so large that

$$(1-e^{-\theta})f'(q+2\varepsilon_0)+e^{-\theta}\gamma\leq f'(q+2\varepsilon_0)+\varepsilon_0\leq 1-\varepsilon_0$$

(see (2.11)). Since the left hand side of (2.16) is bounded by the sum over $l_1\geq 1,\dots,l_n\geq 1$, $i_1\leq l_1,\dots,i_n\leq l_n$ of the left hand side of (2.20), we obtain from

(2.20)–(2.22)

$$\begin{aligned}
&\mathbb{P}\{\min_{\langle i_1,\dots,i_n\rangle\in T} \Gamma(i_1,\dots,i_n) \le C_1 n\} \\
&\quad\le e^{\theta C_1 n} \sum_{i_1\ge 1,\dots,i_n\ge 1} \quad \sum_{l_1\ge i_1,\dots,l_n\ge i_n} p_{l_1}\cdots p_{l_1} \\
&\qquad\qquad\qquad\times \prod_{r=1}^{n-1}\{(q+2\varepsilon_0)^{l_{r+1}-1} + (1-(q+2\varepsilon_0)^{l_{r+1}-1})e^{-\theta}\} \\
&\quad\le e^{\theta C_1 n}(1-\varepsilon_0)^{n-1}.
\end{aligned}$$

(2.16) follows if we take C_1 small enough. □

Lemma 2. *For all $\varepsilon > 0$*

$$\mathbb{P}\{R^\varepsilon(T) = \infty\} = \mathbb{P}\{R(T) = \infty\} = q.$$

Proof. We prove the second equality. This proof works for any choice of F, and thus implies the first equality as well, since adding ε to each resistance has the same effect on the distribution of $R(T)$ as changing $F(x)$ to $F(x-\varepsilon)$.

By the rules for combining resistances (see Grimmett and Kesten (1983) Section 2; also Figure 1 below),

$$\begin{aligned}
\{R(T)\}^{-1} &= \sum_{1\le j\le Z_1} \{R(T^j)\}^{-1} \\
&= \sum_{1\le j\le Z_1} \{R(e(j)) + R(T(j))\}^{-1},
\end{aligned} \tag{2.23}$$

where $e(j)$ denotes the edge between $\langle 0\rangle$ and $\langle j\rangle$. ($T(j)$ is defined just after Proposition 1).

In particular $R(T) = \infty$ if and only if $Z_1 = |T_1| = 0$ or $R(T(j)) = \infty$ for each $j \le Z_1$. In view of (2.13) this implies

$$\begin{aligned}
\mathbb{P}\{R(T) = \infty\} &= \sum_{n\ge 0} \mathbb{P}\{Z_1 = n\}\mathbb{P}\{R(T(j)) = \infty \text{ for } 1 \le j \le n\} \\
&= \sum_{n\ge 0} \mathbb{P}\{Z_1 = n\}(\mathbb{P}\{R(T) = \infty\})^n.
\end{aligned}$$

Thus $\mathbb{P}\{R(T) = \infty\}$ is a solution of the equation

$$x = f(x). \tag{2.24}$$

As is well known (see Harris (1963) proof of Theorem I.6.1) the only solutions of (2.24) in $[0,1]$ are q and 1. It therefore suffices to show that

$$\mathbb{P}\{R(T) = \infty\} < 1. \tag{2.25}$$

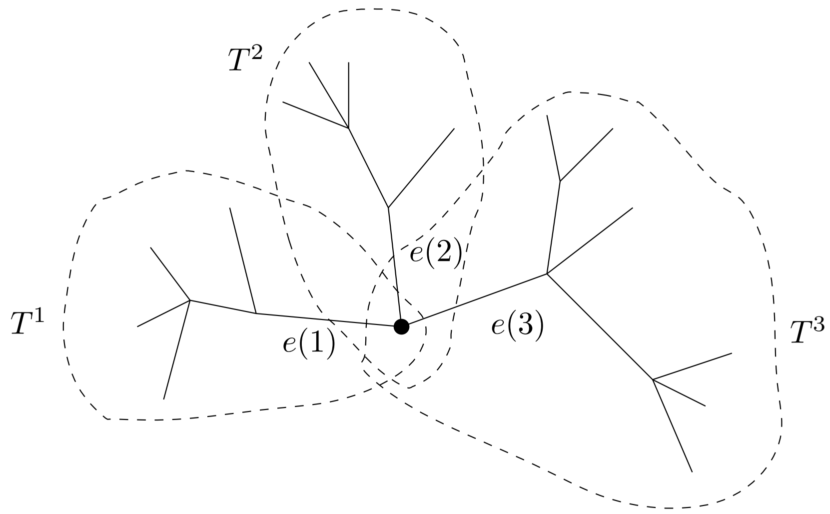

Figure 1. A picture of T when $Z_1 = 3$; the root of T, $\langle 0 \rangle$, is represented by the heavy dot. The subtrees T^j form parallel resistances between $\langle 0 \rangle$ and ∞. In T^j, $e(j)$ and $T(j)$ are in series.

To this end we first choose a constant K such that

$$\gamma F(K) > 1. \tag{2.26}$$

We next consider the subtree $\widetilde{T}$ of T whose vertices are the vertices of T connected to $\langle 0 \rangle$ by a path all of whose edges have a resistance not exceeding K. Of course two vertices of $\widetilde{T}$ are connected by an edge of $\widetilde{T}$ if and only if they are connected by an edge of T. We write $\widetilde{T}_n$ (respectively $\widetilde{T}_{[n]}$) for the part of $\widetilde{T}$ which belongs to T_n (respectively $T_{[n]}$). The children in $\widetilde{T}$ of a vertex $\langle x \rangle$ in $\widetilde{T}$ are precisely those connected to $\langle x \rangle$ by an edge of resistance not exceeding K. $\widetilde{T}$ is therefore the family tree of a branching process $\{\widetilde{Z}_n\}$ with offspring distributions

$$\widetilde{p}_m := \mathbb{P}\{\widetilde{Z}_1 = m\} = \sum_{n \geq m} \mathbb{P}\{Z_1 = n\} \binom{n}{m} F^m(K)(1 - F(K))^{n-m},$$

and mean number of offspring

$$\sum_{m \geq 0} m\widetilde{p}_m = \sum_{n \geq 0} n\mathbb{P}\{Z_1 = n\} F(K) = \gamma F(K) > 1. \tag{2.27}$$

(See (2.26)). Thus the $\widetilde{Z}$ process is supercritical also, and

$$\widetilde{q} := \mathbb{P}\{\widetilde{Z}_n = 0 \text{ eventually}\} < 1.$$

From Theorem I.6.2 of Harris (1963) we conclude that for each $m \geq 1$

$$\mathbb{P}\{\widetilde{Z}_k \geq m\} \to 1 - \widetilde{q} > 0 \quad \text{as } k \to \infty.$$

We can therefore fix κ such that

$$\sum_{n\geq 2} \mathbb{P}\{\widetilde{Z}_\kappa = n\}\{1 - \big(\tfrac{1}{2} + \tfrac{\widetilde{q}}{2}\big)^n - n\big(\tfrac{1}{2} - \tfrac{\widetilde{q}}{2}\big)\big(\tfrac{1}{2} + \tfrac{\widetilde{q}}{2}^{n-1}\} \geq \tfrac{3}{4}(1 - \widetilde{q}). \tag{2.28}$$

With κ fixed in this way we define

$$\begin{aligned} g(x) &= \sum_{n\geq 2} \mathbb{P}(\widetilde{Z}_\kappa = n)\{1 - (1-x)^n - nx(1-x)^{n-1}\} \\ &= \sum_{n\geq 2} \mathbb{P}(\widetilde{Z}_\kappa = n) \sum_{j=2}^{n} \binom{n}{j} x^j (1-x)^{n-j}, \quad 0 \leq x \leq 1, \\ L &= 2\kappa K \end{aligned}$$

and

$$\alpha_n = \mathbb{P}(R(\widetilde{T}_{[\kappa n]}) \leq L) \quad (\alpha_0 = 1),$$

where $R(\widetilde{T}_{[n]})$ is the resistance between $\langle 0 \rangle$ and $\widetilde{T}_n$ in $\widetilde{T}_{[n]}$. We claim that

$$\alpha_n \geq g(\alpha_{n-1}), \quad n \geq 1. \tag{2.29}$$

Before proving (2.29) we show that it quickly implies the lemma. Clearly $g(x)$ is non-decreasing and continuous on $[0,1]$ and (2.28) states that $g(\frac{1}{2}(1-\widetilde{q})) \geq \frac{3}{4}(1-\widetilde{q}) > \frac{1}{2}(1-\widetilde{q})$, while $g(x) \leq 1$. Therefore

$$r := \max\{x \in [0,1] : g(x) \geq x\} > \tfrac{1}{2}(1-\widetilde{q}) > 0.$$

Thus $r = g(r) \leq g(x) < x$ for $x \in (r,1]$, which together with (2.29) and $\alpha_0 = 1$ implies (see Figure 2)

$$\alpha_n \geq g(\alpha_{n-1}) \geq g(g(\ldots(g(1))\ldots)) \to r \quad \text{as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} \mathbb{P}(R(\widetilde{T}_{[\kappa n]}) \leq L) \geq r.$$

By the monotonicity property (2.6) $R(T_{[n]}) \leq R(\widetilde{T}_{[n]})$ so that also $\mathbb{P}(R(T) \leq L) \geq r$. Thus (2.29) will imply (2.25) and the lemma.

Now we prove (2.29). Consider the event that $\widetilde{T}_\kappa$ contains at least two distinct individuals $\langle i_1, \ldots, i_\kappa \rangle$ and $\langle j_1, \ldots, j_\kappa \rangle$ such that the two resistances, between $\langle i_1, \ldots, i_\kappa \rangle$ and $\widetilde{T}_{\kappa n}$, and between $\langle j_1, \ldots, j_\kappa \rangle$ and $\widetilde{T}_{\kappa n}$, in the tree of the $\kappa(n-1)$ generations of descendants of $\langle i_1, \ldots, i_\kappa \rangle$ are both at most L. By virtue of (2.13) the probability of this event is

(2.30)

$$\sum_{n\geq 2} \mathbb{P}(\widetilde{Z}_\kappa = n) \sum_{j=2}^{n} \binom{n}{j} (\mathbb{P}\{R(\widetilde{T}_{[\kappa(n-1)]} \leq L)\})^j (1 - P\{R(\widetilde{T}_{[\kappa(n-1)]} \leq L)\})^{n-j}$$
$$= g(\alpha_{n-1}).$$

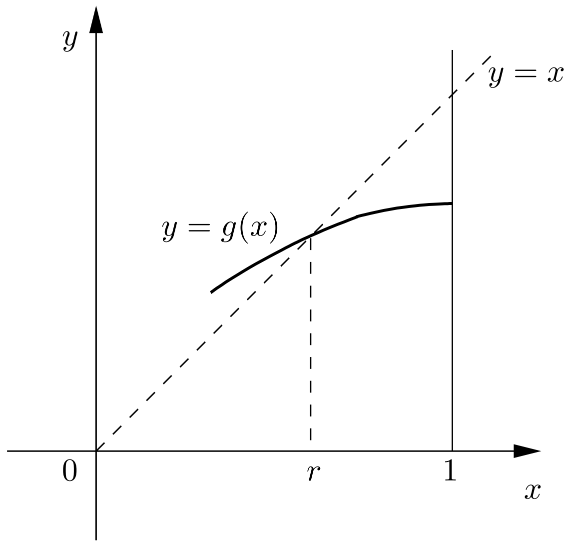

Figure 2.

Moreover, if $i_1 = j_1, \dots, i_s = j_s$, $i_{s+1} \neq j_{s+1}$, then the network consisting of the edges from $\langle 0 \rangle$ to $\langle i_1, \dots, i_s \rangle$ and the two parallel connections from $\langle i_1, \dots, i_s \rangle$ to $\widetilde{T}_{\kappa n}$ via $\langle i_1, \dots, i_\kappa \rangle$ and the tree of its descendants and via $\langle j_1, \dots, j_\kappa \rangle = \langle i_1, \dots, i_s, j_{s+1}, \dots, j_\kappa \rangle$ and the tree of its descendants is at most (see Figure 3)

$$sK + \tfrac{1}{2}\{(\kappa - s)K + L\} \leq L;$$

recall that each edge of $\widetilde{T}$ has resistance $\leq K$, and $L = 2\kappa K$. Thus $R(T_{[\kappa n]}) \leq L$ whenever $\langle i_1, \dots, i_\kappa \rangle$ and $\langle j_1, \dots, j_\kappa \rangle$ exist as above. Consequently (2.29) follows from the value in (2.30) for the probability of the existence of such $\langle i_1, \dots, i_\kappa \rangle$ and $\langle j_1, \dots, j_\kappa \rangle$. $\square$

We need some more notation for the next lemma. Note that this lemma does not involve random quantities. Let t be a rooted labeled tree, with root $\langle 0 \rangle$ and vertices labeled $\langle i_1, \dots, i_n \rangle$, just as described for T in the introduction. Assume that to each edge e of t, a resistance $r(e) < \infty$ has been assigned. If $\langle x \rangle = \langle i_1, \dots, i_n \rangle$ is a vertex of t, then set

$$(2.31)\quad \rho(x) = \{\text{resistance of the unique path in } t \text{ from } \langle 0 \rangle \text{ to } \langle x \rangle\}$$
$$= \sum_{k=1}^{n} \{\text{resistance of edge between } \langle i_1, \dots, i_{k-1} \rangle \text{ and } \langle i_1, \dots, i_k \rangle\}$$

(for $k = 1$, $\langle i_1, \dots, i_{k-1} \rangle = \langle 0 \rangle$). In accordance with previous notation we write $r(t(x))$ for the resistance of the tree $t(x)$ consisting of $\langle x \rangle$ and all its descendants and connecting edges between them. As before, $r(t(x))$ is the limit (as $m \to \infty$) of the resistance in $t(x)$, between $\langle x \rangle$ and its descendants in the $(n+m)$th generation

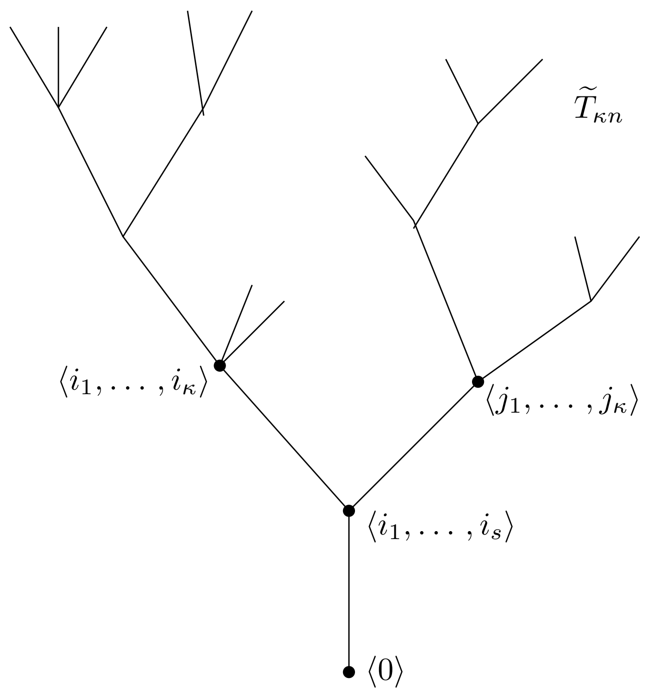

Figure 3.

of t, i.e., between $\langle x \rangle = \langle i_1, \ldots, i_n \rangle$ and the collection of vertices of the form $\langle i_1, \ldots, i_n, j_1, \ldots, j_m \rangle$. Finally, if $\langle y \rangle \in t$ its *equivalence class* $\langle \widehat{y} \rangle$ is the collection of vertices of t connected to $\langle y \rangle$ by paths of zero resistance (see the beginning of this section).

Lemma 3. *Let t be a rooted labeled tree as above. Let $\{X_\nu\}_{\nu \ge 0}$ be a Markov chain with state space the collection of equivalence classes $\{\langle \widehat{y} \rangle : \langle y \rangle \in t\}$ and transition probability matrix*

$$P(\langle y \rangle, \langle z \rangle) = \left\{ \sum_y \frac{1}{r(e)} \right\}^{-1} \sum\nolimits_{y,z} \frac{1}{r(e)}, \quad \langle \widehat{y} \rangle \neq \langle \widehat{z} \rangle, \tag{2.32}$$

where $\sum_y$ runs over all edges e of t with one endpoint in $\langle \widehat{y} \rangle$ and one outside $\langle \widehat{y} \rangle$, while $\sum_{y,z}$ runs only over those edges with one endpoint in $\langle \widehat{y} \rangle$ and the other in $\langle \widehat{z} \rangle$ (compare (2.2)). Then[2]

(2.33)
$$\mathbb{P}(X.\text{ never reaches } \langle \widehat{0} \rangle \mid X_0 = \langle \widehat{x} \rangle) = \mathbb{P}(X.\text{ reaches } \infty \text{ before } \langle \widehat{0} \rangle \mid X_0 = \langle \widehat{x} \rangle)$$
$$\ge \frac{\rho(x)}{\rho(x) + r(t(x))}, \quad \langle \widehat{x} \rangle \neq \langle \widehat{0} \rangle,$$

[2]The second member of (2.33) stands of course for $\lim_{n \to \infty} \mathbb{P}(X.\text{ reaches the } n\text{th generation of } t \text{ before } \langle \widehat{0} \rangle \mid X_0 = \langle \widehat{x} \rangle\}$. We shall use the more intuitive expression of (2.33) even in proofs without formally going through taking the limit as $n \to \infty$.

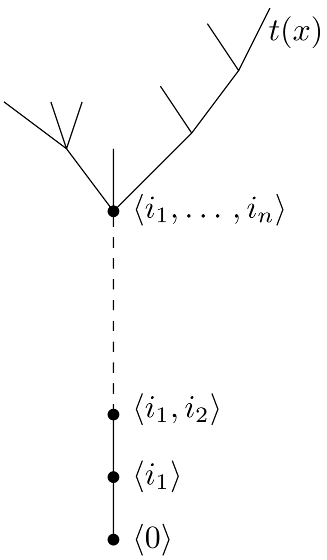

Figure 4. The tree t_*.

provided $\rho(x) + r(t(x)) > 0$.

Proof. For simplicity we only consider the case where all $r(e) > 0$ so that each equivalence class $\langle \widehat{y}\rangle$ consists of one vertex only. Since the identification of vertices in the same equivalence class turns t into another tree[3] with vertices $\langle \widehat{y}\rangle$, it is not hard to extend the argument to the general case where $r(e) = 0$ is allowed.

Now let $\langle x\rangle = \langle i_1, \dots, i_n\rangle$ and consider the tree t_*, whose vertices are only $\langle 0\rangle, \langle i_1\rangle, \langle i_1, i_2\rangle, \dots, \langle i_1, \dots, i_n\rangle = \langle x\rangle$ and all the descendants of $\langle x\rangle$ (see Figure 4). In this tree the resistance between $\langle 0\rangle$ and $\langle x\rangle$ is precisely $\rho(x)$, and the resistance between $\langle x\rangle$ and ∞ is precisely $r(t(x))$. Let $\{X_{*\nu}\}_{\nu\ge 0}$ be the Markov chain on t_* which is analogous to X_ν, and has transition probability matrix

$$P_*(\langle y\rangle, \langle z\rangle) = \left\{\sum\nolimits_{*y} \frac{1}{r(e)}\right\}^{-1} \left\{\sum\nolimits_{*yz} \frac{1}{r(e)}\right\}, \quad \langle z\rangle \neq \langle y\rangle,$$

where $\sum_{*y}$ (respectively $\sum_{*yz}$) runs over all edges of t_* with one endpoint at $\langle y\rangle$ and the other endpoint different from $\langle y\rangle$ (respectively at $\langle z\rangle$). From the relation between potentials and hitting probabilities (see (2.3)) we see that $\mathbb{P}_*\{X_{*\cdot}$ reaches ∞ before $\langle 0\rangle \mid X_{*0} = \langle x\rangle\} = V_*(\langle x\rangle)$, where V_* is the potential in the network t_* when $\langle 0\rangle$ (respectively ∞) is given potential zero (respectively one). But the resistance between $\langle x\rangle$ and $\langle 0\rangle$ in t_* equals $\rho(x)$, and the resistance between $\langle x\rangle$ and ∞ equals $r(t(x))$. Standard computations based on Kirchhoff's laws now show $V_*(\langle x\rangle) = (\rho(x) + r(t(x))^{-1}\rho(x)$, or equivalently

$$\text{(2.34)} \qquad \mathbb{P}_*\{X_{*\cdot} \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_{*0} = \langle x\rangle\} = \frac{\rho(x)}{\rho(x) + r(t(x))}.$$

---

[3]Actually this is an abuse of terminology since we may have multiple edges between a pair $\langle \widehat{y}\rangle$ and $\langle \widehat{z}\rangle$.

The right hand sides of (2.33) and (2.34) are the same. On the other hand, t_* is formed by removing from t all descendants of $\langle i_1, \dots, i_k \rangle$ other than $\langle i_1, \dots, i_{k+1} \rangle$ for $k = 0, 1, \dots, n-1$. Now consider the successive times ν_i at which $X_.$ visits t_* at a different point than at the last visit to t_*. Formally, $\nu_0 = 0$,

$$\nu_{i+1} = \inf\{\nu > \nu_i : X_\nu \in t_*, X_\nu \neq X_{\nu_i}\}.$$

If $X_{\nu_j} = \langle i_1, \dots, i_k \rangle \in t_*$ with $k < n$, then $X_.$ may make several excursions into $\bigcup_{j \neq i_{k+1}} t(i_1, \dots, i_k, j)$ before it visits a point $\langle x \rangle \in t_*$ which differs from $\langle i_1, \dots, i_k \rangle$. However, $X_.$ can reach such a vertex in t_* in one step only from $\langle i_1, \dots, i_k \rangle$. It must first return from an excursion into $t(i_1, \dots, i_k, j)$, $j \neq i_{k+1}$, to $\langle i_1, \dots, i_k \rangle$ before it can reach $\langle i_1, \dots, i_{k-1} \rangle$ or $\langle i_1, \dots, i_{k+1} \rangle$. From this it is not hard to see that, given $X_0 = \langle x \rangle \in t_*$ and $\nu_l < \infty$, the distribution[4] of $X_0, X_{\nu_1}, \dots, X_{\nu_l}$ is the same as that of $X_{*0}, X_{*1}, \dots, X_{*l}$.[5] From this it is practically obvious that

$$\begin{aligned}
\mathbb{P}\{X_. &\text{ reaches } \langle 0 \rangle \text{ at some time} \mid X_0 = \langle x \rangle\} \\
&= \mathbb{P}\{X_{\nu_i} = \langle 0 \rangle \text{ for some } i \mid X_0 = \langle x \rangle\} \\
&\leq \mathbb{P}_*\{X_{*\nu} = \langle 0 \rangle \text{ at some time} \mid X_{*0} = \langle x \rangle\}.
\end{aligned}$$

Consequently

$$\mathbb{P}\{X_. \text{ never reaches } \langle 0 \rangle \mid X_0 = \langle x \rangle\} \geq \mathbb{P}_*\{X_{*.} \text{ never reaches } \langle 0 \rangle \mid X_{*0} = \langle x \rangle\}.$$

In view of (2.34) this is equivalent to (2.33). □

We now prove Proposition 1 in two lemmas, separating the cases $\gamma F(0) \leq 1$ and $\gamma F(0) > 1$,

Lemma 4. *If*

$$\gamma F(0) \leq 1,$$

then (2.12) holds.

Proof. First define $\widetilde{Z}_n$ and $\widetilde{T}$ as in Lemma 2 with $K = 0$. Thus $\widetilde{Z}_n$ is the branching process of nodes which are connected to $\langle 0 \rangle$ by paths of zero resistance. As in (2.27), the mean number of offspring per individual for this branching process is $\gamma F(0)$. But now $\gamma F(0) \leq 1$ so that the $\widetilde{Z}_n$ process dies out eventually w.p.1. (see Harris (1963) Theorem I.6.1). This means that a.s. there is no infinite path in T — the family tree of Z_n — all of whose edges have zero resistance.

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[4]Correction added in 2001: This sentence should not assert equidistribution, but instead that $\mathbb{P}(X_{\nu_i} = \langle x_i \rangle,\ 0 \leq i \leq l \mid X_0 = \langle x \rangle) \leq \mathbb{P}_*(X_{*i} = \langle x_i \rangle,\ 0 \leq i \leq l \mid X_{*0} = \langle x \rangle)$.

[5]$\{X_{*\nu}\}$ is almost the imbedded chain of $\{X_\nu\}$ on t_*, but only observed at times when the state changes.

For $\langle x\rangle = \langle i_1,\dots,i_n\rangle \in T$ define $\rho(x)$ as in (2.31) (with t replaced by T). Recall that the superscript in $R^\epsilon(\cdot)$ indicates that the resistances of all edges have been increased by ϵ. Now set

$$R^+(T(i_1,\dots,i_n)) = \lim_{\epsilon\downarrow 0} R^\epsilon(T(i_1,\dots,i_n)).$$

This limit exists by the monotonicity property (2.6). The principal estimate which we need is as follows: with probability 1, for each infinite sequence $\{i_k\}_{k\ge 1}$ such that $\langle i_1,\dots,i_n\rangle \in T$ for all n one has

$$\liminf_{n\to\infty} \frac{R^+(T(i_1,\dots,i_n))}{\rho(\langle i_1,\dots,i_n\rangle)} = 0. \tag{2.35}$$

Write $e(i_1,\dots,i_n)$ for the edge between $\langle i_1,\dots,i_{n-1}\rangle$ and $\langle i_1,\dots,i_n\rangle$ if $n>1$, and $e(i_1)$ for the edge between $\langle 0\rangle$ and $\langle i_1\rangle$. (2.35) is fairly easy for paths in T which satisfy

$$\sum_{k=1}^{\infty} R(e(i_1,\dots,i_k)) = \infty. \tag{2.36}$$

For such a path we first choose $\epsilon_0>0$ such that (2.11) holds. By Lemma 2 we can then find an $L<\infty$ such that

$$\mathbb{P}\{R^1(T) > L\} \le q+\epsilon_0.$$

Then

$$\begin{aligned}\mathbb{P}\{R^1(T^j) > 2L \mid \langle j\rangle \in T_1\} &\le \mathbb{P}\{R^1(e(j)) > L\} + \mathbb{P}\{R^1(T(j)) > L \mid \langle j\rangle \in T_1\}\\ &\le \mathbb{P}\{R^1(e(1)) > L\} + q + \epsilon_0,\end{aligned}$$

since $R^1(T^j) = R^1(e(j)) + R^1(T(j))$ whenever $\langle j\rangle\in T_1$ (compare (2.23)). Without loss of generality we can therefore choose L large enough such that

$$\mathbb{P}\{R^1(T^j) > 2L \mid \langle j\rangle \in T_1\} \le q+2\varepsilon_0. \tag{2.37}$$

We now apply Lemma 1 with the following choice for $\mathcal{P}$: T has property $\mathcal{P}$ if and only if $R^1(T)\le 2L$. Then by (2.15), w.p.1, for each infinite path $\langle i_1, i_2,\dots\rangle$ in T there exist infinitely many n and $j_{n+1}\ne i_{n+1}$ with

$$R^1(T^{j_{n+1}}(i_1,\dots,i_n)) \le 2L. \tag{2.38}$$

but $T^{j_{n+1}}(i_1,\dots,i_n)$ is a subtree of $T(i_1,\dots,i_n)$ so that (2.38) and the monotonicity property (2.6) imply

$$R^1(T(i_1,\dots,i_n)) \le R^1(T^{j_{n+1}}(i_1,\dots,i_n)) \le 2L.$$

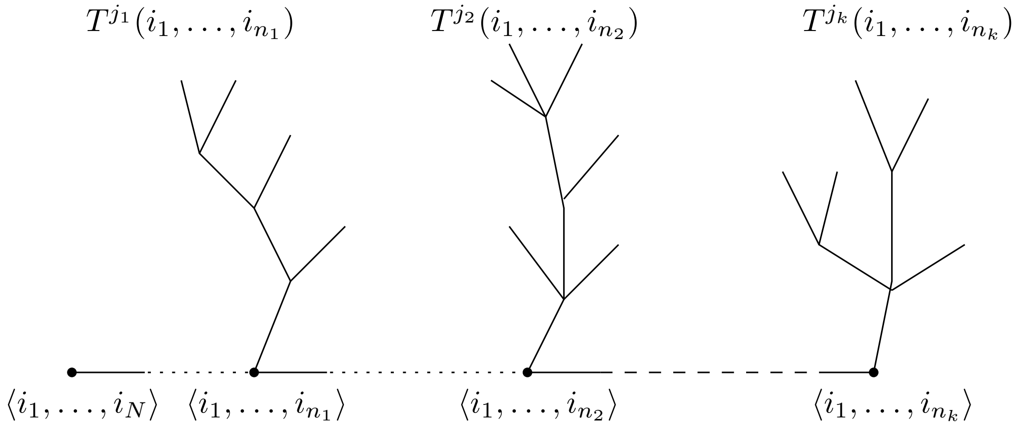

Figure 5.

Moreover, by virtue of (2.36)

$$\rho(\langle i_1,\dots,i_n\rangle)=\sum_{k=1}^{n}R(e(i_1,\dots,i_k))\to\infty,$$

so that (2.35) holds under the condition (2.36) (note that $R^+\le R^1$ by (2.6) again).

If (2.36) fails for some path $i_1,i_2,\dots,$ then

$$\sum_{k=1}^{\infty}R(e(i_1,\dots,i_k))<\infty \tag{2.39}$$

for this path. Such paths can actually have positive probability; see Bramson (1978). In this case we can find a.s. for each $\delta>0$ an N such that

$$\sum_{k=N+1}^{\infty}R(e(i_1,\dots,i_k))\le\delta\rho(\langle i_1,\dots,i_N\rangle)=\delta\sum_{k=1}^{N}R(e(i_1,\dots,i_k)) \tag{2.40}$$

(since a.s. the last sum is strictly positive for large N by the first paragraph of the proof). Also, by the preceding argument we can find a.s. $N<n_1<n_2<\cdots$ and $j_l\ne i_{n_l+1}$ such that

$$R^1(T^{j_l}(i_1,\dots,i_{n_l}))\le 2L. \tag{2.41}$$

In this case $T(i_1,\dots,i_N)$ contains the tree consisting of the path $\langle i_1,\dots,i_N\rangle$, $\langle i_1,\dots,i_{N+1}\rangle,\dots,\langle i_1,\dots,i_{n_k}\rangle$ together with the trees $T^{j_l}(i_1,\dots,i_{n_l})$, $l=1,\dots,k$ which are attached to this path. A schematic diagram of this graph is given in Figure 5. Since each $T^{j_l}(i_1,\dots,i_{n_l})$ satisfies (2.41) we also have

$$R^+(T^{j_l}(i_1,\dots,i_{n_l}))\le 2L,\quad j=1,\dots,k.$$

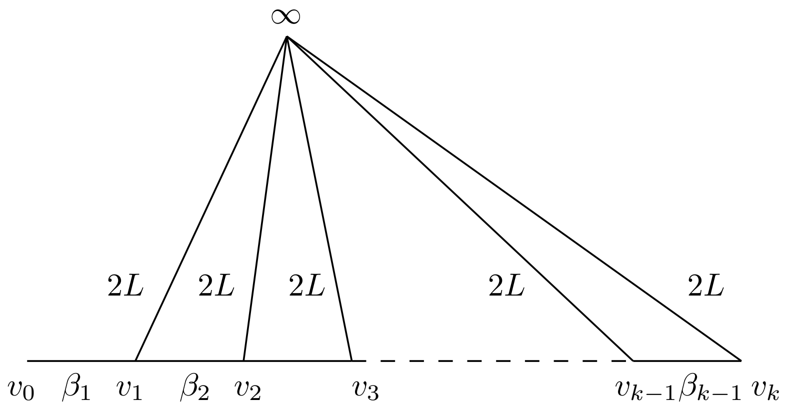

Figure 6. The number next to an edge gives the resistance of that edge.

Therefore $R^+(T(i_1,\dots,i_N))$ is at most equal to the resistance between v_0 and ∞ in the network of Figure 6, where the resistance of the edge between v_i and ∞ is $2L$, and the edge between v_{l-1} and v_l has resistance

$$\beta_l := \sum_{r=n_{l-1}+1}^{n_l} R(e(i_1,\dots,i_r)) \quad (\text{with } n_0 = N).$$

A simple inductive argument (add a vertex on the left and a resistance $2L$ between v_0 and ∞) shows that the resistance between v_0 and ∞ is at most

$$\text{(2.42)} \qquad \begin{aligned} \beta_1 + \cdots + \beta_k + \frac{2L}{k} &\le \sum_{r=N+1}^{\infty} R(e(i_1,\dots,i_r)) + \frac{2L}{k} \\ &\le \delta\rho(\langle i_1,\dots,i_N\rangle) + \frac{2L}{k} \qquad (\text{see (2.40)}). \end{aligned}$$

Since this estimate can be proved for each k, it follows that for each $\delta > 0$ we can find an N with

$$R^+(T(i_1,\dots,i_N)) \le 2\delta\rho(\langle i_1,\dots,i_N\rangle).$$

This proves (2.35) in all cases.

We now prove (2.12) from (2.35). Fix $\delta > 0$. We claim that there exists a finite collection $\mathcal{C} = \mathcal{C}(\delta)$ of vertices of T with the following properties:

(2.43) Each $\langle i_1,\dots,i_n\rangle \in \mathcal{C}$ satisfies: $T(i_1,\dots,i_n)$ is infinite,
$\rho(\langle i_1,\dots,i_n\rangle) > 0$ and
$R^+(T(i_1,\dots,i_n)) \le \delta\rho(\langle i_1,\dots,i_n\rangle)$.

(2.44) Any path from $\langle 0\rangle$ to ∞ in T contains some vertex in $\mathcal{C}$.

Note that the first paragraph of the proof and (2.35) show that w.p.1 each infinite path $\langle 0, i_1, i_2, \dots\rangle$ in T contains a first vertex $\langle i_1,\dots,i_n\rangle$ with the properties listed in (2.43). Take for $\mathcal{C}$ the collection of all these first vertices $\langle i_1, \dots, i_n\rangle$ obtainable in this way as $\langle 0, i_1, i_2, \dots\rangle$ varies over the infinite paths in T. This $\mathcal{C}$ has the properties (2.43), (2.44) and we merely have to verify that $\mathcal{C}$ is finite. But, if $\mathcal{C}$ were infinite, then there would have to exist an infinite sequence of infinite paths $\langle 0, i_1^k, i_2^k, \dots\rangle \in T$, $k = 1, 2, \dots$, such that, for all $j \le k$, one of the properties listed in (2.43) fails for $\langle i_1^k, \dots, i_j^k\rangle$. By a diagonal selection we could then find an infinite path $\langle 0, i_1, i_2, \dots\rangle \in T$ such that, for each j, $\langle i_1, \dots, i_j\rangle$ lacks one of the properties in (2.43). Since we already saw that no such sequence exists $\mathcal{C}$ must be finite.

Since $\mathcal{C}$ is finite we can find an $\varepsilon_1 > 0$ such that for each $\varepsilon \le \varepsilon_1$ and each $\langle i_1, \dots, i_n\rangle \in \mathcal{C}$

$$R^\varepsilon(T(i_1, \dots, i_n)) \le 2\delta\rho(\langle i_1, \dots, i_n\rangle) \le 2\delta\rho^\varepsilon(\langle i_1, \dots, i_n\rangle). \tag{2.45}$$

Fix any $\varepsilon \le \varepsilon_1$ and let X_ν be the Markov chain defined in Lemma 3 for $t = T$ and $r(e) = R^\varepsilon(e)$. Thus all equivalence classes consist of a single vertex now. Then (see (2.4))

$$\{R^\varepsilon(T)\}^{-1} = \sum_{i \in T_1} \frac{V(\langle i\rangle)}{R^\varepsilon(e(i))},$$

where $V(\langle i\rangle)$ is the potential of $\langle i\rangle$ when $\langle 0\rangle$ is given the potential zero and[6] ∞ is given potential one (by means of an external voltage source). Note that $\langle 0\rangle \notin \mathcal{C}$ by construction. It follows therefore from (2.44) that if $X_0 = \langle i\rangle \in T_1$, then X_ν cannot reach ∞ without passing through $\mathcal{C}$. Consequently,

$$\begin{aligned} V(\langle i\rangle) &= \mathbb{P}\{X_\cdot \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\} \\ &= \sum_{\langle x\rangle \in \mathcal{C}} \mathbb{P}\{X_\cdot \text{ reaches } \mathcal{C} \text{ before } \langle 0\rangle \text{ and hits } \mathcal{C} \text{ first in } \langle x\rangle \mid X_0 = \langle i\rangle\} \\ &\qquad\qquad \times \mathbb{P}\{X_\cdot \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle x\rangle\}. \end{aligned}$$

By Lemma 3 and (2.45) we therefore have

$$\begin{aligned} \{R^\varepsilon(T)\}^{-1} &\ge \frac{1}{1+2\delta} \sum_{i \in T_1} \frac{1}{R^\varepsilon(e(i))} \\ &\quad \times \sum_{\langle x\rangle \in \mathcal{C}} \mathbb{P}\{X_\cdot \text{ reaches } \mathcal{C} \text{ before } \langle 0\rangle \text{ and hits } \mathcal{C} \text{ first in } \langle x\rangle \mid X_0 = \langle i\rangle\}. \end{aligned} \tag{2.46}$$

Let $\overline{T}$ be the finite subtree of T which contains $\langle 0\rangle$ and all vertices $\langle i_1, \dots, i_k\rangle$ which have no predecessor $\langle i_1, \dots, i_l\rangle$ with $l < k$ belonging to $\mathcal{C}$. Then it follows again by (2.3) and (2.4) that the double sum in the right hand side of (2.46) is

[6] As with $R^\varepsilon(T)$ this has to be interpreted as a statement about a limit. Give 0 potential 0 and T_N potential 1 and let $N \to \infty$. The potential $V(\langle i\rangle)$ is decreasing in N and hence has a limit, as one can see for instance from (2.3).

just the reciprocal of the resistance between $\langle 0\rangle$ and $\mathcal{C}$ in $\overline{T}$, still when each edge e is given the resistance $R^{\varepsilon}(e)$. However, $\overline{T}$ is a finite network, so that resistances in $\overline{T}$ are continuous in ε as ε tends to zero. This finally gives

$$\begin{aligned} R^+(T) = \lim_{\varepsilon\downarrow 0} R^{\varepsilon}(T) &\le (1+2\delta)\times\{\text{resistance between } \langle 0\rangle \text{ and } \mathcal{C} \text{ in } \overline{T}\} \\ &\le (1+2\delta)R(T). \end{aligned}$$

The last inequality is again an easy consequence of the monotonicity property (2.6). Since, also by (2.6), $R^+(T)\ge R(T)$ we now obtain (2.12) by letting $\delta\downarrow 0$. □

Lemma 5. *If*

$$\gamma F(0) > 1$$

then (2.12) holds.

Proof. The proof of (2.35) remains valid for any path $i_1, i_2, \dots$ with

$$\sum_{k=1}^{\infty} R(e(i_1,\dots,i_k)) = \lim_{n\to\infty}\rho(\langle i_1,\dots,i_n\rangle) > 0. \tag{2.47}$$

Consequently, the proof of Lemma 4 goes through unchanged on any realization T of the family tree and its resistances for which (2.47) holds for all infinite paths in the tree, or equivalently for any realization in which the equivalence class $\widehat{\langle 0\rangle}$ is finite. The only new complication is that for $\gamma F(0)>1$ the branching process $\widetilde{Z}_n$ of Lemma 4 is supercritical and has a positive probability of never dying out (Harris (1963), Theorem I.6.1). If $\widetilde{Z}_n$ does not die out, then $\widehat{\langle 0\rangle}$ is infinite. In this case $\langle 0\rangle$ is connected to ∞ by a path of zero resistance and $R(T)=0$. We merely have to show that also $R^+(T)=0$ a.s. on the set of realizations which contain a path of zero resistance between $\langle 0\rangle$ and ∞. This, however, follows also by the arguments of Lemma 4. With probability one for any path $i_1, i_2, \dots$ of zero resistance there exist infinitely many n and $j_{n+1}\ne i_{n+1}$ for which (2.38) holds. The middle expression in (2.42) with $N=0$ is therefore still an upper bound for $R^+(T)$ whenever such a path $i_1, i_2, \dots$ of zero resistance exists. But this says $R^+(T)\le 2L/k$ for any k or $R^+(T)=0$ a.s. on the set of trees for which (2.47) fails for some path. Thus the lemma holds in all cases. □

Lemmas 4 and 5 together prove Proposition 1.

We shall need an improved version of Lemma 3 for the case where the resistance assigned to e is at least $\varepsilon>0$. (In particular this will hold if we assign the value $R^{\varepsilon}(e)$ to e.) In this situation the equivalence classes considered in Lemma 3 all consist of one vertex only. For each realization t and $r(\cdot)$ of T and its resistances we can therefore consider the Markov chain $\{X_\nu^m\}_{\nu\ge 0} = \{X_\nu^m(t_{[m]}, r(\cdot))\}$

with state space the vertices in $t_{[m]}$ and transition probability matrix
(2.48)

$$P(\langle y\rangle, \langle z\rangle) = \begin{cases} \left\{\sum_y \frac{1}{r(e)}\right\}^{-1} \frac{1}{r(y,z)} & \text{if } \langle y\rangle \text{ and } \langle z\rangle \text{ are adjacent in } t_{[m]}, \\ 0 & \text{otherwise.} \end{cases}$$

Here $r(y,z)$ is the resistance of the edge between y and z, and $\sum_y$ is the sum over all edges e in $t_{[m]}$ which are incident to y. For $s < m$ let

$$\mathcal{A}_s = \{\langle y\rangle \in t_s : \langle y\rangle \text{ has at least one descendant in } t_m, \text{ the } m\text{th generation of } t_{[m]}\}.$$

Further, for $\langle x\rangle \in t_{s-1}$ set

$$\pi(x, t_{[m]}, r, s) = \mathbb{P}\{X^m_\cdot \text{ reaches } \langle 0\rangle \text{ before it reaches } \mathcal{A}_s \mid X^m_0 = \langle x\rangle\}$$

and

$$\Pi(t_{[m]}, r) = \Pi(t_{[m]}, r, s) = \max_{\langle x\rangle \in t_{s-1}} \pi(x, t_{[m]}, r, s). \tag{2.49}$$

Thus, Π measures the probability for X^m_ν to go from t_{s-1} through $t_{[m]}$ to the root of $t_{[m]}$ without passing through $\mathcal{A}_s$. When $t_{[m]}$ and r are taken random again, then Π is also a random variable.

Lemma 6. *If $R(e) \geq \varepsilon$ w.p.1, then there exist constants $0 < C_1, C_2, L < \infty$ (independent of ε, s and m) such that*

$$\mathbb{P}\left\{\mathcal{A}_s \neq \emptyset \text{ and } \Pi(T_{[m]}, R, s) > \left(\frac{2L}{2L+\varepsilon}\right)^{C_1 s}\right\} \leq e^{-C_2 s} \quad \text{for } s \geq 2. \tag{2.50}$$

Proof. Let $\langle x\rangle = \langle i_1, \dots, i_{s-1}\rangle \in t_{s-1}$. Then, if $X^m_0 = \langle x\rangle$, X^m_ν cannot reach $\langle 0\rangle$ without passing through $\langle i_1, \dots, i_k\rangle$ for each $1 \leq k \leq s-1$. Let τ_k be the first time $X^m_\cdot$ reaches $\langle i_1, \dots, i_k\rangle$ ($\tau_{s-1} = 0$). Then for $\tau_k \leq \nu < \tau_{k-1}$, X_ν takes only values in the collection of descendants of $\langle i_1, \dots, i_{k-1}\rangle$ in $T_{[m]}$ (this is the collection of vertices $\langle i_1, \dots, i_{k-1}, j_k, \dots, j_l\rangle$, $k \leq l \leq m$). These are vertices of $T(\langle i_1, \dots, i_{k-1}\rangle)$, and therefore between τ_k and τ_{k-1}, X^m_ν is a Markov chain on $T(\langle i_1, \dots, i_{k-1}\rangle)$. In fact, until $X^m_\cdot$ reaches T_m for the first time we can view $\{X^m_\nu\}$ as a realization of the Markov chain $\{X_\nu\}$ of Lemma 3. (The equivalence classes $\langle\widehat{x}\rangle$ now consist of single points $\langle x\rangle$ only.) Observe now that if $X^m_\cdot$ or $X_\cdot$ starts from some vertex $\langle i_1, \dots, i_k\rangle \in T(i_1, \dots, i_{k-1})$ with $k < s < m$, then neither

$X^m_\cdot$ nor $X_\cdot$ can reach T_m (and a fortiori neither can reach ∞) without passing through some point of $\mathcal{A}_s$. Therefore

(2.51)

$$\begin{aligned}&\mathbb{P}\{X^m_\cdot \text{ does not reach } \mathcal{A}_s \text{ between } \tau_k \text{ and } \tau_{k-1} \mid \tau_k, X^m_0, X^m_1, \dots, X^m_{\tau_k}\}\\ &\qquad = \mathbb{P}\{X_\cdot \text{ reaches } \langle i_1, \dots, i_{k-1}\rangle \text{ before it reaches } \mathcal{A}_s \mid X_0 = \langle i_1, \dots, i_k\rangle\}\\ &\qquad \le \mathbb{P}\{X_\cdot \text{ reaches } \langle i_1, \dots, i_{k-1}\rangle \text{ before it reaches } \infty\\ &\qquad\qquad\qquad \text{in } T(i_1, \dots, i_{k-1}) \mid X_0 = \langle i_1, \dots, i_k\rangle\}.\end{aligned}$$

As in (2.34) the last probability equals

$$\frac{R(T(i_1, \dots, i_k))}{R(e(i_1, \dots, i_k)) + R(T(i_1, \dots, i_k))} \tag{2.52}$$

($e(i_1, \dots, i_k)$ again denotes the edge between $\langle i_1, \dots, i_{k-1}\rangle$ and $\langle i_1, \dots, i_k\rangle$). In particular, if there exists some $j_{k+1} \ne i_{k+1}$ such that

$$R(T^{j_{k+1}}(i_1, \dots, i_k)) \le 2L,$$

then (2.52) is at most $2L/(2L + \varepsilon)$. Thus, if

$$\begin{aligned}\Delta(i_1, \dots, i_s) = \Big\{&\text{number of } k \in [1, s-1] \text{ for which there exists a}\\ &j_{k+1} \ne i_{k+1} \text{ with } R(T^{j_{k+1}}(i_1, \dots, i_k)) \le 2L\Big\},\end{aligned}$$

then we obtain from the strong Markov property

$$\begin{aligned}&\mathbb{P}\{X^m_\cdot \text{ reaches } \langle 0\rangle \text{ before it reaches } \mathcal{A}_s \mid T, R, \mathcal{A}_s \ne \emptyset, X^m_0 = \langle i_1, \dots, i_{s-1}\rangle\}\\ &\quad \le \prod_{k=1}^{s-1} \mathbb{P}\{X_\cdot \text{ does not reach } \mathcal{A}_s \text{ between } \tau_k \text{ and } \tau_{k-1} \mid T, R, \mathcal{A}_s \ne \emptyset, \tau_k, X^m_0, \dots, X^m_{\tau_k}\}\\ &\quad \le \left(\frac{2L}{2L+\varepsilon}\right)^{\Delta(i_1, \dots, i_s)}.\end{aligned}$$

In view of the definition of Π this implies

$$\Pi(T_{[m]}, R, s) \le \left(\frac{2L}{2L+\varepsilon}\right)^{\Delta_s} \text{ on } \{\mathcal{A}_s \ne \emptyset\}, \tag{2.53}$$

where

$$\Delta_s = \min_{\langle i_1, \dots, i_s\rangle \in T_s} \Delta(i_1, \dots, i_s).$$

Now take L so large that

$$\mathbb{P}\{R(T^j) > 2L \mid \langle j\rangle \in T_1\} \le q + 2\varepsilon_0$$

(this can be done as shown in (2.37)) and apply Lemma 1 with $\mathcal{P}$ the property of trees t that $R(t) \le 2L$. By virtue of (2.16) there exist $0 < C_1, C_2 < \infty$ (which depend only on $\mathcal{P}$ and hence only on L) such that

$$\mathbb{P}\{\mathcal{A}_s \ne \emptyset \text{ and } \Delta_s \le C_1 s\} \le e^{-C_2 s} \quad \text{for } s \ge 2.$$

The lemma therefore follows from (2.53). $\square$

3. Proofs of Theorems 2 and 3

We begin with the proof of Theorem 2. This proof is quite similar to that of Lemma 5 in Grimmett and Kesten (1983). The result will follow fairly easily from considerations about the random subgraphs τ^0 and τ^∞ of K_{n+2} whose vertices are defined to be those which are connected to 0 and ∞ respectively, by conducting paths. We call an edge e of K_{n+2} *conducting* if $R(e) < \infty$; a *path* is called *conducting* if all its edges are conducting. According to (1.1), $\mathbb{P}\{e \text{ is conducting}\} = \gamma(n)/n$ for each $e \in K_{n+2}$, and given that e is conducting $R(e)$ has conditional distribution function F. The number of conducting edges in K_{n+2} incident to a fixed vertex v has the binomial distribution $B(n+1, \gamma(n)/n)$, ($(n+1)$ trials with success probability $\gamma(n)/n$ for each trial). Now let $\tau_0^0 = \{0\}$ (respectively $\tau_0^\infty = \{\infty\}$) and let τ_k^0 (respectively τ_k^∞) be the set of vertices of K_{n+2} which can be connected to 0 (respectively ∞) by a path of k conducting edges, but not by a shorter conducting path. Clearly the τ_k^0, $k \geq 0$, are disjoint, and each vertex in τ_k^0 is connected by one conducting edge to some vertex in τ_{k-1}^0. There may be several vertices in τ_{k-1}^0 for which this holds. However, there is never a conducting edge connecting a vertex in τ_j^0 with a vertex in τ_k^0 when $|k-j| \geq 2$. Similar statements hold for the τ_k^∞. The first lemma of this section states that the graph consisting of the vertices $\bigcup \tau_k^0$ and the conducting edges between them converges in some distributional sense as $n \to \infty$ to a family tree of a Bienaymé–Galton–Watson branching process. Let $T = T^\gamma$ be the family tree of such a branching process, whose offspring distribution is a Poisson distribution with mean γ as described in the Introduction. Also T_n and $T_{[n]}$ are as described in the Introduction. Similarly $\tau_{[k]}^0$ and $\tau_{[k]}^\infty$ are the graphs with vertex sets $\bigcup_{m=0}^k \tau_m^0$ and $\bigcup_{m=0}^k \tau_m^\infty$, respectively, and edge sets the sets of conducting edges between these vertices. For a fixed rooted labeled tree t consisting of a root ρ and k generations, the statement $\tau_{[k]}^0 = t$ means that there exists a graph isomorphism between $\tau_{[k]}^0$ and t in which 0 corresponds to ρ. A similar definition holds for $\tau_{[k]}^\infty$ or $T_{[k]}$.

Lemma 7. *If (1.1) holds and $\gamma(n) \to \gamma < \infty$ then for any fixed rooted labeled tree t of k generations*

$$\mathbb{P}\{\tau_{[k]}^0 = t\} = \mathbb{P}\{\tau_{[k]}^\infty = t\} \to \mathbb{P}\{T_{[k]}^\gamma = t\} \tag{3.1}$$

as $n \to \infty$. More generally, if t_1 and t_2 are two fixed rooted labeled trees with k generations then

$$\mathbb{P}\{\tau_{[k]}^0 \text{ is disjoint from } \tau_{[k]}^\infty\} \to 1 \text{ as } n \to \infty \tag{3.2}$$

and

$$\mathbb{P}\{\tau_{[k]}^0 = t_1, \tau_{[k]}^\infty = t_2\} \to \mathbb{P}\{T_{[k]}^\gamma = t_1\}\mathbb{P}\{T_{[k]}^\gamma = t_2\} \tag{3.3}$$

as $n \to \infty$.

Proof. We prove

$$\mathbb{P}\{\tau^0_{[k]} = t\} \to \mathbb{P}\{T^\gamma_{[k]} = t\} \tag{3.4}$$

if $\gamma(n) \to \gamma < \infty$. Since $\tau^0_{[k]}$ and $\tau^\infty_{[k]}$ clearly have the same distribution this will prove (3.1). It will be clear how to generalize the argument to obtain (3.2) and (3.3).

To prove (3.4) consider for any set A of vertices of K_{n+2} and a vertex x of K_{n+2}, the number of vertices outside A connected by a conducting edge to x. Denote this random number by $N(x, A)$. Then, under (1.1), $N(x, A)$ has a $B(n + 2 - |A \cup \{x\}|, \gamma(n)/n)$ distribution. Thus if A varies with n such that $|A_n|/n \to 0$, and $\gamma(n) \to \gamma$, then by the familiar Poisson limit for the binomial distribution

$$\mathbb{P}\{N(x, A_n) = l\} \to e^{-\gamma}\frac{\gamma^l}{l!}, \quad l = 0, 1, \ldots. \tag{3.5}$$

Now let t be a rooted labeled tree of one generation. If the size of the first generation of t equals l, then

$$\mathbb{P}\{T^\gamma_{[1]} = t\} = \mathbb{P}\{|T^\gamma_1| = l\} = e^{-\gamma}\frac{\gamma^l}{l!},$$

because $|T^\gamma_1|$ has a Poisson distribution with mean γ. Also $\tau^0_{[1]}$ consists of the root 0 and a random number of vertices connected to 0 by a single conducting edge. This number is precisely $N(0, \{0\})$, so that by (3.3)

$$\begin{aligned}\mathbb{P}\{\tau^0_{[1]} = t\} &= \mathbb{P}\{|\tau^0_1| = l\} = \mathbb{P}\{N(0, \{0\}) = l\}\\ &\to e^{\gamma}\frac{\gamma^l}{l!}.\end{aligned}$$

Thus (3.4) holds for $k = 1$. Of course it also holds for $k = 0$.

We now prove (3.4) by induction on the number of generations of t. Let t' be a rooted labeled tree of $(k + 1)$ generations. Let t be the subtree of the first k generations, and denote the vertices in the kth generation by $v_1, v_2, \ldots, v_M$. Finally, let $\nu(v_i)$ be the number of vertices in the $(k+1)$th generation of t' which are connected by an edge to v_i. (Thus in total $\nu(v_i) + 1$ edges are incident to v_i in t'.) Our induction hypothesis is that (3.4) holds for the given k and t. To prove that (3.4) also holds when k is replaced by $k + 1$ and t by t' we start with the trivial relation

$$\mathbb{P}\{T^\gamma_{[k+1]} = t'\} = \mathbb{P}\{T^\gamma_{[k]} = t\}\mathbb{P}\{T^\gamma_{[k+1]} = t' \mid T^\gamma_{[k]} = t\}.$$

If $T^\gamma_{[k]} = t$ then there exists some isomorphism I between t and $T^\gamma_{[k]}$. Denote by $\langle v_i \rangle$ the vertex of $T^\gamma_{[k]}$ which is the image of v_i under I. I can be extended to an isomorphism between t' and $T^\gamma_{[k+1]}$ if and only if $\langle v_i \rangle$ has exactly $\nu(v_i)$ children in T^γ_{k+1}, $i = 1, \ldots, M$. The latter event has probability

$$\prod_{i=1}^{M} e^{-\gamma} \frac{\gamma^{\nu(v_i)}}{\nu(v_i)!}, \tag{3.6}$$

because the numbers of children of the $\langle v_i \rangle$ are independent Poisson variables with mean γ. If $T^\gamma_{[k]} = t$ there may be several choices for I, but each one can be extended to an isomorphism of $T^\gamma_{[k+1]}$ and t' if and only if each $\langle v_i \rangle \in T^\gamma_k$ has the correct number of children (which depends on I). Denote by $\lambda(t)$ the number of distinct assignments of children to each $\langle v_i \rangle \in T^\gamma_k$ which will make $T^\gamma_{[k+1]}$ isomorphic to t'. The conditional probability, given $T^\gamma_{[k]}$, of the occurrence of a specific assignment of numbers of children is given by (3.6). This is true for each of the possible assignments, since (3.6) depends only on t, and not on I. Consequently

$$\mathbb{P}\{T^\gamma_{[k+1]} = t' \mid T^\gamma_{[k]} = t\} = \lambda(t) \prod_{i=1}^{M} e^{-\gamma} \frac{\gamma^{\nu(v_i)}}{\nu(v_i)!}. \tag{3.7}$$

We can analyze

$$\mathbb{P}\{\tau^0_{[k+1]} = t' \mid \tau^0_{[k]} = t\} \tag{3.8}$$

in a similar way. Let J be an isomorphism between t and $\tau^0_{[k]}$ and let x_i be the image in τ^0_k of v_i. Denote by A the set of vertices of K_{n+2} which belong to $\tau^0_{[k]}$. Then J can be extended to an isomorphism of t' and $\tau^0_{[k+1]}$ if and only if

(3.9) x_i is connected by a conducting edge to exactly $\nu(v_i)$ vertices of K_{n+2} outside A, but no $x \in K_{n+2} \setminus A$ is connected by conducting edges to two of the vertices $x_1, \ldots, x_M$.

Denote by B_i the collection of vertices outside A which are connected by a conducting edge to x_i. Then (3.9) is just the event

$$\{|B_i| = \nu(v_i), 1 \le i \le M, \text{ and the } B_i \text{ are disjoint}\}.$$

Thus, if we set $A_0 = A$, $A_i = A_0 \cup B_1 \cup \cdots \cup B_i$, then the conditional probability of (3.9) given $\tau^0_{[k]} = t$ can be written as

$$\begin{aligned} &\prod_{i=1}^{M} \mathbb{P}\Big\{|B_i| = \nu(v_i), B_i \subset K_{n+2} \setminus A_{i-1} \,\Big|\, \tau^0_{[k]} = t, \\ &\qquad\qquad |B_j| = \nu(v_j), 1 \le j \le i-1 \text{ and } B_1, \ldots, B_{i-1} \text{ are disjoint}\Big\}. \end{aligned} \tag{3.10}$$

Figure 7. All vertices in the dashed rectangle are shortcircuited; this illustrates the construction in the proof of Lemma 8, with $k = 3$.

Finally, knowledge of $\tau^0_{[k]}$ and $B_1, \dots, B_{i-1}$ gives no information about edges between x_i and $K_{n+2} \setminus A$. Therefore the ith factor in (3.10) equals

$$\binom{n+2-|A_{i-1}|}{\nu(v_i)} \left(\frac{\gamma(n)}{n}\right)^{\nu(v_i)} \left(1 - \frac{\gamma(n)}{n}\right)^{n+2-|A|-\nu(v_i)}. \tag{3.11}$$

As in (3.5) the limit of (3.11) as $n \to \infty$ equals

$$e^{-\gamma} \frac{\gamma^{\nu(v_i)}}{\nu(v_i)!}$$

so that (3.10) converges to (3.6). It follows that (3.8) converges to the right hand side of (3.7). This, together with the induction hypothesis implies

$$\mathbb{P}\{\tau^0_{[k+1]} = t'\} \to \mathbb{P}\{T^\gamma_{[k+1]} = t'\}.$$

This completes the induction step for the proof of (3.4). As mentioned before, the proofs of (3.2) and (3.3) follow along similar lines when $\gamma(n) \to \gamma < \infty$. □

Lemma 8. *Assume (1.1). If $\gamma(n) \to \gamma < \infty$, then*

$$\limsup_{n\to\infty} \mathbb{P}\{R_n \le x\} \le \mathbb{P}\{R'(\gamma) + R''(\gamma) \le x\}, \tag{3.12}$$

at each continuity point of the right hand side, where $R'(\gamma)$ and $R''(\gamma)$ are independent random variables, each with the distribution of $R(T^\gamma)$.

Proof. Assume that for some n and a certain realization of the resistances and fixed k, $\tau^0_{[k]}$ and $\tau^\infty_{[k]}$ are disjoint. By the monotonicity property (2.6), R_n, the resistance between 0 and ∞ in K_{n+2}, is then at least equal to the resistance between 0 and ∞ in the network consisting of $\tau^0_{[k]} \cup \tau^\infty_{[k]}$ and shortcircuits between all pairs of vertices $v, w \in K_{n+2} \setminus \tau^0_{[k-1]} \cup \tau^\infty_{[k-1]}$. Indeed the resistance of each edge in this network is less than or equal to the resistance assigned to it in the original network on K_{n+2}, since by the definition of τ^0 and τ^∞ there are no conducting edges in the original network between $\tau^0_{[k-1]} \cup \tau^\infty_{[k-1]}$ and $K_{n+2} \setminus \tau^0_{[k-1]} \cup \tau^\infty_{[k-1]}$, except for the edges between τ^0_{k-1} and τ^0_k and between τ^∞_{k-1} and τ^∞_k.

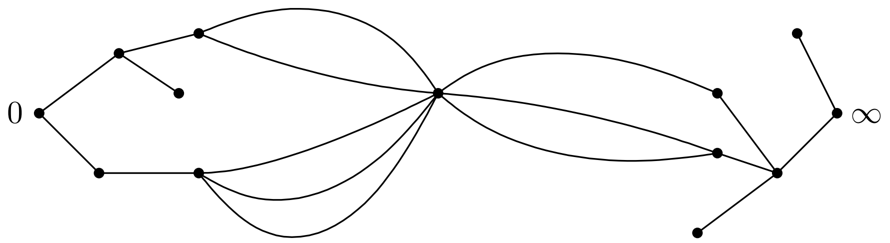

Figure 8. The network of Figure 7 is equivalent to the above network.

Now the network with the shortcircuits inserted is equivalent to $\tau^0_{[k]}$ and $\tau^\infty_{[k]}$ in series, after all vertices in $\tau^0_k \cup \tau^\infty_k$ are identified as a single vertex (see Figure 7 for $k = 3$ and Figure 8).

Denote by r^0_k the resistance between 0 and τ^0_k in $\tau^0_{[k]}$ when all vertices in τ^0_k are identified (or shortcircuited). Define r^∞_k similarly by replacing 0 by ∞. Then the resistance between 0 and ∞ in the network with short circuits is $r^0_k + r^\infty_k$. By Lemma 7 the probability that $\tau^0_{[k]}$ and $\tau^\infty_{[k]}$ are disjoint tends to 1, while $\tau^0_{[k]}$ and $\tau^\infty_{[k]}$ converge in distribution to two independent trees with the distribution of $T^\gamma_{[k]}$. Moreover, the fact that an edge belongs to $\tau^0_{[k]} \cup \tau^\infty_{[k]}$ says no more about its resistance than that this resistance is finite. Thus the resistance of each of the edges of $\tau^0_{[k]} \cup \tau^\infty_{[k]}$ has the distribution function F. In addition these resistances are independent. Therefore (r^0_k, r^∞_k) converges in distribution to $(R'_k(\gamma)), R''_k(\gamma))$, where R'_k, R''_k are independent, each with the distribution of

$$R(T^\gamma_{[k]}) = \{\text{resistance between } \langle 0\rangle \text{ and } T^\gamma_k \text{ in } T^\gamma_{[k]}\}.$$

In view of the above

$$\limsup \mathbb{P}\{R_n \le x\} \le \limsup \mathbb{P}\{r^0_k + r^\infty_k \le x\} = \mathbb{P}\{R'_k(\gamma) + R''_k(\gamma) \le x\},$$

at each continuity point of the last member. But $R(T^\gamma) = \lim_{k\to\infty} R(T^\gamma_{[k]})$ by definition, so that (3.12) follows. □

Note that Lemma 8 proves "one half" of Theorem 3. It shows that R_n is asymptotically at least as large in distribution as $R'(\gamma) + R''(\gamma)$. Also, a much simplified version of the proof of Lemma 8 implies Theorem 2, as we now show.

Proof of Theorem 2. $R_n = \infty$ if 0 is not connected to ∞ by a conducting path. Thus, by Lemma 7,

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{P}\{R_n = \infty\} &\ge \liminf_{n\to\infty} \mathbb{P}\{\text{for some } k < \infty, \tau^0_k = \emptyset \text{ and } \infty \notin \tau^0_{[k]}\}\\ &\ge \mathbb{P}\{T^\gamma_k = \emptyset \text{ for some } k\}.\end{aligned}$$

However, the last probability is just the extinction probability of the branching process $\{Z_n\}$, and this probability equals 1 if $\gamma \le 1$ (see Harris (1963) Theorem I.6.1). □

To obtain an upper bound for R_n in Theorem 3 we need to show that $\tau^0_{[k]}$ is close in distribution to $T^{\gamma}_{[k]}$ not only for fixed k and large n, but even for k a suitable multiple of $\log n$. Since we only want an upper bound for R_n, it suffices (as we shall see) to show that for k a suitable multiple of $\log n$ and for fixed $\delta < \gamma$, $\tau^0_{[k]}$ is stochastically larger than $T^{\delta}_{[k]}$. We shall do this by "coupling". For the remainder of these notes we assume (1.4) and take $1 < \delta < \gamma$. We shall construct on one probability space $\tau^0_{[k]}$ and two other graphs $\widetilde{\tau}^0_{[k]}$ and $\widetilde{T}_{[k]} = \widetilde{T}^{\delta}_{[k]}$, such that $\widetilde{T}_{[k]}$ has the same distribution as $T^{\delta}_{[k]}$ and such that $\widetilde{\tau}^0_{[k]}$ and $\widetilde{T}_{[k]}$ are trees with root at 0 and such that with high probability

$$\widetilde{T}_{[k]} \subset \widetilde{\tau}^0_{[k]} \subset \tau^0_{[k]}. \tag{3.13}$$

$\tau^0_{[k]}$ has already been constructed in the beginning of this section. We construct $\widetilde{\tau}^0_{[\cdot]}$ as a subgraph of $K_{n+2} \setminus \{\infty\}$ in stages. We set $B_0 = \{0\} = \widetilde{\tau}^0_0$. $\widetilde{\tau}^0_{[0]}$ is the graph which consists of the vertex 0 only. At stage l, $\widetilde{\tau}^0_{[j]}$ will have been constructed for $j \le l$, such that these graphs are an increasing family of trees in $K_{n+2} \setminus \{\infty\}$. Let B_j denote the set of vertices of $\widetilde{\tau}^0_{[j]}$, and let $i_1 < \cdots < i_r$ be the vertices in $B_l \setminus B_{l-1}$. (If $l \ge 1$, then $i_1, \dots, i_r \in \{1, \dots, n\}$.) We then construct $\widetilde{\tau}^0_{[l+1]}$ by choosing disjoint subsets of $\{1, 2, \dots, n\} \setminus B_l$ for the vertices which will be connected by an edge in $\widetilde{\tau}^0_{[l+1]}$ to $i_1, \dots, i_r$. These choices too are made successively. Let $C_l(p)$ denote the union of B_l and all vertices of $\{1, \dots, n\}$ which have been chosen already to be connected to $i_1, i_2, \dots, i_{p-1}$. Thus $C_l(1) = B_l$. We now add to the vertex set of $\widetilde{\tau}^0_{[l+1]}$ all vertices in $\{1, \dots, n\} \setminus C_l(p)$ which are connected by a *conducting* edge to i_p. The edges between these vertices and i_p are added to the edge set of $\widetilde{\tau}^0_{[l+1]}$. $\widetilde{\tau}^0_{[l+1]}$ is the graph obtained after all these additions have been performed for $i_1, \dots, i_r$. It is clear from the construction that $\widetilde{\tau}^0_{[l]}$ is a tree for each l, since each time we only add vertices which have not been used before and only one edge between each new vertex and the old vertices. Comparison with the construction of $\tau^0_{[l]}$ also shows immediately that $\widetilde{\tau}^0_{[l]}$ is a subgraph of $\tau^0_{[l]}$.

To construct $\widetilde{T}_{[\cdot]}$ it is convenient to view K_{n+2} as a subgraph of K_∞, which is the complete graph with vertices $0, \infty, 1, 2, \dots$. (Recall that K_{n+2} has vertices $0, \infty, 1, 2, \dots, n$.) We shall use an auxiliary family of random variables $\{Y(j), U(j) : j \ge 0\}$, with each Y a Poisson variable with mean δ ($1 < \delta < \gamma$ is a fixed number) and each U uniformly distributed on $[0, 1]$. All these variables are taken independent of each other and independent of all resistances and of all $\widetilde{\tau}^0_{[l]}$. Again we construct $\widetilde{T}_{[k]}$ in stages. Set $D_0 = \{0\} = \widetilde{T}_0$ and take for $\widetilde{T}_{[0]}$ the graph consisting of the vertex 0 only. Assume we have already chosen $\widetilde{T}_{[j]}$, $j \le l$ as subgraphs of K_∞ such that for each $j \le l$ each vertex of $\widetilde{T}_{[j]}$ either is a vertex of $\widetilde{\tau}^0_{[j]}$, or belongs to $\{n+1, n+2, \dots\} \subset K_\infty \setminus K_{n+2}$. We then choose $\widetilde{T}_{l+1}$ as follows. Let D_l be the vertex set of $\widetilde{T}_{[l]}$, and let $j_1 < j_2 < \cdots < j_s$ be the vertices in $D_l \setminus D_{l-1}$. Again we add successively disjoint sets of vertices and connect them by edges to $j_1, j_2, \dots, j_s$, respectively, to form $\widetilde{T}_{[l+1]}$. Denote by $E_l(p)$ the union of D_l and all the vertices which have already been connected to $j_1, \dots, j_{p-1}$; $E_l(1) = D_l$. We now choose the vertices connected to j_p. First consider the case where $j_p \le n$. By our inductive assumption j_p is then a vertex of $\widetilde{\tau}^0_{[l]}$, since it belongs to $\widetilde{T}_{[l]}$ as well as to $\{1, \dots, n\}$. Let $j_p = i_\nu$ and let $r_1 < r_2 < \dots < r_q$ be the vertices of $\widetilde{\tau}^0_{[l+1]}$ which are connected by an edge of $\widetilde{\tau}^0_{[l+1]}$ to $i_\nu = j_p$. By construction all $r_i \in \{1, \dots, n\} \setminus C_l(\nu)$. Note that $q = 0$ is possible, so that there may not be any vertices of this kind. Put $\beta(-1) = 0$ and for $x \ge 0$

$$\begin{aligned}(3.14)\quad \beta(x) &= \beta(x; n, |C_l(\nu)|)\\ &= \sum_{j \le x} \binom{n+1-|C_l(\nu)|}{j} \left(\frac{\gamma(n)}{n}\right)^j \left(1 - \frac{\gamma(n)}{n}\right)^{n+1-|C_l(\nu)|-j}\end{aligned}$$

and

$$\pi(x) = \pi(x; \delta) = \sum_{j \le x} e^{-\delta} \frac{\delta^j}{j!}.$$

With U the previously chosen uniform random variable we take

$$\begin{aligned}(3.15)\qquad & V(j_p) = \beta(q-1) + U(j_p)[\beta(q) - \beta(q-1)],\\ & u = \pi^{-1}(V(j_p)) := \{\text{smallest integer } m \text{ for which } \pi(m) \ge V(j_p)\}.\end{aligned}$$

If $u \le q$, then we add to $\widetilde{T}_{[l+1]}$ the vertices $r_1, \dots, r_u$ plus the edges between $r_1, \dots, r_u$ and $j_p = i_\nu$ (all these edges are also edges of $\widetilde{\tau}^0_{[l+1]}$). If $u > q$, then we add $r_1, \dots, r_q$ plus the edges between these and j_p, and in addition choose $u - q$ vertices from $\{n+1, n+2, \dots\} \setminus E_l(p)$ and add these vertices also to $\widetilde{T}_{[l+1]}$, together with an edge from each of them to j_p. Thus in each case u new vertices are connected to j_p in $\widetilde{T}_{[l+1]}$. Finally, if $j_p > n$, then we add $Y(j_p)$ vertices from $\{n+1, n+2, \dots\} \setminus E_l(p)$ to $\widetilde{T}_{[l+1]}$ and an edge from each of these vertices to j_p ($Y(\cdot)$ is the previously chosen Poisson variable). $\widetilde{T}_{[l+1]}$ is the graph obtained when the above construction is completed for all $j_1, \dots, j_s$.

Lemma 9. *Let $1 < \delta < \gamma$ be fixed and let*[7]

$$(3.16)\qquad m = m_n = \left\lfloor \frac{3}{4} \frac{\log n}{\log \gamma} \right\rfloor.$$

Then $\widetilde{T}^\delta_{[m]}$ is a tree with the same distribution as $T^\delta_{[m]}$. Also $\widetilde{\tau}^0_{[m]}$ is a tree, and as $n \to \infty$,

$$(3.17)\qquad \mathbb{P}\{\widetilde{T}^\delta_{[m]} \subset \widetilde{\tau}^0_{[m]} \subset \tau^0_{[m]}\} \to 1.$$

[7] $\lfloor a \rfloor$ denotes the largest integer $\le a$.

Proof. To show that $\widetilde{T}^{\delta}_{[m]}$ is a tree we merely have to observe that at each stage of its construction we add vertices which have not been used before and one edge from each new vertex to one old vertex. Thus at no stage can a circuit arise in any $\widetilde{T}_{[l]}$.

To show that $\widetilde{T}_{[m]}$ has the same distribution as $T^{\delta}_{[m]}$ we must show that for $l < m$ the "number of children" of each vertex of $\widetilde{T}_{[l]}$ in $\widetilde{T}_{[l+1]}$ has a Poisson distribution with mean δ, and that all these numbers are independent. Use the notation of the construction preceding this lemma. Let j_p be a vertex in $\widetilde{T}_{[l]}$. If $j_p > n$ then its children (i.e., vertices of $\widetilde{T}_{[l+1]} \setminus \widetilde{T}_{[l]}$ connected to j_p) are precisely $Y(j_p)$ vertices from $\{n+1, n+2, \dots\} \setminus E_l(p)$. Since $Y(j_p)$ was a Poisson variable with mean δ independent of all other Y and U, there is nothing to prove in this case. Now consider a $j_p \in \{1, \dots, n\}$ with $j_p = i_\nu$ as above. In this case j_p has $u = \pi^{-1}(V(j_p))$ children. The distribution of u is given by

$$\mathbb{P}\{u \le r \mid \widetilde{\tau}^0_{[j]}, \widetilde{T}_{[j]}, j \le l, C_l(\nu)\} = \mathbb{P}\{V(j_p) \le \pi(r) \mid \widetilde{\tau}^0_{[j]}, \widetilde{T}_{[j]}, j \le l, C_l(\nu)\}. \tag{3.18}$$

Recall the definition of V in (3.15) and note that q in this formula is just the number of conducting edges between $j_p = i_\nu$ and vertices in $\{1, \dots, n\} \setminus C_l(\nu)$. Given $\widetilde{\tau}^0_{[j]}$, $\widetilde{T}_{[j]}$, $j \le l$, and the sets $C_l(\nu)$, $E_l(p)$, the conditional distribution of q is binomial $B(n+1-|C_l(\nu)|, \gamma(n)/n)$. In particular, if $0 < \gamma(n) < n$ and $q_0 < n$ is a fixed integer, then

$$\mathbb{P}\{\beta(q) \le \beta(q_0) \mid \widetilde{\tau}^0_{[j]}, \widetilde{T}_{[j]}, j \le l, C_l(\nu)\} = \mathbb{P}\{q \le q_0 \mid \widetilde{\tau}^0_{[j]}, \widetilde{T}_{[j]}, j \le l, C_l(\nu)\}. \tag{3.19}$$

If we take

$$q_0 = q_0(r) = \{\text{largest integer } s \text{ with } \beta(s) \le \pi(r)\}$$

then we obtain from (3.15), (3.18) and (3.19)

$$\begin{aligned}
&\mathbb{P}\{u \le r \mid \widetilde{\tau}^0_{[j]}, \widetilde{T}_{[j]}, j \le l, C_l(\nu)\} \\
&\qquad = \mathbb{P}\{q \le q_0 \mid \widetilde{\tau}^0_{[j]}, \widetilde{T}_{[j]}, j \le l, C_l(\nu)\} \\
&\qquad\qquad + \mathbb{P}\{q = q_0 + 1 \mid \widetilde{\tau}^0_{[j]}, \widetilde{T}_{[j]}, j \le l, C_l(\nu)\}\mathbb{P}\left\{U(j_p) \le \frac{\pi(r) - \beta(q_0)}{\beta(q_0+1) - \beta(q_0)}\right\} \\
&\qquad = \beta(q_0) + \pi(r) - \beta(q_0) = \pi(r).
\end{aligned}$$

Thus the number of children of any vertex in $\widetilde{T}_{[l]}$ indeed has a Poisson distribution with mean δ, as desired. A slightly closer look at the above argument shows also that the numbers of children of each of the vertices in $\widetilde{T}_{[l]}$ are independent, so that the first claim of the lemma follows.

We already observed that $\widetilde{\tau}^0_{[m]}$ is a subtree of $\tau^0_{[m]}$ by construction, so that we only need to prove

$$\mathbb{P}\{\widetilde{T}_{[m]} \subset \widetilde{\tau}^0_{[m]}\} \to 1 \tag{3.20}$$

for (3.17). First observe that $\widetilde{T}_{[m]} \subset \widetilde{\tau}^0_{[m]}$ fails only if there exists some $l < m$ and a vertex j_p of $\widetilde{T}_{[l]}$ which equals a vertex i_ν of $\widetilde{\tau}^0_{[l]}$ such that $u > q$ where u is the number of children of j_p in $\tilde{T}_{[l+1]}$ and q is the number of children of i_ν in $\widetilde{\tau}^0_{[l+1]}$. By our construction this requires $\pi(q;\delta) < \beta(q; n, |C_l(\nu)|)$; see (3.15). Next we obtain a lower bound for $\beta(q)$. The expected number of vertices in any subset of $\{1, \dots, n\}$ connected by a conducting edge to any fixed vertex of K_{n+2} is at most $n \cdot \gamma(n)/n = \gamma(n)$. It follows from this that

$$\mathbb{E}\{|\widetilde{\tau}^0_{l+1}| \mid \widetilde{\tau}^0_j, j \le l\} \le \gamma(n)|\widetilde{\tau}^0_l|$$

and (see (3.16))

$$\mathbb{E}|\widetilde{\tau}^0_{[m]}| \le \sum_{l=0}^{m} \{\gamma(n)\}^l \le \frac{\gamma(n)}{\gamma(n)-1} n^{3/4}.$$

Therefore

$$\mathbb{P}\{|\widetilde{\tau}^0_{[m]}| > n^{7/8}\} \le \frac{\gamma(n)}{\gamma(n)-1} n^{-1/8} \to 0. \tag{3.21}$$

If $|\widetilde{\tau}^0_{[m]}| \le n^{7/8}$, then also $|C_l(\nu)| \le |\widetilde{\tau}^0_{[m]}| \le n^{7/8}$ for all $C_l(\nu)$ used in the construction of $\widetilde{\tau}^0_{[m]}$. Therefore, if we set

$$\widetilde{\gamma}(n) = \frac{n+1-n^{7/8}}{n}\gamma(n),$$

then for any $x \le n^{1/16}$ and some constant C

$$\begin{aligned}
\beta(x) &= \beta(x; n, |C_l(\nu)|) \\
&\le \sum_{j \le x} \binom{n+1-n^{7/8}}{j} \left(\frac{\gamma(n)}{n}\right)^j \left(1 - \frac{\gamma(n)}{n}\right)^{n+1-n^{7/8}-j} \\
&\le (1 + Cn^{-15/16}) \sum_{j \le x} e^{-\widetilde{\gamma}(n)} \frac{\{\widetilde{\gamma}(n)\}^j}{j!} \\
&\le \pi(x; \widetilde{\gamma}(n)) + Cn^{-15/16} = \int_{\widetilde{\gamma}(n)}^{\infty} e^{-z} \frac{z^x}{x!}\, dz + Cn^{-15/16}.
\end{aligned}$$

Moreover, for $x \ge \widetilde{\gamma}(n)$

$$\int_{\widetilde{\gamma}(n)}^{\infty} e^{-z} \frac{z^x}{x!}\, dz + Cn^{-15/16} \le \int_{\delta}^{\infty} e^{-z} \frac{z^x}{x!}\, dz = \pi(x;\delta)$$

as long as

$$\int_{\delta}^{\widetilde{\gamma}(n)} e^{-z}\frac{z^x}{x!}\,dz \geq (\widetilde{\gamma}(n)-\delta)e^{-\delta}\frac{\delta^x}{x!} \geq Cn^{-15/16}.$$

Therefore, if we define

$$s(n) = \left\{\text{smallest } s \text{ with } e^{-\delta}\frac{\delta^s}{s!} < \frac{2Cn^{-15/16}}{\gamma-\delta}\right\}, \tag{3.22}$$

then for sufficiently large n, $\pi(q;\delta) < \beta(q;n,|C_l(\nu)|)$ can occur only if[8] $q \geq s(n) \wedge n^{1/16}$. It follows that $\widetilde{T}_{[m]} \subset \widetilde{\tau}^0_{[m]}$ whenever $|\widetilde{\tau}^0_{[m]}| \leq n^{7/8}$ and all vertices in $\widetilde{T}_{[m-1]}$ have fewer than $s(n) \wedge n^{1/16}$ children. Thus, by virtue of (3.21)

$$\begin{aligned}\mathbb{P}\{&\widetilde{T}_{[m]} \text{ is not a subgraph of } \widetilde{\tau}^0_{[m]}\}\\ &\leq \frac{\gamma(n)}{\gamma(n)-1}n^{-1/8}\\ &\qquad + \mathbb{P}\{\text{some vertex of } \widetilde{T}_{[m-1]} \text{ has at least } s(n)\wedge n^{1/16} \text{ children}\}.\end{aligned} \tag{3.23}$$

Finally we use the fact that $\widetilde{T}_{[m]}$ has the same distribution as $T^{\delta}_{[m]}$, so that from standard branching process formulae (see Harris (1963) Theorem I.5.1)

$$\mathbb{P}\{|T^{\delta}_{[m]}| \geq A\delta^m\} \leq \frac{1}{A\delta^m}\mathbb{E}|T^{\delta}_{[m]}| \leq \frac{\delta}{A(\delta-1)}.$$

For the right hand side of (3.23) we therefore find for large n the estimate

$$\begin{aligned}&\frac{\gamma(n)}{\gamma(n)-1}n^{-1/8} + \mathbb{P}\{|T^{\delta}_{[m]}| \geq A\delta^m\}\\ &\quad + \mathbb{P}\{\text{one of } A\delta^m \text{ independent Poisson variables, mean } \delta, \text{ is at least } s(n)\wedge n^{1/16}\}\\ &\qquad \leq \frac{\gamma(n)}{\gamma(n)-1}n^{-1/8} + \frac{\delta}{A(\delta-1)} + A\delta^m \sum_{k\geq s(n)\wedge n^{1/16}} e^{-\delta}\frac{\delta^k}{k!}.\end{aligned} \tag{3.24}$$

Finally, for $t \geq 2\delta$

$$\sum_{k\geq t} e^{-\delta}\frac{\delta^k}{k!} \leq \left(1-\frac{\delta}{t}\right)^{-1} e^{-\delta}\frac{\delta^t}{t!} \leq 2e^{-\delta}\frac{\delta^t}{t!}$$

[8] $a \wedge b = \min\{a, b\}$

so that by virtue of (3.22) and (3.16)

$$\begin{aligned}A\delta^m \sum_{k\geq s(n)} e^{-\delta}\frac{\delta^k}{k!} &\leq 2A\delta^m e^{-\delta}\frac{\delta^{s(n)}}{s(n)!}\\ &\leq 2A\delta^m \frac{2C}{\gamma-\delta} n^{-15/16}\\ &= \mathrm{O}(n^{3/4-15/16}) = \mathrm{O}(n^{-3/16}).\end{aligned}$$

Obviously, also

$$A\delta^m \sum_{k\geq n^{1/16}} e^{-\delta}\frac{\delta^k}{k!} \to 0 \quad \text{ as } n\to\infty.$$

Thus the right hand side of (3.24) can be made as small as desired by choosing first A and then n large. (3.20) and (3.17) follow. □

In the same way as we constructed $\widetilde{\tau}^0_{[m]}$ we can construct a subtree $\widetilde{\tau}^\infty_{[m]}$ of $\tau^\infty_{[m]}$. We want $\widetilde{\tau}^\infty_{[m]}$ disjoint from $\widetilde{\tau}^0_{[m]}$. This can be achieved by first constructing $\widetilde{\tau}^0_{[m]}$ and then choosing for $\widetilde{\tau}^\infty_{[m]}$ only vertices of $K_{n+2}\setminus\widetilde{\tau}^0_{[m]}$. We can then construct a random tree $\widehat{T}_{[m]}$ which has the same relation to $\widetilde{\tau}^\infty_{[m]}$ as $\widetilde{T}_{[m]}$ to $\widetilde{\tau}^0_{[m]}$. We shall, however, use Poisson and uniform variables for $\widehat{T}$ which are independent of the $Y(\cdot)$ and $U(\cdot)$ used in the construction of $\widetilde{T}_{[m]}$. Also we shall choose all vertices of $\widehat{T}_{[m]}$ disjoint from those of $\widetilde{T}_{[m]}$. It is then not hard to show that $\widetilde{T}_{[m]}$ and $\widehat{T}_{[m]}$ are disjoint trees, which are independent of each other, each with the distribution of $T^\delta_{[m]}$. Moreover,

$$\mathbb{P}\{T^\delta_{[m]}\subset\widetilde{\tau}^0_{[m]}\subset\tau^0_{[m]} \text{ and } \widehat{T}^\delta_{[m]}\subset\widetilde{\tau}^\infty_{[m]}\subset\tau^\infty_{[m]}\}\to 1 \quad \text{as } n\to\infty. \tag{3.25}$$

The next lemma is almost immediate from (3.25), but we need some more notation. Let T' and T'' be independent disjoint trees, each with the distribution of T^δ. $T'_m, T'_{[m]}, T''_m$ and $T''_{[m]}$ have the obvious meaning. The vertices of T' and T'' will be labeled $\langle 0\rangle', \langle i_1,\dots,i_n\rangle'$ and $\langle 0\rangle'', \langle i_1,\dots,i_n\rangle''$ in the usual way. All edges of T' and T'' are assigned a random resistance, chosen according to the distribution F. All these resistances are assumed independent. For each k we can form a network $N(n,k)$ consisting of $T'_{[k]}, T''_{[k]}$ and a resistance between each pair of vertices v', v'' with $v'\in T'_k$, $v''\in T''_k$. The latter resistances are chosen independent of each other and of T', T'' and the resistances in $T'\cup T''$. Each of the resistances between T'_k and T''_k is chosen according to (1.1). Of course this is equivalent to connecting each given pair v', v'' only with probability $\gamma(n)/n$, but with F for the conditional distribution function of the resistance between them, when it is given that they are connected. We put

$$\rho(n,k) = \{\text{resistance between } \langle 0\rangle' \text{ and } \langle 0\rangle'' \text{ in } N(n,k)\}.$$

Lemma 10. *Let $1 < \delta < \gamma$ and let $m = m_n$ be as in (3.16). Then*

$$\mathbb{P}\{R_n \le x\} \ge \mathbb{P}\{\rho(n, m_n) \le x\} + \mathrm{o}(1), \tag{3.26}$$

where $\mathrm{o}(1) \to 0$ *as* $n \to \infty$.

Proof. The construction in stages of $\widetilde{\tau}^0_{[m]}$, $\widetilde{T}_{[m]}$, $\widetilde{\tau}^\infty_{[m]}$ and $\widehat{T}_{[m]}$ is such that it gives no information about the resistances of edges of K_{n+2} between the last generation of $\widetilde{\tau}^0_{[m]}$ (i.e., $\widetilde{\tau}^0_{[m]} \setminus \widetilde{\tau}^0_{[m-1]}$) and the last generation of $\widetilde{\tau}^\infty_{[m]}$ (i.e., $\tau^\infty_{[m]} \setminus \tau^\infty_{[m-1]}$). Consequently, conditionally on $\widetilde{\tau}^0_{[m]}$, $\widetilde{\tau}^\infty_{[m]}$, $\widetilde{T}_{[m]}$, $\widehat{T}_{[m]}$, all these edges have independent resistances, each with distribution given in (1.1). We also insert an edge between any pair of vertices v, w with $v \in \widetilde{T}_m := \widetilde{T}_{[m]} \setminus \widetilde{T}_{[m-1]}$ and $w \in \widehat{T}_m := \widehat{T}_{[m]} \setminus \widehat{T}_{[m-1]}$ and not both v and w in K_{n+2}. These edges are also given resistances with the distribution (1.1), and we take all these resistances independent of each other and of the ones in K_{n+2}. Finally we choose further independent resistances with distribution function F for all edges in $\widetilde{T}_{[m]} \cup \widehat{T}_{[m]}$ which are not edges of K_{n+2}. Recall that all edges of $\widetilde{\tau}^0_{[m]}$ and $\widetilde{\tau}^\infty_{[m]}$ are conducting by construction, and hence have conditional distribution F for their resistance. Thus, the resistance, $\widetilde{\rho}$ say, between 0 and ∞ in the network consisting of $\widetilde{T}_{[m]}$, $\widehat{T}_{[m]}$ and the edges between $\widetilde{T}_m$ and $\widehat{T}_m$ has precisely the distribution of $\rho(n, m)$. Moreover, when $\widetilde{T}_{[m]}$ and $\widehat{T}_{[m]}$ are subgraphs of $\widetilde{\tau}^0_{[m]}$ and $\widetilde{\tau}^\infty_{[m]}$, respectively, then this network is part of K_{n+2}. Since the resistance between 0 and ∞ in any sub-network of K_{n+2} is at least R_n (by the monotonicity property (2.6)) we have $\mathbb{P}\{\widetilde{\rho} \ge R_n\} \to 1$ (by (3.25)). (3.26) follows because $\widetilde{\rho}$ and $\rho(n, m)$ have the same distribution. $\square$

We shall now show that $\rho(n, m_n)$ converges in distribution to $R'(\delta) + R''(\delta)$. Except for the proof of (3.72) which occurs almost at the end of these notes we make no further use of the fact that the offspring distribution in the branching process is a Poisson distribution.

Lemma 11. *Assume that*

$$F(K) = 1 \quad \textit{for some} \quad K < \infty. \tag{3.27}$$

Then there exist constants $0 < C_3, C_4 < \infty$ such that for $0 \le \varepsilon < \frac{1}{3}$ and $k \ge \frac{1+2\varepsilon}{2\log\delta}\log n$ and n sufficiently large

(3.28)
$$\begin{aligned}
&\mathbb{P}\{\rho(n, k) \le (2k+1)K \mid |T'_k| \ne 0, |T''_k| \ne 0\} \\
&\qquad = \mathbb{P}\{\exists \text{ conducting path between } \langle 0\rangle' \text{ and } \langle 0\rangle'' \text{ in } N(n,k) \mid |T'_k| \ne 0, |T''_k| \ne 0\} \\
&\qquad \ge 1 - C_3 n^{-C_4\varepsilon}.
\end{aligned}$$

Proof. Let

$$f_l(s) = f_l^\delta(s) = \mathbb{E}s^{|T_l^\delta|}, \quad 0 \le s \le 1.$$

Then, for $0 < s \le 1$

$$\mathbb{P}\{0 < |T_l^\delta| \le l\} \le s^{-l}\mathbb{E}\{s^{|T_l^\delta|}; |T_l^\delta| \ne 0\} = \frac{f_l(s) - f_l(0)}{s^l}.$$

However, $\{|T_l^\delta|\}_{l\ge 0}$ is a supercritical branching process, so that by Cor. I.11.1 in Athreya and Ney (1972) there exists a $\lambda = \lambda(\delta) < 1$ such that (in the notation of Athreya and Ney (1972)) for all $0 \le s < 1$

$$\lim_{l\to\infty} \frac{f_l(s) - f_l(0)}{\lambda^l} = Q(s) - Q(0) < \infty.$$

Thus, if we fix $\lambda < s < 1$, then there exists some l_0 such that for $l \ge l_0$

$$\mathbb{P}\{0 < |T_l^\delta| \le l\} \le 2\{Q(s) - Q(0)\}\left(\frac{\lambda}{s}\right)^l. \tag{3.29}$$

Next we observe that

$$\lim_{r\to\infty} \frac{|T_r^\delta|}{\delta^r} = W \quad \text{exists with probability 1,}$$

and

$$\mathbb{P}\{W > 0\} = \mathbb{P}\{|T_r^\delta| \text{ is never zero}\} = 1 - q(\delta) > 0 \tag{3.30}$$

(see Harris (1963) Theorems I.8.1 and I.8.3). Thus, there exists an $\alpha = \alpha(\delta) > 0$ such that

$$\mathbb{P}\{|T_r^\delta| \ge \alpha\delta^r \text{ for all } r \ge 0\} \ge \tfrac{1}{2}(1 - q(\delta)).$$

Each $\langle i_1, \dots, i_l\rangle$ in T_l^δ has a certain number of descendants in T_k^δ $(k > l)$. By the branching property these numbers for different $\langle i_1, \dots, i_l\rangle$ are independent and have the same distribution as $|T_{k-l}^\delta|$. Therefore,

$$\begin{aligned}
&\mathbb{P}\{|T_k^\delta| \le \alpha\delta^{k-l} \mid T_{[l]}^\delta\} \\
&\quad \le \mathbb{P}\{\text{each individual } \langle i_1, \dots, i_l\rangle \text{ in } T_l^\delta \text{ has fewer than } \alpha\delta^{k-l} \text{ children in } T_k^\delta \mid T_{[l]}^\delta\} \\
&\quad \le \left\{1 - \tfrac{1}{2}(1 - q(\delta))\right\}^{|T_l^\delta|} = \left\{\frac{1 + q(\delta)}{2}\right\}^{|T_l^\delta|}.
\end{aligned}$$

It follows that for each $l < k$

$$\begin{aligned}
&\mathbb{P}\{0 < |T_k^\delta| < \alpha\delta^{k-l}\} \\
&\qquad \le \mathbb{P}\{0 < |T_l^\delta| \le l\} + \mathbb{E}\Big\{\mathbb{P}\{|T_k^\delta| < \alpha\delta^{k-l} \mid T_{[l]}^\delta\}; |T_l^\delta| > l\Big\} \\
&\qquad \le 2\{Q(s) - Q(0)\}\left(\frac{\lambda}{s}\right)^l + \left\{\frac{1 + q(\delta)}{2}\right\}^l.
\end{aligned}$$

If we choose $l \sim \varepsilon k$, then we see that for some $\beta < 1$

$$\mathbb{P}\{0 < |T_k^\delta| < \alpha\delta^{(1-\varepsilon)k}\} \le \{2Q(s) - 2Q(0) + 1\}\beta^{\varepsilon k}.$$

Since T_k' and T_k'' are independent, each with the same distribution as T_k^δ, we conclude (use (3.30)) that

$$\begin{aligned}\mathbb{P}\{|T_k'| < \alpha\delta^{(1-\varepsilon)k} \text{ or } |T_k''| < \alpha\delta^{(1-\varepsilon)k} \mid |T_k'| \neq 0, |T_k''| \neq 0\}& \\ \le \{4Q(s) - 4Q(0) + 2\}\{1 - q(\delta)\}^{-2}\beta^{\varepsilon k}&.\end{aligned}$$

Finally, we observe that the conditional probability, given $T_{[k]}', T_{[k]}''$, that there does not exist any conducting edge in $N(n,k)$ between T_k' and T_k'' is

$$\left(1 - \frac{\gamma(n)}{n}\right)^{|T_k'|\cdot|T_k''|} \le \exp\left(-\frac{\gamma(n)}{n}|T_k'| \cdot |T_k''|\right).$$

Whenever $T_k' \neq 0$, and $T_k'' \neq 0$ and there is a conducting edge between T_k' and T_k'', then $\langle 0\rangle'$ is connected to $\langle 0\rangle''$ along a path from $\langle 0\rangle'$ to T_k' (in $T_{[k]}'$), then to T_k'' and then to $\langle 0\rangle''$ (in $T_{[k]}''$) . This path contains $(2k+1)$ edges, and since each conducting edge has resistance at most K (by (3.27)) $\rho(n,k) \le (2k+1)K$ in this situation. Thus the first equality in (3.28) holds and

$$\begin{aligned}&\mathbb{P}\{\text{there is no conducting path in } N(n,k) \text{ between } \langle 0\rangle' \text{ and } \langle 0\rangle'' \mid |T_k'| \neq 0, |T_k''| \neq 0\} \\ &\qquad \le \mathbb{P}\{|T_k'| < \alpha\delta^{(1-\varepsilon)k} \text{ or } |T_k''| < \alpha\delta^{(1-\varepsilon)k} \mid |T_k'| \neq 0, |T_k''| \neq 0\} \\ &\qquad\qquad + \exp\left(-\frac{\gamma(n)}{n}\alpha^2\delta^{2(1-\varepsilon)k}\right) \\ &\qquad \le \{4Q(s) - 4Q(0) + 2\}\{1 - q(\delta)\}^{-2}\ \beta^{\varepsilon k} + \exp\left(-\frac{\gamma(n)}{n}\alpha^2\delta^{2(1-\varepsilon)k}\right).\end{aligned}$$

(3.28) follows when $2(1-\varepsilon)k \ge (1-\varepsilon)(1+2\varepsilon)\log n/\log\delta$. □

For the time being we maintain the extra assumption (3.27). We use Lemma 11 to replace the random network $N(n, m_n)$ by another random network, $M(n)$, which with high probability has a resistance at least equal to $\rho(n, m_n)$. $M(n)$ is constructed as follows (see Figure 9). Let $s = s_n = \lfloor\sqrt{\log n}\rfloor$. Form $T_{[m_n]}'$ and $T_{[m_n]}''$. If $T_s' \neq 0$ and $T_s'' \neq 0$, then connect each pair of vertices $\langle i_1, \ldots, i_s\rangle' \in T_s'$ and $\langle j_1, \ldots, j_s\rangle'' \in T_s''$ which have descendants in T_{m_n}' and T_{m_n}'', respectively, by a resistance of size

$$\{|T_s'| + |T_s''|\}\frac{K}{\log\gamma}\log n. \tag{3.31}$$

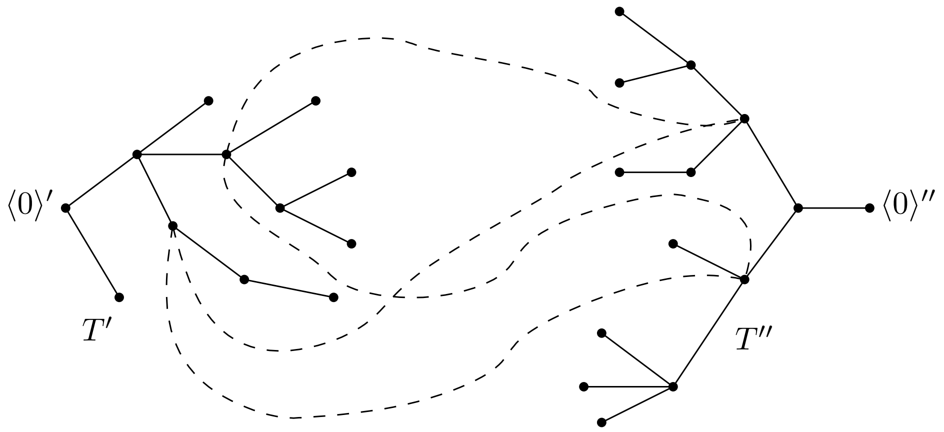

Figure 9. A schematic representation of $M(n)$ with $s = 2$, $m = 4$. The dashed curves represent the connections of resistance $\{|T'_s| + |T''_s|\}\frac{K}{\log\gamma}\log n$. These dashed connections have no interior points in common (when realized in space instead of in the plane).

We shall write $\mathcal{A}'_s$ (respectively $\mathcal{A}''_s$) for the collection of vertices $\langle i_1, \dots, i_s\rangle' \in T'_s$ (respectively $\langle j_1, \dots, j_s\rangle'' \in T''_s$) which have descendants in T'_{m_n} (respectively T''_{m_n}).

This describes the network $M(n)$. We denote the resistance between $\langle 0\rangle'$ and $\langle 0\rangle''$ in $M(n)$ by $\mathcal{R}(n, K)$ ($\mathcal{R}(n, K) = \infty$ if T'_s or T''_s is empty, or even if $\mathcal{A}'_s$ or $\mathcal{A}''_s$ is empty).

Lemma 12. *If* $1 < \delta \le \gamma$ *is such that*

$$\frac{\log\gamma}{\log\delta} < \frac{6}{5}, \tag{3.32}$$

and if (3.27) holds, then

$$\mathbb{P}\{\rho(n, m_n) \le \mathcal{R}(n, K)\} \to 1, \tag{3.33}$$

and

$$\mathbb{P}\{R_n \le x\} \ge \mathbb{P}\{\mathcal{R}(n, K) \le x\} + \mathrm{o}(1). \tag{3.34}$$

where $\mathrm{o}(1) \to 0$ *as* $n \to \infty$.

Proof. Let $\langle i_1, \dots, i_s\rangle' \in T'_s$, $\langle j_1, \dots, j_s\rangle'' \in T''_s$ with $s = s_n = \lfloor\sqrt{\log n}\rfloor$. We apply Lemma 11 to the sub-network of $N(n, m_n)$ consisting of the tree of descendants of $\langle i_1, \dots, i_s\rangle'$ in $T'_{[m_n]}$, the tree of descendants of $\langle j_1, \dots, j_s\rangle''$ in $T''_{[m_n]}$ and the edges between the last generations of these trees. Conditionally on $T'_{[s]}, T''_{[s]}$, this network has the same distribution as $N(n, k)$ with $k = k_n := m_n - s_n$. Therefore, the resistance between $\langle i_1, \dots, i_s\rangle'$ and $\langle j_1, \dots, j_s\rangle''$ in this network has the same distribution as $\rho(n, k_n)$. In particular, given that $\langle i_1, \dots, i_s\rangle' \in \mathcal{A}'_s$, $\langle j_1, \dots, j_s\rangle'' \in \mathcal{A}''_s$, the conditional probability that there is a path between $\langle i_1, \dots, i_s \rangle'$ and $\langle j_1, \dots, j_s \rangle''$ of resistance $\le (2k_n + 1)K$ in the above network is at least (for sufficiently large n)

$$1 - C_3 n^{-C_4/8}, \tag{3.35}$$

by virtue of (3.28) (with $\varepsilon = 1/8$) and the fact that

$$\begin{aligned} k_n = m_n - s_n &\sim \frac{3}{4} \frac{\log n}{\log \gamma} \quad \text{(see (3.16))} \\ &> \frac{5}{8} \frac{\log n}{\log \delta} \quad \text{(by (3.32))}. \end{aligned}$$

Note that if the above path between $\langle i_1, \dots, i_s \rangle'$ and $\langle j_1, \dots, j_s \rangle''$ exists, then it is made up entirely from edges outside $T'_{[s]}$ or $T''_{[s]}$. In fact it is built up from edges in the trees $T'(\langle i_1, \dots, i_s \rangle')$ and $T''(\langle j_1, \dots, j_s \rangle'')$ of the descendants of $\langle i_1, \dots, i_s \rangle'$ and $\langle j_1, \dots, j_s \rangle''$, respectively, plus an edge between T'_m and T''_m which does not belong to T' or to T''. Let us denote by $\mathcal{C} = \mathcal{C}_{m_n}$ the collection of conducting edges between T'_{m_n} and T''_{m_n} in $N(n, m_n)$. If there is a conducting path connecting $\langle i_1, \dots, i_s \rangle'$ and $\langle j_1, \dots, j_s \rangle''$, then it contains one edge from $\mathcal{C}$, and this edge connects a descendant of $\langle i_1, \dots, i_s \rangle'$ and a descendant of $\langle j, \dots, j_s \rangle''$. Therefore, for different pairs $\langle i_1, \dots, i_s \rangle'$, $\langle j_1, \dots, j_s \rangle''$ different edges from $\mathcal{C}$ will be used.

Now consider the event

$E_n := \{$ each pair $\langle i_1, \dots, i_s \rangle' \in T'$ and $\langle j_1, \dots, j_s \rangle'' \in T''$ which have descendants in T'_m and T''_m, respectively, are connected in $N(n, m_n)$ by a conducting path in $T'(\langle i_1, \dots, i_s \rangle') \cup T''(\langle j_1, \dots, j_s \rangle'') \cup \mathcal{C}$ of length $(2k_n + 1)\}$.

By the estimate (3.35) and the fact that $\mathcal{A}'_s \subset T'_s$, $\mathcal{A}''_s \subset T''_s$ we have

$$\mathbb{P}\{E_n \mid T'_{[s]}, T''_{[s]}\} \ge 1 - |T'_s| \cdot |T''_s| C_3 n^{-C_4/8}.$$

Since (see Harris (1963) Theorem I.5.1)

$$\mathbb{E}|T'_s| = \mathbb{E}|T''_s| = \delta^s = \mathrm{o}(n^{C_4/16}),$$

it follows that

$$\begin{aligned} \mathbb{P}\{E_n\} &\ge 1 - \mathbb{E}\{|T'_s| \cdot |T''_s| C_3 n^{-C_4/8}\} \\ &\ge 1 - \delta^{2s} C_3 n^{-C_4/8} \to 1 \quad \text{as } n \to \infty. \end{aligned}$$

(3.33) now follows easily from this and the monotonicity property (2.6). Indeed for $\langle i_1, \dots, i_s \rangle' \in T'_s$ we may replace any edge in $T'(\langle i_1, \dots, i_s \rangle')$, with resistance r say, by $|T''_s|$ parallel edges of resistance $|T''_s| r$ without changing the resistance between any pair of vertices in T'_s and T''_s. Similarly we may replace any edge in

$T''_s(\langle j_1,\dots,j_r\rangle'')$ by $|T'_s|$ parallel edges whose resistance is $|T'_s|$ times the resistance of the original edge. After this has been done, we can, on the event E_n, connect each pair $\langle i_1,\dots,i_s\rangle' \in \mathcal{A}'_s$, $\langle j_1,\dots,j_s\rangle'' \in \mathcal{A}''_s$ by a path of length $(2k_n+1)$, such that the different paths have no edges in common. Indeed each edge in $T'(\langle i_1,\dots,i_s\rangle')$ has been split into $|T''_s| \geq |\mathcal{A}''_s|$ parallel edges and we can use a different one of these parallel edges to connect $\langle i_1,\dots,i_s\rangle'$ to different $\langle j_1,\dots,j_s\rangle''$. The new paths each consist of k_n edges of resistance $\leq |T''_s|K$, an edge of $\mathcal{C}$ of resistance $\leq K$ and k_n edges of resistance $\leq |T'_s|K$, all of these edges being in series. The resistance of such a path is therefore at most

$$\{k_n(|T'_s|+|T''_s|)+1\}K \leq \{|T'_s|+|T''_s|\}\frac{K}{\log\gamma}\log n.$$

The new paths between all the pairs $\langle i_1,\dots,i_s\rangle' \in \mathcal{A}''_s$ and $\langle j_1,\dots,j_s\rangle'' \in \mathcal{A}''_s$ are edge-disjoint, but they still have vertices in common in $T'_{[m_n]}$ and in $T''_{[m_n]}$. However, by (2.6) these contacts between different paths can only reduce the resistance between $\langle 0\rangle'$ and $\langle 0\rangle''$. Therefore, on E_n, the resistance between $\langle 0\rangle'$ and $\langle 0\rangle''$ in $N(n,m_n)$ is at most the resistance between $\langle 0\rangle'$ and $\langle 0\rangle''$ in $M(n)$. This proves (3.33). (3.34) follows from (3.33) and (3.26). $\square$

Apart from removing some truncations later on, the only estimate left is one which shows that $\mathcal{R}(n,K)$ is essentially equal to the sum of the resistance of $T'_{[s]}$ and $T''_{[s]}$. This is done in the next lemma by showing that all vertices in $\mathcal{A}'_s$ and $\mathcal{A}''_s$ have almost the same potential.

Lemma 13. *Assume that for some* $0<\varepsilon<K<\infty$

$$F(\varepsilon-)=0, \quad F(K)=1. \tag{3.36}$$

Then

$$\mathbb{P}\{\mathcal{R}(n,K)\leq x\} \to \mathbb{P}\{R'(\delta)+R''(\delta)\leq x\} \tag{3.37}$$

at each continuity point of the right hand side.

Proof. We shall only need

$$\liminf_{n\to\infty}\mathbb{P}\{\mathcal{R}(n,K)\leq x\} \geq \mathbb{P}\{R'(\delta)+R''(\delta)\leq x\}. \tag{3.38}$$

We therefore only prove (3.38) and leave the (easy) other half of (3.37) to the interested reader. First we note that $\mathcal{R}(n,K)=\infty$ if $\mathcal{A}'_s=\emptyset$ or $\mathcal{A}''_s=\emptyset$, or equivalently, if $T'_{m_n}=\emptyset$ or $T''_{m_n}=\emptyset$. But as $n\to\infty$ also $m_n\to\infty$ and

$$\begin{aligned}\mathbb{P}\{T'_{m_n}=\emptyset\}=\mathbb{P}\{T''_{m_n}=\emptyset\}&\to\mathbb{P}\{T^\delta \text{ is finite}\}=\mathbb{P}\{Z \text{ dies out}\}\\ &=q(\delta)=\mathbb{P}\{R'(\delta)=\infty\}=\mathbb{P}\{R''(\delta)=\infty\},\end{aligned}$$

where Z is the branching process corresponding to T^δ (apply Lemma 2 to T^δ). We therefore should show (at continuity points x of the right hand side of (3.37))

$$\liminf_{n\to\infty} \mathbb{P}\{\mathcal{R}(n,K) \le x \mid \mathcal{A}'_s \neq \emptyset, \mathcal{A}''_s \neq \emptyset\} \\ \ge \mathbb{P}\{R'(\delta) + R''(\delta) \le x \mid T' \text{ and } T'' \text{ are infinite}\}. \tag{3.39}$$

Now assume that $\mathcal{A}'_s \neq \emptyset$, $\mathcal{A}''_s \neq \emptyset$ and let $\{X_\nu\}_{\nu\ge 0}$ be a Markov chain on the vertices of $T'_{[s_n]} \cup T''_{[s_n]}$ with transition probability matrix

$$P(\langle y\rangle, \langle z\rangle) = \left\{\sum_y \frac{1}{R(e)}\right\}^{-1} \frac{1}{R(\langle y\rangle, \langle z\rangle)} \tag{3.40}$$

whenever $\langle y\rangle$ and $\langle z\rangle$ are neighbors in $M(n)$ (by this we mean that $\langle y\rangle$ and $\langle z\rangle$ both belong to $T'_{[s_n]}$ or both to $T''_{[s_n]}$, and are neighbors in $T_{[s_n]}$, respectively $T''_{[s_n]}$, or one belongs to $\mathcal{A}'_{s_n}$ and the other to $\mathcal{A}''_{s_n}$). In (3.40) $R(\langle y\rangle, \langle z\rangle)$ denotes the resistance of the edge between $\langle y\rangle$ and $\langle z\rangle$ (if $\langle y\rangle$ and $\langle z\rangle$ are neighbors), and $\sum_y$ runs over all edges of $M(n)$ with one endpoint at y. By (2.3) and (2.4) $\mathcal{R}(n,K)$, the resistance between $\langle 0\rangle'$ and $\langle 0\rangle''$ in $M(n)$, equals

$$\left\{\sum_{\langle i_1\rangle\in T'} \frac{1}{R(\langle 0\rangle', \langle i_1\rangle')} \mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0 = \langle i_1\rangle'\}\right\}^{-1}. \tag{3.41}$$

Furthermore, it is probabilistically evident that

$$\begin{aligned}&\mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0 = \langle i_1\rangle'\} \\ &\quad = \sum_{\langle x\rangle\in\mathcal{A}'_s} \mathbb{P}\{X_\cdot \text{ reaches } \mathcal{A}'_s \text{ before } \langle 0\rangle' \text{ and does so first at } \langle x\rangle' \mid X_0 = \langle i_1\rangle'\} \\ &\qquad\qquad \times \mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0 = \langle x\rangle'\}.\end{aligned} \tag{3.42}$$

Assume that we can prove the existence of a (random) sequence of numbers P'_n such that

$$\sup_{\langle x\rangle'\in\mathcal{A}_s'} |\mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0 = \langle x\rangle'\} - P'_n| \to 0 \quad (n\to\infty) \tag{3.43}$$

in probability on the event

$$F_n := \{\mathcal{A}'_{s_n} \neq \emptyset, \mathcal{A}''_{s_n} \neq \emptyset\}. \tag{3.44}$$

(Of course this means that the probability of the subset of F_n on which (3.43) fails tends to 0.) Then (3.41)–(3.43) yield

$$\begin{aligned}&\mathcal{R}^{-1}(n,K) = (P'_n + \mathrm{o}_n(1)) \\ &\qquad \times \sum_{\langle i_1\rangle'\in T'} \frac{1}{R(\langle 0\rangle', \langle i_1\rangle')} \mathbb{P}\{X_\cdot \text{ reaches } \mathcal{A}_s' \text{ before } \langle 0\rangle' \mid X_0 = \langle i_1\rangle'\}\end{aligned} \tag{3.45}$$

where $o_n(1) \to 0$ in probability on F_n as $n \to \infty$. Again by (2.3) and (2.4) the sum in the right hand side of (3.45) equals the reciprocal of the resistance between $\langle 0\rangle'$ and $\mathcal{A}'_s$ in $T'_{[s]}$. For the time being we denote the latter resistance by $\mathcal{R}'_n$. With this notation (3.45) can be written as[9]

$$P'_n - \frac{\mathcal{R}'_n}{\mathcal{R}(n,K)} \to 0 \quad \text{in probability on } F_n. \tag{3.46}$$

If we can prove (3.43), then by interchanging the roles of T' and T'' we can also prove the existence of a P''_n such that

$$\sup_{\langle y\rangle'' \in \mathcal{A}''_s} |\mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle' \text{ before } \langle 0\rangle'' \mid X_0 = \langle y\rangle''\} - P''_n| \to 0 \tag{3.47}$$

in probability on F_n, and

$$P''_n - \frac{\mathcal{R}''_n}{\mathcal{R}(n,K)} \to 0 \quad \text{in probability on } F_n, \tag{3.48}$$

where $\mathcal{R}''_n$ is the resistance between $\langle 0\rangle''$ and $\mathcal{A}''_s$ in $T''_{[s]}$. Furthermore, (3.43) and (3.47) together imply on F_n, uniformly in $\langle x\rangle' \in \mathcal{A}'_s$,

$$\begin{aligned}
&\mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0 = \langle x\rangle'\} \\
&\quad = P'_n + o_n(1) \\
&\quad = \sum_{\langle y\rangle'' \in \mathcal{A}''_s} \mathbb{P}\{X_\cdot \text{ reaches } \mathcal{A}''_s \text{ before } \langle 0\rangle' \text{ and does so first at } \langle y\rangle'' \mid X_0 = \langle x\rangle'\} \\
&\qquad\qquad \times \mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0 = \langle y\rangle''\} \\
&\quad = \mathbb{P}\{X_\cdot \text{ reaches } \mathcal{A}''_s \text{ before } \langle 0\rangle' \mid X_0 = \langle x'\rangle\}(1 - P''_n + o_n(1)).
\end{aligned} \tag{3.49}$$

We shall next prove that

$$\min_{\langle x\rangle' \in \mathcal{A}'_s} \mathbb{P}\{X_\cdot \text{ reaches } \mathcal{A}''_s \text{ before } \langle 0\rangle' \mid X_0 = \langle x\rangle'\} \to 1 \quad \text{in probability on } F_n. \tag{3.50}$$

(Once one has (3.43) and (3.47) it is not hard to obtain (3.50) by equating the current flowing into $\mathcal{A}'_s$ and the current flowing out of $\mathcal{A}'_s$. However, we need (3.50) to prove (3.43) so we must prove it directly.) The construction of $M(n)$ is such that the resistance of any edge between any pair of vertices $\langle u\rangle' \in \mathcal{A}'_s$ and $\langle v\rangle'' \in \mathcal{A}''_s$ has the same value, namely the value in (3.31). But there are $|\mathcal{A}''_s|$ edges between any $\langle u\rangle' \in \mathcal{A}'_s$ and the set $\mathcal{A}''_s$, and the only other edge incident

[9] Note that $\mathcal{R}'_n$ and $\mathcal{R}(n,K)$ cannot be zero under (3.36).

to $\langle u\rangle'$ is an edge from $\langle u\rangle'$ to T'_{s-1} with resistance between ε and K (by (3.36)). Therefore for any $\langle u\rangle' \in \mathcal{A}'_s$.

$$P\{\langle u\rangle', \langle v\rangle''\} \text{ has the same value for all } \langle v\rangle'' \in \mathcal{A}''_s, \tag{3.51}$$

and the value in (3.51) satisfies
(3.52)

$$\sum_{\langle v\rangle''\in\mathcal{A}''_s} P\{\langle u\rangle', \langle v\rangle''\} \geq \frac{|\mathcal{A}''_s|\log\gamma}{K\{|T'_s|+|T''_s|\}\log n}\left\{\frac{|\mathcal{A}''_s|\log\gamma}{K\{|T'_s|+|T''_s|\}\log n}+\frac{1}{\varepsilon}\right\}^{-1}.$$

A decomposition with respect to the last visit of $X_.$ to $\mathcal{A}'_s$ before it hits $\langle 0\rangle'$ or $\mathcal{A}''_s$ yields.

(3.53)

$$\begin{aligned}
&\mathbb{P}\{X_. \text{ reaches } \langle 0\rangle' \text{ before } \mathcal{A}''_s \mid X_0 = \langle x\rangle'\}\\
&\quad= \sum_{\langle u\rangle'\in\mathcal{A}'_s} \mathbb{E}\{\text{number of visits to } \langle u\rangle' \text{ before reaching } \langle 0\rangle' \text{ or } \mathcal{A}''_s \mid X_0 = \langle x\rangle'\}\\
&\qquad\qquad \times \mathbb{P}\{X_. \text{ reaches } \langle 0\rangle' \text{ without returning to } \mathcal{A}'_s \mid X_0 = \langle u\rangle'\}.
\end{aligned}$$

Similarly,

(3.54)

$$\begin{aligned}
1 &= \mathbb{P}\{X_. \text{ reaches } \langle 0\rangle' \text{ or } \mathcal{A}''_s \text{ sometime} \mid X_0 = \langle x\rangle'\}\\
&= \sum_{\langle u\rangle'\in\mathcal{A}'_s} \mathbb{E}\{\text{number of visits to } \langle u\rangle' \text{ before reaching } \langle 0\rangle' \text{ or } \mathcal{A}''_s \mid X_0 = \langle x\rangle'\}\\
&\qquad \times \mathbb{P}\{X_. \text{ reaches } \langle 0\rangle' \text{ or } \mathcal{A}''_s \text{ without returning to } \mathcal{A}'_s \mid X_0 = \langle u\rangle'\}.
\end{aligned}$$

Dividing (3.53) by (3.54) we see that

(3.55)

$$\begin{aligned}
&\mathbb{P}\{X_. \text{ reaches } \langle 0\rangle' \text{ before } \mathcal{A}''_s \mid X_0 = \langle x\rangle'\}\\
&\quad\leq \max_{\langle u\rangle'\in\mathcal{A}'_s} \frac{\mathbb{P}\{X_. \text{ reaches } \langle 0\rangle' \text{ before } \mathcal{A}'_s \mid X_0 = \langle u\rangle'\}}{\mathbb{P}\{X_. \text{ reaches } \langle 0\rangle' \text{ or } \mathcal{A}''_s \text{ before returning to } \mathcal{A}'_s \mid X_0 = \langle u\rangle'\}}.
\end{aligned}$$

To estimate (3.55) note first that

$$\mathcal{A}''_s \subset T''_s;\ \text{hence } |\mathcal{A}''_s| \leq |T''_s|. \tag{3.56}$$

Also, there exist random variables W', W'' such that

$$\frac{|T'_s|}{\delta^s} \to W',\ \frac{|T''_s|}{\delta^s} \to W'' \quad \text{w.p.1},$$

and

$$W' > 0 \text{ a.e., on the set } \{T'_p \neq \emptyset \text{ for all } p\}$$
$$\text{and } W'' > 0 \text{ a.e., on the set } \{T''_p \neq \emptyset \text{ for all } p\}$$

(see Harris (1963) Theorem I.8.1 and Remark I.8.1). Thus $|T'_s|$ and $|T''_s|$ are both of order δ^s on most of the set

$$\{T'_s \neq \emptyset, T''_s \neq \emptyset\}.$$

Moreover, by definition of $\mathcal{A}'_s$,

$$\begin{aligned}\mathbb{P}\{\langle u\rangle' \in \mathcal{A}'_s \mid T'_{[s]}, \langle u\rangle' \in T'_s\} &= \mathbb{P}\{\langle u\rangle' \text{ has descendants in } T'_{[m]} \mid \langle u\rangle' \in T'_s\} \\ &= \mathbb{P}\{T'_{m-s} \neq \emptyset\} \\ &\to \mathbb{P}\{T'_p \neq \emptyset \text{ for all } p\} = 1 - q(\delta) > 0\end{aligned}$$

(compare (2.8) and (2.9)). If $T'_s = \{\langle u_1\rangle', \dots, \langle u_t\rangle'\}$, then by the branching property, the events $\langle u_i\rangle' \in \mathcal{A}'_s$, $i = 1, \dots, t$, are conditionally independent, given $\langle u_i\rangle' \in T'_s$, $1 \le i \le t$. It follows from these observations that

$$I[T'_s \neq \emptyset]\left\{\frac{|\mathcal{A}_s|}{|T'_s|} - (1 - q(\delta))\right\} \to 0 \quad \text{in probability.}$$

The same relation holds when $\mathcal{A}'_s$ and T'_s are replaced by $\mathcal{A}''_s$ and T''_s. Since $\mathcal{A}''_s \neq \emptyset$ implies $T''_s \neq \emptyset$, and similarly for $\mathcal{A}'_s$ and T'_s we find also that (cf. (3.56)).

$$\frac{|\mathcal{A}'_s|}{|T'_s|} \text{ and } \frac{|\mathcal{A}''_s|}{|T''_s|} \to 1 - q(\delta) \quad \text{in probability on the set } F_n \text{ of (3.44)}$$

and

$$\frac{|\mathcal{A}''_s|}{|T'_s| + |T''_s|} \to \frac{(1 - q(\delta))W''}{W' + W''} \quad \text{in probability on the set } F_n. \tag{3.57}$$

We return to (3.55). First we estimate its denominator.

$$\begin{aligned}&\min_{\langle u\rangle' \in \mathcal{A}'_s} \mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle' \text{ or } \mathcal{A}''_s \text{ before returning to } \mathcal{A}'_s \mid X_0 = \langle u\rangle'\} \\ &\qquad \ge \sum_{\langle v\rangle'' \in \mathcal{A}''_s} P\{\langle u\rangle', \langle v\rangle''\}.\end{aligned} \tag{3.58}$$

By virtue of (3.52) and (3.57), for every $\eta > 0$ and all large n there exists a $\kappa(\eta) > 0$ such that the subset of F_n on which the right hand side of (3.58) is less than

$$\frac{\varepsilon}{\kappa(\eta)\log n}$$

has probability $\leq \eta$. On the other hand, for all $\langle u\rangle' \in \mathcal{A}'_s$ the numerator in the right hand side of (3.55) is bounded above by

$$\sup_{w\in T'_{s-1}} \pi(\langle w\rangle', T'_{[m]}, R, s) = \Pi(T'_{[m]}, R, s)$$

(by (2.49)). Therefore, by Lemma 6, the numerator in the right hand side of (3.55) is at most (for large n)

$$\left(\frac{2L}{2L+\varepsilon}\right)^{C_1 s_n}$$

on the set F_n minus a subset of probability at most $\exp(-C_2 s_n)$. Consequently, for large n, on the set F_n, minus a set of probability at most 2η, (3.55) is at most

$$\left(\frac{2L}{2L+\varepsilon}\right)^{C_1 s_n} \quad \frac{\kappa(\eta)\log n}{\varepsilon} \to 0$$

(as $n \to \infty$, since $s_n = \sqrt{\log n}$). Since η is arbitrary we finally proved that the right hand side of (3.55) tends to 0 in probability on F_n. This implies (3.50).

From here on it is easy to complete the proof of the lemma (still under the assumptions (3.43) and (3.47)). Firstly, (3.49) and (3.50) together imply

$$P'_n + P''_n \to 1 \quad \text{in probability on } F_n. \tag{3.59}$$

In turn (3.59), (3.46), and (3.48) together show

$$\frac{\mathcal{R}'_n + \mathcal{R}''_n}{\mathcal{R}(n,K)} \to 1 \quad \text{in probability on } F_n. \tag{3.60}$$

It follows from (2.3), (2.4) and the definition $\mathcal{A}'_s$ that $R(T'_{[s_n]}) \leq \mathcal{R}' \leq R(T'_{[m_n]})$. A similar inequality holds for $\mathcal{R}''$ so that

$$\mathcal{R}'_n \to R(T'),\ \mathcal{R}''_n \to R(T'') \quad \text{w.p.1.}$$

Since $R(T')$ and $R(T'')$ both have the distribution of $R(T^\delta)$, that is of $R'(\delta)$ and $R''(\delta)$, (3.60) implies (3.39) and the lemma. The proof has therefore been reduced to proving (3.43) and (3.47).

(3.43) can now be proved quickly from (3.52) and (3.51). Indeed when $X_\cdot$ starts at $\langle x\rangle' \in \mathcal{A}'_s$ it cannot reach $\langle 0\rangle''$ without passing through $\mathcal{A}''_s$. Let τ be the first time $X_\cdot$ visits $\mathcal{A}''_s$. Then a decomposition with respect to the values of $\tau - 1$, $X_{\tau-1}$ and X_τ gives

(3.61)

$$\begin{aligned}
&\mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0 = \langle x\rangle'\} \\
&\quad = \sum_{n=1}^{\infty} \sum_{\langle u\rangle'\in\mathcal{A}'_s} \sum_{\langle v\rangle''\in\mathcal{A}''_s} \mathbb{P}\{X_n = \langle u\rangle', X_j \neq \langle 0\rangle', X_j \notin \mathcal{A}''_s, 0 \leq j \leq n\} \\
&\qquad\qquad \times P\{\langle u\rangle', \langle v\rangle''\}\mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0 = \langle v\rangle''\}.
\end{aligned}$$

By virtue of (3.51)

$$\begin{aligned}\sum_{\langle v\rangle''\in\mathcal{A}''_s} & P\{\langle u\rangle',\langle v\rangle''\}\mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0=\langle v\rangle''\} \\ &= \frac{1}{|\mathcal{A}''_s|}\sum_{\langle v\rangle''\in\mathcal{A}''_s} P\{\langle u\rangle',\langle v\rangle''\} \\ &\qquad \times \sum_{\langle w\rangle''\in\mathcal{A}''_s}\mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0=\langle w\rangle''\}.\end{aligned}$$

Thus, (3.61) equals

$$\begin{aligned}&\mathbb{P}\{X_\cdot \text{ reaches } \mathcal{A}''_s \text{ before } \langle 0\rangle' \mid X_0=\langle x\rangle'\} \\ &\qquad \times \frac{1}{|\mathcal{A}''_s|}\sum_{\langle w\rangle''\in\mathcal{A}''_s}\mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0=\langle w\rangle''\},\end{aligned}$$

which, together with (3.50) implies (3.43) with

$$P'_n = \frac{1}{|\mathcal{A}''_s|}\sum_{\langle w\rangle''\in\mathcal{A}''_s}\mathbb{P}\{X_\cdot \text{ reaches } \langle 0\rangle'' \text{ before } \langle 0\rangle' \mid X_0=\langle w\rangle''\}.$$

This proves (3.43), and as observed before, (3.47) follows by interchanging the roles T' and T'' in the proof of (3.43). □

As a result of Lemmas 12 and 13 we have

$$\liminf_{n\to\infty}\mathbb{P}\{R_n\le x\}\ge\mathbb{P}\{R'(\delta)+R''(\delta)\le x\} \tag{3.62}$$

at each continuity point of the right hand side, whenever (3.32) and (3.36) hold. Now set

$$R(e,\varepsilon,K)=\begin{cases} R^\varepsilon(e)=R(e)+\varepsilon & \text{if } R^\varepsilon\le K \\ \infty & \text{if } R^\varepsilon>K.\end{cases} \tag{3.63}$$

When $R(e)$ is replaced by $R(e,\varepsilon,K)$ we shall write $R_n(\varepsilon,K)$ (respectively $R(\delta;\varepsilon,K)$ or $R(T^\delta;\varepsilon,K)$) for the resistance between 0 and ∞ in K_{n+2} (respectively between $\langle 0\rangle$ and ∞ in T^δ). (3.62) applies when $R(e)$ is replaced by $R(e,\varepsilon,K)$ (provided (3.32) holds). This replacement only increases resistances. Consequently

$$\begin{aligned}\liminf_{n\to\infty}\mathbb{P}\{R_n\le x\} &\ge \liminf_{n\to\infty}\mathbb{P}\{R_n(\varepsilon,K)\le x\} \\ &\ge \mathbb{P}\{R'(\delta;\varepsilon,K)+R''(\delta;\varepsilon,K)\le x\}\end{aligned}$$

for each $0<\varepsilon<K<\infty$ and δ satisfying (3.32). Here, of course, $R'(\delta;\varepsilon,K)$ and $R''(\delta;\varepsilon,K)$ are independent copies of $R(\delta;\varepsilon,k)$. It follows that

$$\liminf_{n\to\infty}\mathbb{P}\{R_n\le x\}\ge\lim_{\varepsilon\downarrow 0}\lim_{\delta\uparrow\gamma}\lim_{K\to\infty}\mathbb{P}\{R'(\delta;\varepsilon,K)+R''(\delta;\varepsilon,K)\le x\}.$$

Theorem 3 will therefore be a consequence of Lemma 8 and the next lemma in which we remove the "truncations".

Lemma 14. *As $K \to \infty$, $\delta \uparrow \gamma$ and $\varepsilon \downarrow 0$ (in this order) $R(\delta; \varepsilon, K)$ converges in distribution to $R(\gamma) = R(T^\gamma)$.*

Proof. For any realization $t, r(\cdot)$ of T^δ and $R(\cdot)$ we consider a Markov chain $\{X_\nu\} = \{X_\nu(\varepsilon, K, t, r)\}$ on t with transition probability matrix

$$(3.64)\quad \begin{aligned} P(\langle y\rangle, \langle z\rangle) &= P(\langle y\rangle, \langle z\rangle; \varepsilon, K, t, r) \\ &= \left\{\sum_y \frac{1}{r(e;\varepsilon,K)}\right\}^{-1} \frac{1}{r(y,z;\varepsilon,K)}, \quad y, z \text{ adjacent on } t, \end{aligned}$$

where again $\sum_y$ runs over all edges e incident to y, $r(e;\varepsilon,K)$ is defined by (3.63) with R replaced by r and $r(y,z;\varepsilon,K)$ is $r(e;\varepsilon,K)$ for e the edge between y and z. As before we take $P(\langle y\rangle, \langle z\rangle) = 0$ if y and z are not adjacent in t. By (2.3) and (2.4)

$$\begin{aligned} &\{R(\delta;\varepsilon,K)\}^{-1} \\ &\quad = \sum_{\langle i\rangle \in T_1^\delta} \frac{1}{R(e(i);\varepsilon,K)} \mathbb{P}\{X_\nu(\varepsilon, K, T^\delta, R) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\}. \end{aligned}$$

First we show that we can take $K = \infty$, i.e., we prove

$$(3.65)\quad \begin{aligned} &\mathbb{P}\{X_\nu(\varepsilon, K, T^\delta, R) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\} \\ &\qquad \to \mathbb{P}\{X_\nu(\varepsilon, \infty, T^\delta, R) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\} \end{aligned}$$

in probability as $K \to \infty$. Here $X_\nu(\varepsilon, \infty, T^\delta, R)$ is defined by (3.64) and (3.63) with $K = \infty$, i.e., the resistance of e is taken as $R^\varepsilon(e)$. Note that when we refer to "convergence in probability" in (3.65) we view both sides as random variables, namely as functions of T^δ and $R(\cdot)$. Of course both sides are zero when T^δ is finite so that we can restrict ourselves to that part of the probability space where T^δ is infinite. On this event we define

$$\mathcal{B}_s = \mathcal{B}_s^\delta = \{\langle x\rangle \in T_s^\delta : T^\delta(x), \text{ the tree of descendants of } \langle x\rangle, \text{ is infinite}\}.$$

Clearly for any realization t, r of T, R with t infinite

$$\begin{aligned} &\mathbb{P}\{X_\nu(\varepsilon, K, t, r) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\} \\ &\qquad \le \mathbb{P}\{X_\nu(\varepsilon, K, t, r) \text{ reaches } \mathcal{B}_s \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\}. \end{aligned}$$

Also, for $s \ge 1$, X_ν is contained in a finite graph until it reaches $\mathcal{B}_s$, so that

$$\begin{aligned} &\lim_{K\to\infty} \mathbb{P}\{X_\nu(\varepsilon, K, t, r) \text{ reaches } \mathcal{B}_s \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\} \\ &\qquad = \mathbb{P}\{X_\nu(\varepsilon, \infty, t, r) \text{ reaches } \mathcal{B}_s \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\}. \end{aligned}$$

Therefore

$$\begin{aligned}(3.66)\quad \limsup_{K\to\infty} &\mathbb{P}\{X_\nu(\varepsilon, K, T^\delta, r) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\} \\ &\le \lim_{s\to\infty} \mathbb{P}\{X_\nu(\varepsilon, \infty, T^\delta, r) \text{ reaches } \mathcal{B}_s \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\} \\ &= \mathbb{P}\{X_\nu(\varepsilon, \infty, T^\delta, R) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\}.\end{aligned}$$

On the other hand

(3.67)

$$\begin{aligned}&\mathbb{P}\{X_\nu(\varepsilon, K, T^\delta, R) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\} \\ &\quad = \sum_{\langle x\rangle\in\mathcal{B}_s} \mathbb{P}\{X_\nu(\varepsilon, K, T^\delta, R) \text{ reaches } \mathcal{B}_s \text{ before } \langle 0\rangle \text{ and} \\ &\qquad\qquad \text{reaches } \mathcal{B}_s \text{ first at } \langle x\rangle \mid X_0 = \langle i\rangle\} \\ &\qquad\qquad \times \mathbb{P}\{X_\nu(\varepsilon, K, T^\delta, R) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = x\}.\end{aligned}$$

By Lemma 3 the last factor under the sum in the right hand side of (3.67) is at least

$$\frac{\rho(x;\varepsilon,K)}{\rho(x;\varepsilon,K) + R(T^\delta(x);\varepsilon,K)}, \tag{3.68}$$

where for $\langle x\rangle = \langle i_1, \dots, i_s\rangle \in T_s^\delta$

$$\rho(x;\varepsilon,K) = \sum_{j=1}^{s} R(e(i_1,\dots,i_j);\varepsilon,K) \ge s\varepsilon.$$

Consequently (3.68) is at least

$$1 - \min\left\{\frac{R(T^\delta(x);\varepsilon,K)}{s\varepsilon}, 1\right\},$$

and for each $\eta \in (0,1)$

$$\begin{aligned}&\mathbb{P}\{X_\nu(\varepsilon, K, T^\delta, R) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\} \\ &\quad \ge (1-\eta)\mathbb{P}\{X_\nu(\varepsilon, K, T^\delta, R) \text{ reaches } \mathcal{B}_s \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\} \\ &\qquad\qquad - \mathbb{P}\{R(T^\delta(Y_s);\varepsilon,K) \ge \eta\varepsilon s \mid X_0 = \langle i\rangle\},\end{aligned}$$

where Y_s denotes the position where $X_.$ hits $\mathcal{B}_s$ first, provided X_0 does hit $\mathcal{B}_s$; if $X_.$ does not hit $\mathcal{B}_s$ then $R(T^\delta(Y_s);\varepsilon,K)$ is taken to be zero. But until $X_.$ hits $\mathcal{B}_s$ it cannot enter $T^\delta(Y_s)$ and so knowledge of Y_s contains no information on $T^\delta(Y_s)$, nor on its resistances, except that $T^\delta(Y_s)$ must be infinite (that is the meaning of $Y_s \in \mathcal{B}_s$). Therefore the conditional distribution of $R(T^\delta(Y_s);\varepsilon,K)$ given $T^\delta_{[s]}$, $\mathcal{B}_s$ and Y_s, is simply the conditional distribution of $R(T^\delta;\varepsilon,K)$ given that T^δ is infinite. Thus, to prove (3.65) we merely have to prove that for each $\sigma > 0$ there exists a $K_\sigma < \infty$ and $y_\sigma < \infty$ such that

$$\mathbb{P}\{R(T^\delta;\varepsilon,K) \ge y \mid T^\delta \text{ is infinite}\} \le 2\sigma \quad \text{for all } K \ge K_\sigma \text{ and } y \ge y_\sigma. \tag{3.69}$$

To prove (3.69) first choose $\widehat{K}$ such that

$$\delta \mathbb{P}\{R(e;\varepsilon,\widehat{K}) < \infty\} = \delta F(\widehat{K}-\varepsilon) > 1.$$

Now consider a $K > \widehat{K}$. $R(T^\delta;\varepsilon,K)$ is the resistance between $\langle 0\rangle$ and ∞ in the tree obtained by removing from T^δ each edge e with $R(e) > K-\varepsilon$ and by replacing $R(e)$ by $R^\varepsilon(e)$ on the other edges. If with an edge from $\langle i_0,\dots,i_n\rangle$ to $\langle i_0,\dots,i_{n+1}\rangle$ which gets removed — because its resistance exceeds $(K-\varepsilon)$ — we also remove all of $T^\delta(i_0,\dots,i_{n+1})$, then the resulting tree is just the family tree $\widetilde{T}$ of a branching process whose offspring distribution puts mass

$$\widetilde{p}_m := \sum_{n\ge m} \mathbb{P}\{|T_1^\delta| = n\}\binom{n}{m} F^m(K-\varepsilon)(1-F(K-\varepsilon))^{n-m}$$

on m (compare with $\widetilde{T}$ and $\widetilde{p}_m$ in Lemma 2). The mean of this offspring distribution is

$$\mathbb{E}|T_1^\delta| F(K-\varepsilon) = \delta F(K-\varepsilon) \ge \delta F(\widehat{K}-\varepsilon) > 1.$$

Consequently, by Lemma 2

$$\mathbb{P}\{R(T^\delta;\varepsilon,K) = \infty\} = \mathbb{P}\{\widetilde{T} \text{ is finite}\} = \widetilde{q}(K), \tag{3.70}$$

where $\widetilde{q} = \widetilde{q}(K)$ is the unique root in $[0,1)$ of

$$x = \mathbb{E}x^{|\widetilde{T}_1|} = \sum_{m=0}^\infty \widetilde{p}_m x^m.$$

Now, since $\widetilde{T}$ is a subgraph of T^δ,

(3.71)

$$\begin{aligned}
&\mathbb{P}\{R(T^\delta;\varepsilon,K) \ge y \mid T^\delta \text{ is infinite}\}\\
&\quad = \frac{1}{1-q(\delta)} \mathbb{P}\{R^\varepsilon(\widetilde{T}) \ge y \text{ and } T^\delta \text{ is infinite}\}\\
&\quad = \frac{1}{1-q(\delta)} \Big[\mathbb{P}\{\widetilde{T} \text{ is finite but } T^\delta \text{ is infinite}\} + \mathbb{P}\{R^\varepsilon(\widetilde{T}) \ge y \text{ and } \widetilde{T} \text{ is infinite}\}\Big]\\
&\quad = \frac{\widetilde{q}(K) - q(\delta)}{1-q(\delta)} + \frac{1-\widetilde{q}(K)}{1-q(\delta)} \mathbb{P}\{R^\varepsilon(\widetilde{T}) \ge y \mid \widetilde{T} \text{ is infinite}\}.
\end{aligned}$$

Since

$$\lim_{K\to\infty} \mathbb{E}x^{|\widetilde{T}_1|} = \mathbb{E}x^{|T_1^\delta|} \quad \text{uniformly on } [0,1],$$

one easily sees that

$$\lim_{K\to\infty} \widetilde{q}(K) = q(\delta).$$

We can therefore first choose $K_\sigma > \widehat{K}$ large to make

$$\frac{\widetilde{q}(K_\sigma) - q(\delta)}{1 - q(\delta)} \le \sigma,$$

and then choose y so large that the second term in the last member of (3.71) with $K = K_\sigma$ is at most σ (by (3.70)). Since $R(T^\delta; \varepsilon, K) \le R(T^\delta; \varepsilon, K_\sigma)$ for $K \ge K_\sigma$ (by (2.6)) this proves (3.69), and consequently also (3.65).

(3.65) allows us to take $K = \infty$. Next we should prove

(3.72)
$$\begin{aligned}\mathbb{P}\{X_\nu(\varepsilon, \infty, T^\delta, R) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\}\\ \to \mathbb{P}\{X_\nu(\varepsilon, \infty, T^\gamma, R) \text{ reaches } \infty \text{ before } \langle 0\rangle \mid X_0 = \langle i\rangle\}\end{aligned}$$

in probability as $\delta \uparrow \gamma$. This will allow us to prove that we may take $\delta = \gamma$. We do not give the details for (3.72). It is very similar to the proof of (3.65) if we take account of the fact that we realize T^δ for all $\delta < \gamma$ and T^γ on the same probability space such that $T^\delta \subset T^\gamma$ and $T^\delta_{[s]} \uparrow T^\gamma_{[s]}$ as $\delta \uparrow \gamma$. To make this construction we merely have to choose a uniform variable $U(e)$ for each edge e in T^γ, and then for the construction of T^δ remove e (and its successors) if and only if $U(e) > \delta/\gamma$. One easily sees from the fact that the number of edges from $\langle i_1, \dots, i_n\rangle$ to T^γ_{n+1} has a Poisson distribution with mean γ that the resulting tree has the same distribution as T^δ.

Once we have (3.65) and (3.72) we obtain that

$$\lim_{\delta\uparrow\gamma} \lim_{K\to\infty} R(\delta; \varepsilon, K) = R^\varepsilon(T^\gamma) \quad \text{in distribution.}$$

Finally, we may let $\varepsilon \downarrow 0$ by virtue of Proposition 1. □

4. Some further remarks concerning the proof of Theorem 1

Theorem 1 in Grimmett and Kesten (1983) asserts that

(4.1) $$\gamma(n) R_n \to 2\left\{\int_{[0,\infty)} x^{-1}\, dF(x)\right\}^{-1} \quad \text{in probability}$$

if $\gamma(n) \to \infty$ as $n \to \infty$. In view of Theorem 4 of Grimmett and Kesten (1983) and a passage to subsequences we may restrict ourselves to the case where

(4.2) $$\gamma(n) \to \infty \quad \text{but} \quad \frac{\log \gamma(n)}{\log n} \to 0.$$

Also we only have to prove an upper bound for R_n, since Lemma 5 in Grimmett and Kesten (1983) provides the necessary lower bound.

To obtain an upper bound for R_n, let $\eta > 0$ be a small number and $K < \infty$ such that

$$F(K) \geq 1 - \eta.$$

Define m_n by (3.16) with $\gamma = \gamma(n)$ and set $s_n = (m_n)^{1/2}$. One can then show that with probability tending to 1 (as $n \to \infty$) the following events (4.3)–(4.5) do occur:

(4.3) $\tau^0_{[m_n]}$ and $\tau^\infty_{[m_n]}$ as constructed in the beginning of Sect. 3 contain subtrees $\widetilde{T}$ and $\widehat{T}$ with roots at 0 and ∞, respectively. $\widetilde{T}$ and $\widehat{T}$ are disjoint, and each has m_n generations. Each vertex of $\widetilde{T}$ which is not in the m_nth generation of $\widetilde{T}$ has exactly $\lfloor (1-2\eta)\gamma(n) \rfloor$ children in $\widetilde{T}$. The last statement remains true when $\widetilde{T}$ is everywhere replaced by $\widehat{T}$.

(4.4) Each edge in $\widetilde{T}$ not incident to 0 and each edge in $\widehat{T}$ not incident to ∞ has resistance at most K.

(4.5) For each vertex x of the s_nth generation of $\widetilde{T}$ and each vertex y of the s_nth generation of $\widehat{T}$ there exists a path from x to y of $(2m_n - 2s_n + 1)$ edges of K_{n+2} of resistance $\leq K$. This path consists of three pieces: (i) a path of $(m_n - s_n)$ edges in $\widetilde{T}$ in the tree of descendants of x, going from x to a vertex u in the m_nth generation of $\widetilde{T}$; (ii) a single edge from u to a vertex v of the m_nth generation of $\widehat{T}$ (this edge does not belong to $\widetilde{T} \cup \widehat{T}$); (iii) a path of $(m_n - s_n)$ edges in $\widehat{T}$ from v to y in the tree of descendants of y. For distinct pairs x, y the edges between the m_nth generations of $\widetilde{T}$ and $\widehat{T}$ in the corresponding paths are different.

The proofs of (4.3)–(4.5) are analogous to those of Lemmas 7, 10 and 11. Note that (4.3)–(4.5) contain no statement about the resistances of the edges incident to 0 in $\widetilde{T}$ and the edges incident to ∞ in $\widehat{T}$, except that they are edges of $\tau^0_{[m_n]}$ and $\tau^\infty_{[m_n]}$. Accordingly, the conditional distribution of the resistances of these edges is simply F. Since we are only looking for an upper bound for R_n we may raise all other resistances in $\widetilde{T}$ and $\widehat{T}$, and those between the m_nth generation of $\widetilde{T}$ and the m_nth generation of $\widehat{T}$ to K. From here on the proof proceeds exactly as in Lemmas 12 and 13 with the following replacements for $T'_{[s]}$, $T''_{[s]}$ and $M(n)$. $T'_{[s]}$ and $T''_{[s]}$ each are trees — rooted at 0 and ∞, respectively — of $s = s_n$ generations, in which each vertex except those of the s_nth generation has exactly $\lfloor (1-2\eta)\gamma(n) \rfloor$ children. Each edge not incident to one of the roots has resistance K, while all edges incident to one of the roots have independent resistances, chosen according to the distribution function F. $M(n)$ consists of $T'_{[s]}$, $T''_{[s]}$ plus an edge between each pair of vertices x, y with x in the sth generation of T' and y in the sth generation of T''. The latter edges are distinct and each has resistance

$$2m_n K\{\lfloor (1-2\eta)\gamma(n) \rfloor\}^{s_n}. \tag{4.6}$$

(4.6) takes the place of (3.31). With these replacements the proofs of Lemma 12 and 13 need no significant changes. $\mathcal{A}'_s$ ($\mathcal{A}''_s$) is simply replaced by the full sth generation of T' (T'') and no appeal to Lemma 6 is needed, since now the analogue of $\pi(\langle x\rangle, T_{[m]}, R, s)$ becomes

$$\mathbb{P}\{X.\text{ reaches } \langle 0\rangle \text{ before } T_s \mid X_0 = \langle x\rangle\} \tag{4.7}$$

for $\langle x\rangle \in T_{s-1}$, and this last probability is always $\leq (2L_n/(2L_n+K))^{s-1}$, where $2L_n$ is the resistance of an infinite tree in which each vertex has $\lfloor(1-2\eta)\gamma(n)\rfloor$ children and each edge has resistance K (compare proof of Lemma 6; actually $L_n \to 0$ as $n \to \infty$, but that is not important). It is also easy to see that for each K

$$(1-2\eta)\gamma(n)\{\text{resistance between 0 and } T'_s \text{ in } T'_{[s]}\} \to \left\{\int \frac{1}{x}\,dF(x)\right\}^{-1} \quad \text{in probability as } n \to \infty.$$

It is not necessary to remove truncations — as done in Lemma 14 — in the present case. In particular Prop. 1 is not needed when (4.2) holds. We merely have to take the limit as $\eta \downarrow 0$ in the preceding estimates.

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(GRG, current address) Statistical Laboratory, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, United Kingdom
E-mail: g.r.grimmett@statslab.cam.ac.uk
URL: http://www.statslab.cam.ac.uk/~grg/

(HK) Mathematics Department, Cornell University, Ithaca NY 14853, USA
E-mail: kesten@math.cornell.edu

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