SHARP THRESHOLDS FOR THE RANDOM-CLUSTER AND ISING MODELS

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Abstract. A sharp-threshold theorem is proved for box-crossing probabilities on the square lattice. The models in question are the random-cluster model near the self-dual point $p_{\mathrm{sd}}(q) = \sqrt{q}/(1+\sqrt{q})$, the Ising model with external field, and the coloured random-cluster model. The principal technique is an extension of the influence theorem for monotonic probability measures applied to increasing events with no assumption of symmetry.

1. Introduction

The method of 'sharp threshold' has been fruitful in probabilistic combinatorics (see [20, 27] for recent reviews). It provides a fairly robust tool for showing the existence of a sharp threshold for certain processes governed by independent random variables. Its most compelling demonstration so far in the field of physical systems has been the proof in [9] that the critical probability of site percolation on the Voronoi tessellation generated by a Poisson process on $\mathbb{R}^2$ equals $\frac{1}{2}$.

Each of the applications alluded to above involves a product measure. It was shown in [16] that the method may be extended to non-product probability measures satisfying the FKG lattice condition. The target of this note is to present two applications of such a sharp-threshold theorem to measures arising in statistical physics, namely those of the random-cluster model and the Ising model. In each case, the event in question is the existence of a crossing of a large box, by an open path in the case of the random-cluster model, and by a single-spin path in the case of the Ising model. A related but more tentative and less complete result has been obtained in [16] in the first case, and the second case has been studied already in [23, 24] and [7].

Our methods for the Ising model can be applied to a more general model termed here the coloured random-cluster model (CRCM), see Section 8. This model is related to the so-called fractional Potts model of [26], and the fuzzy Potts model and the divide-and-colour model of [5, 13, 21, 22].

Date: 6 March 2009, revised 9 February 2010.

2000 *Mathematics Subject Classification.* 60K35, 82B20, 60E15.

Key words and phrases. Random-cluster model, Potts model, Ising model, percolation, box-crossing, influence, sharp threshold, coloured random-cluster model, fuzzy Potts model.

The sharp-threshold theorem used here is an extension of that given for product measure in [15, 37], and it makes use of the results of [16]. It is stated, with an outline of the proof, in Section 5. The distinction of the current sharp-threshold theorem is that it makes no assumption of symmetry on either the event or measure in question. Instead, one needs to estimate the maximum influence of the various components, and it turns out that this may be done in a manner which is very idiomatic for the models in question. The sharp-threshold theorem presented here may find further applications in the study of dependent random variables.

2. The models

2.1. **The random-cluster model.** The random-cluster model on a connected graph G has two parameters: an edge-weight p and a cluster-weight q. See Section 3 for a formal definition. When $q \geq 1$ and G is infinite, there is a critical value $p_c(q)$ that separates the subcritical phase of the model (when $p < p_c(q)$ and there exist no infinite clusters) and the supercritical phase. It has long been conjectured that, when G is the square lattice $\mathbb{Z}^2$,

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}, \qquad q \geq 1. \tag{2.1}$$

This has been proved rigorously in three famous cases. When $q = 1$, the random-cluster model is bond percolation, and the exact calculation $p_c(1) = \frac{1}{2}$ was shown by Kesten [28]. When $q = 2$, the model is intimately related to the Ising model, and the calculation of $p_c(2)$ is equivalent to that of Onsager and others concerning the Ising critical temperature (see [1, 3] for a modern treatment of the Ising model). Formula (2.1) has been proved for sufficiently large values of q (currently $q \geq 21.61$) in the context of the proof of first-order phase transition, see [19, 29, 30, 31]. We recall that, when $q \in \{2, 3, \dots\}$, the critical temperature T_c of the q-state Potts model on a graph G satisfies

$$p_c(q) = 1 - e^{-1/T_c}. \tag{2.2}$$

A fairly full account of the random-cluster model, and its relation to the Potts model, may be found in [19].

Conjecture (2.1) is widely accepted. Physicists have proceeded beyond a 'mere' calculation of the critical point, and have explored the behaviour of the process at and near this value. For example, it is believed that there is a continuous (second-order) phase transition if $1 \leq q < 4$, and a discontinuous (first-order) transition when $q > 4$, see [6]. Amongst recent progress, we highlight the stochastic Löwner evolution process $\mathrm{SLE}_{16/3}$ associated with the cluster boundaries in the critical case when $q = 2$ and $p = \sqrt{2}/(1+\sqrt{2})$, see [36, 35].

The expression in (2.1) arises as follows through the use of planar duality. When the underlying graph G is planar, it possesses a (Whitney) dual graph G_{d}. The random-cluster model on G with parameters p, q may be related to a dual random-cluster model on G_{d} with parameters p_{d}, q, where

$$\frac{p_{\mathrm{d}}}{1-p_{\mathrm{d}}} = \frac{q(1-p)}{p}. \tag{2.3}$$

The mapping $p \mapsto p_{\mathrm{d}}$ has a fixed point $p = p_{\mathrm{sd}}(q)$, where

$$p_{\mathrm{sd}}(q) := \frac{\sqrt{q}}{1+\sqrt{q}}$$

is termed the *self-dual point*. The value $p = p_{\mathrm{sd}}(q)$ is especially interesting when G and G_{d} are isomorphic, as in the case of the square lattice $\mathbb{Z}^2$. See [19, Chap. 6]. We note for future use that

$$p < p_{\mathrm{sd}}(q) \quad \text{if and only if} \quad p_{\mathrm{d}} > p_{\mathrm{sd}}(q). \tag{2.4}$$

Henceforth, we take $G = \mathbb{Z}^2$. The inequality

$$p_{\mathrm{c}}(q) \geq p_{\mathrm{sd}}(q), \qquad q \geq 1, \tag{2.5}$$

was proved in [17, 38] using Zhang's argument (see [18, p. 289]). Two further steps would be enough to imply the complementary inequality $p_{\mathrm{c}}(q) \leq p_{\mathrm{sd}}(q)$: firstly, that the probability of crossing a box $[-m, m]^2$ approaches 1 as $m \to \infty$, when $p > p_{\mathrm{sd}}(q)$; and secondly, that this implies the existence of an infinite cluster. The first of these two claims is proved in Theorem 3.4.

Kesten's proof for percolation, [28], may be viewed as a proof of the first claim in the special case $q = 1$. The second claim follows for percolation by RSW-type arguments, see [32, 33, 34] and [18, Sect. 11.7]. Heavy use is made in these works of the fact that the percolation measure is a product measure, and this is where the difficulty lies for the random-cluster measure.

We prove our main theorem (Theorem 3.4 below) by the method of influence and sharp threshold developed for product measures in [15, 25]. This was adapted in [16] to monotonic measures applied to increasing events, subject to a certain hypothesis of symmetry. We show in Section 5 how this hypothesis may be removed, and we apply the subsequent inequality in Section 6 to the probability of a box-crossing, thereby extending to general q the corresponding argument of [10].

2.2. **Ising model.** We shall consider the Ising model on the square lattice $\mathbb{Z}^2$ with edge-interaction parameter β and external field h. See Section 4 for the relevant definitions. Write β_{c} for the critical value of β when $h = 0$, so that

$$1 - e^{-2\beta_{\mathrm{c}}} = p_{\mathrm{sd}}(2)$$

where $p_{\mathrm{sd}}(2)$ is given as in (2.1). Two notions of connectivity are required: the usual connectivity relation $\leftrightarrow$ on $\mathbb{Z}^2$ viewed as a graph, and the relation $\leftrightarrow_*$, termed $*$-connectivity, and obtained by adding diagonals to each unit face of $\mathbb{Z}^2$. Let $\pi_{\beta,h}$ denote the Ising measure on $\mathbb{Z}^2$ with parameters β, h.

Higuchi proved in [23, 24] that, when $\beta \in (0, \beta_c)$, there exists a critical value $h_c = h_c(\beta)$ of the external field such that:

(a) $h_c(\beta) > 0$,
(b) when $h > h_c$, there exists $\pi_{\beta,h}$-almost-surely an infinite $+$ cluster of $\mathbb{Z}^2$, and the radius of the $*$-connected $-$ cluster at the origin has exponential tail,
(c) when $0 < h < h_c$, there exists $\pi_{\beta,h}$-almost-surely an infinite $*$-connected $-$ cluster of $\mathbb{Z}^2$, and the radius of the $+$ cluster at the origin has exponential tail.

A further approach to Higuchi's theorem has been given recently by van den Berg [7]. A key technique of the last paper is a sharp-threshold theorem of Talagrand [37] for product measures. The Ising measure $\pi_{\Lambda,\beta,h}$ on a box Λ is of course not a product measure, and so it was necessary to encode it in terms of a family of independent random variables. We show here that the influence theorem of [16] may be extended and applied directly to the Ising model to obtain the necessary sharp threshold result. (The paper [7] contains results for certain other models encodable in terms of product measures, and these appear to be beyond the scope of the current method.)

2.3. **Coloured random-cluster model.** The Ising model with external field is a special case of a class of systems that have been studied by a number of authors, and which we term *coloured random-cluster models* (CRCM). Sharp-threshold results may be obtained for such systems also. Readers are referred to Section 8 for an account of the CRCM and the associated results.

3. Box-crossings in the random-cluster model

The random-cluster measure is given as follows on a finite graph $G = (V, E)$. The configuration space is $\Omega = \{0, 1\}^E$. For $\omega \in \Omega$, we write $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ for the set of 'open' edges, and $k(\omega)$ for the number of connected components in the open graph $(V, \eta(\omega))$. Let $p \in [0, 1]$, $q \in (0, \infty)$, and let $\phi_{p,q}$ be the probability measure on Ω given by

$$\phi_{p,q}(\omega) = \frac{1}{Z}\left\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\}q^{k(\omega)}, \qquad \omega \in \Omega, \tag{3.1}$$

where $Z = Z_{G,p,q}$ is the normalizing constant. We shall assume throughout this paper that $q \geq 1$, so that $\phi_{p,q}$ satisfies the so-called FKG lattice condition

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \geq \mu(\omega_1)\mu(\omega_2), \qquad \omega_1, \omega_2 \in \Omega. \tag{3.2}$$

Here, as usual,

$$\omega_1 \vee \omega_2(e) = \max\{\omega_1(e), \omega_2(e)\},$$
$$\omega_1 \wedge \omega_2(e) = \min\{\omega_1(e), \omega_2(e)\},$$

for $e \in E$. As a consequence of (3.2), $\phi_{p,q}$ satisfies the FKG inequality. See [19] for the basic properties of the random-cluster model.

Consider the square lattice $\mathbb{Z}^2$ with edge-set $\mathbb{E}$, and let $\Omega = \{0,1\}^{\mathbb{E}}$. Let $\Lambda = \Lambda_n = [-n,n]^2$ be a finite box of $\mathbb{Z}^2$, with edge-set $\mathbb{E}_\Lambda$. For $b \in \{0,1\}$ define

$$\Omega_\Lambda^b = \{\omega \in \Omega : \omega(e) = b \text{ for } e \notin \mathbb{E}_\Lambda\}.$$

On Ω_Λ^b we define a random-cluster measure $\phi_{\Lambda,p,q}^b$ as follows. For $p \in [0,1]$ and $q \in [1,\infty)$, let

$$\phi_{\Lambda,p,q}^b(\omega) = \frac{1}{Z_{\Lambda,p,q}^b} \left\{ \prod_{e \in \mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)}, \quad \omega \in \Omega_\Lambda^b, \tag{3.3}$$

where $k(\omega,\Lambda)$ is the number of clusters of $(\mathbb{Z}^2, \eta(\omega))$ that intersect Λ. The boundary condition $b = 0$ (respectively, $b = 1$) is usually termed 'free' (respectively, 'wired'). It is standard that the weak limits

$$\phi_{p,q}^b = \lim_{n\to\infty} \phi_{\Lambda_n,p,q}^b$$

exist, and that they are translation-invariant, ergodic, and satisfy the FKG inequality. See [19, Chap. 4].

For $A, B \subseteq \mathbb{Z}^2$, we write $A \leftrightarrow B$ if there exists an open path joining some $a \in A$ to some $b \in B$. We write $x \leftrightarrow \infty$ if the vertex x is the endpoint of some infinite open path. The percolation probabilities are given as

$$\theta^b(p,q) = \phi_{p,q}^b(0 \leftrightarrow \infty), \qquad b = 0,1.$$

Since each θ^b is non-decreasing in p, one may define the critical point by

$$p_c(q) = \sup\{p : \theta^1(p,q) = 0\}.$$

It is known that $\phi_{p,q}^0 = \phi_{p,q}^1$ if $p \neq p_{sd}(q)$, and we write $\phi_{p,q}$ for the common value. In particular, $\theta^0(p,q) = \theta^1(p,q)$ for $p \neq p_c(q)$. It is conjectured that $\phi_{p,q}^0 = \phi_{p,q}^1$ when $p = p_c(q)$ and $q \le 4$.

Let $B_k = [0,k] \times [0,k-1]$, and let H_k be the event that B_k possesses an open left–right crossing. That is, H_k is the event that B_k contains an open path having one endvertex on its left side and one on its right side.

Theorem 3.4. *Let $q \ge 1$. We have that*

$$\phi_{p,q}(H_k) \le 2\rho_k^{p_{sd}-p}, \qquad 0 < p < p_{sd}(q), \tag{3.5}$$
$$\phi_{p,q}(H_k) \ge 1 - 2\nu_k^{p-p_{sd}}, \qquad p_{sd}(q) < p < 1, \tag{3.6}$$

for $k \ge 1$, where

$$\rho_k = [2q\eta_k/p]^{c/q}, \qquad \nu_k = [2q\eta_k/p_{\mathrm{d}}]^{c/q}, \tag{3.7}$$

and

$$\eta_k = \phi^0_{p_{\mathrm{sd}}(q),q}(0 \leftrightarrow \partial\Lambda_{k/2}) \to 0 \qquad \text{as } k \to \infty. \tag{3.8}$$

Here, c is an absolute positive constant, and p_{d} satisfies (2.3).

When k is odd, we interpret $\partial\Lambda_{k/2}$ in (3.8) as $\partial\Lambda_{\lfloor k/2\rfloor}$.

In essence, the probability of a square-crossing has a sharp threshold around the self-dual 'pivot' $p_{\mathrm{sd}}(q)$. Related results were proved in [16], but with three relative weaknesses, namely: only non-square rectangles could be handled, the 'pivot' of the threshold theorems was unidentified, and there was no result for *infinite-volume* measures. The above strengthening is obtained by using the threshold Theorem 5.3 which makes no assumption of symmetry on the event or measure in question. The corresponding threshold theorem for product measure leads to a simplification of the arguments of [10] for percolation, see [20, Sect. 5.8].

Since $\phi^0_{\Lambda_n,p,q} \le_{\mathrm{st}} \phi_{p,q} \le_{\mathrm{st}} \phi^1_{\Lambda_n,p,q}$ and H_k is an increasing event, Theorem 3.4 implies certain inequalities for finite-volume probabilities also.

No estimate for the rate at which $\eta_k \to 0$ is implicit in the arguments of this paper, and indeed one of the targets of the current work is to show that no estimate is necessary for sharp threshold. It is expected that $\eta_k \to 0$ at a rate that depends on whether or not the phase transition is continuous: one expects that η_k decays as a power when $1 \le q < 4$, and as an exponential when $q > 4$ (see [19, Sect. 6.4]). This would imply a threshold of order either $1/\log k$ or $1/k$ in (3.5)–(3.6). That the radius R of the open cluster at the origin is $\phi^0_{p_{\mathrm{sd}}(q),q}$-a.s. finite is a consequence of the (a.s.) uniqueness of the infinite open cluster whenever it exists. See [19, Thm 6.17(a)] for a proof of the relevant fact that

$$\theta^0(p_{\mathrm{sd}}(q), q) = 0, \qquad q \ge 1. \tag{3.9}$$

We shall prove a slightly more general result than Theorem 3.4. Let $B_{k,m} = [0,k] \times [0,m]$ and let $H_{k,m}$ be the event that there exists an open left–right crossing of $B_{k,m}$.

Theorem 3.10. *Let $q \ge 1$. We have that*

$$\phi_{p_1,q}(H_{k,m})[1 - \phi_{p_2,q}(H_{k,m}))] \le \rho_k^{p_2-p_1}, \qquad 0 < p_1 < p_2 \le p_{\mathrm{sd}}(q), \tag{3.11}$$

$$\phi_{p_1,q}(H_{k,m})[1 - \phi_{p_2,q}(H_{k,m}))] \le \nu_{m+1}^{p_2-p_1}, \qquad p_{\mathrm{sd}}(q) \le p_1 < p_2 < 1, \tag{3.12}$$

for $k, m \ge 1$, where ρ_k (respectively, ν_k) is given in (3.7) *with $p = p_1$ (respectively, $p = p_2$), and $\phi_{p_{\mathrm{sd}}(q),q}$ is to be interpreted as $\phi^0_{p_{\mathrm{sd}}(q),q}$.*

4. Box-crossings in the Ising model

Let Λ be a box of $\mathbb{Z}^2$. The spin-space of the Ising model on Λ is $\Sigma_\Lambda = \{-1,+1\}^\Lambda$, and the Hamiltonian is

$$H_\Lambda(\sigma) = -\beta \sum_{e=\langle x,y\rangle \in \mathbb{E}_\Lambda} \sigma_x \sigma_y - h \sum_{x\in\Lambda} \sigma_x,$$

where $\beta > 0$, $h \geq 0$. The relevant Ising measure is given by

$$\pi_{\Lambda,\beta,h}(\sigma) \propto e^{-H_\Lambda(\sigma)}, \qquad \sigma \in \Sigma_\Lambda,$$

and it is standard that the (weak) limit measure $\pi_{\beta,h} = \lim_{\Lambda\to\mathbb{Z}^2} \pi_{\Lambda,\beta,h}$ exists. We shall also need the + boundary-condition measure $\pi^+_{\beta,0}$ given as the weak limit of $\pi_{\Lambda,\beta,0}$ conditional on $\sigma_x = +1$ for $x \in \partial\Lambda$. (Here, $\partial\Lambda$ denotes as usual the boundary of Λ, i.e., the set of $x \in \Lambda$ possessing a neighbour not belonging to Λ). By the FKG inequality or otherwise, $\pi^+_{\beta,0}(\sigma_0) \geq 0$, and the critical value of β when $h = 0$ is given by

$$\beta_c = \sup\{\beta : \pi^+_{\beta,0}(\sigma_0) = 0\}.$$

As remarked in Section 2, $1 - e^{-2\beta_c} = p_{sd}(2)$. It is well known that there exists a unique infinite-volume measure for the Ising model on $\mathbb{Z}^2$ if either $h \neq 0$ or $\beta < \beta_c$, and thus $\pi_{\beta,h}$ is this measure. By Holley's Theorem, (see [19, Sect. 2.1], for example), $\pi_{\beta,h}$ is stochastically increasing in h.

Let

$$\theta^+(\beta,h) = \pi_{\beta,h}(0 \overset{+}{\leftrightarrow} \infty), \quad \theta^-(\beta,h) = \pi_{\beta,h}(0 \overset{-}{\leftrightarrow}_* \infty),$$

where the relation $\overset{+}{\leftrightarrow}$ (respectively, $\overset{-}{\leftrightarrow}_*$) means that there exists a path of $\mathbb{Z}^2$ each of whose vertices has state $+1$ (respectively, a $*$-connected path of vertices with state -1). The next theorem states the absence of coexistence of such infinite components, and its proof (given in Section 7) is a simple application of the Zhang argument for percolation (see [18, Sect. 11.3]).

Theorem 4.1. *We have that*

$$\theta^+(\beta,h)\theta^-(\beta,h) = 0, \qquad \beta \geq 0,\ h \geq 0.$$

There exists $h_c = h_c(\beta) \in [0,\infty)$ such that

$$\theta^+(\beta,h) \begin{cases} = 0 & \text{if } 0 \leq h < h_c, \\ > 0 & \text{if } h > h_c. \end{cases}$$

Recall from [23, 24] that $h_c(\beta) > 0$ if and only if $\beta < \beta_c$. It is proved in [24] that

$$\theta^\pm(\beta, h_c(\beta)) = 0, \tag{4.2}$$

but we shall not make use of this fact in the proofs of this paper. Indeed, one of the main purposes of this article is to show how certain sharp-thresholds for box-crossings may be obtained using a minimum of background information on the model in question.

Let $H_{k,m}$ be the event that there exists a left–right + crossing of the box $B_{k,m} = [0,k] \times [0,m]$. Let $x^+ = \max\{x, 0\}$.

Theorem 4.3. *Let $0 \le \beta < \beta_c$ and $R > 0$. There exist $\rho_{i,+} = \rho_{i,+}(\beta)$ and $\rho_{i,-} = \rho_{i,-}(\beta, R)$ satisfying*

$$\rho_{i,+}\rho_{i,-} \to 0 \qquad \text{as } i \to \infty, \tag{4.4}$$

such that: for $0 \le h_1 \le h_c \le h_2 < R$,

$$\pi_{\beta,h_1}(H_{k,m})[1 - \pi_{\beta,h_2}(H_{k,m})] \le \rho_{k,+}^{h_c - h_1}\rho_{m,-}^{h_2-h_c}, \qquad k, m \ge 1. \tag{4.5}$$

The proof of this theorem shows also that

$$\pi_{\beta,h_1}(H_{k,m})[1 - \pi_{\beta,h_2}(H_{k,m})] \le \rho_{k,+}^{h_2-h_1}, \qquad h_1 \le h_2 \le h_c,$$
$$\pi_{\beta,h_1}(H_{k,m})[1 - \pi_{\beta,h_2}(H_{k,m})] \le \rho_{m,-}^{h_2-h_1}, \qquad h_c \le h_1 \le h_2.$$

As in Theorem 3.4, the proof neither uses nor implies any estimate on the rate at which $\rho_{i,\pm} \to 0$. The $\rho_{i,\pm}$ are related to the tails of the radii of the + cluster and the $-$ $*$-cluster at the origin. More explicitly,

$$\rho_{i,+} = \left[2(1+e^{8\beta})\pi_{\beta,h_c}(0 \overset{+}{\leftrightarrow} \partial\Lambda_{i/2})\right]^{B_+}, \tag{4.6}$$

$$\rho_{i,-} = \left[2(1+e^{8\beta+2R})\pi_{\beta,h_c}(0 \overset{-}{\leftrightarrow}_* \partial\Lambda_{i/2})\right]^{B_-}, \tag{4.7}$$

where

$$B^+ = 2c\xi_{\beta,h_c}, \quad B_- = 2c\xi_{\beta,R},$$

and $\xi_{\beta,h}$ is given in the forthcoming (7.4). Equation (4.4) holds by Theorem 4.1 with $h = h_c(\beta)$. It is in fact a consequence of (4.2) that $\rho_{i,\pm} \to 0$ as $i \to \infty$.

5. Influence, and sharp threshold

Let S be a finite set. Let μ be a measure on $\Omega = \{0,1\}^S$ satisfying the FKG lattice condition (3.2), and assume that μ is *positive* in that $\mu(\omega) > 0$ for all $\omega \in \Omega$. It is standard that, for a positive measure μ, (3.2) is equivalent to the condition that μ be *monotone*, which is to say that the one-point conditional measure $\mu(\sigma_x = 1 \mid \sigma_y = \eta_y \text{ for } y \ne x)$ is non-decreasing in η. Furthermore, (3.2) implies that μ is positively associated, in that increasing events are positively correlated. See, for example, [19, Chap. 2].

For $p \in (0,1)$, let μ_p be given by

$$\mu_p(\omega) = \frac{1}{Z_p}\left\{\prod_{s\in S} p^{\omega(s)}(1-p)^{1-\omega(s)}\right\}\mu(\omega), \qquad \omega \in \Omega, \tag{5.1}$$

where Z_p is chosen in such a way that μ_p is a probability measure. It is easy to check that each μ_p satisfies the FKG lattice condition.

Let A be an increasing event, and write 1_A for its indicator function. We define the *(conditional) influence* of the element $s \in S$ on the event A by

$$J_{A,p}(s) = \mu_p(A \mid 1_s = 1) - \mu_p(A \mid 1_s = 0), \qquad s \in S, \tag{5.2}$$

where 1_s is the indicator function that $\omega(s) = 1$. Note that $J_{A,p}(s)$ depends on the choice of μ. The conditional influence is not generally equal to the (absolute) influence of [25],

$$I_{A,p}(s) = \mu_p(1_A(\omega^s) \neq 1_A(\omega_s)),$$

where the configuration ω^s (respectively, ω_s) is that obtained from ω by setting $\omega(s) = 1$ (respectively, $\omega(s) = 0$).

Theorem 5.3. *There exists a constant $c > 0$ such that the following holds. For any such S, μ, and any increasing event $A \neq \varnothing, \Omega$,*

$$\frac{d}{dp}\mu_p(A) \geq \frac{c\xi_p}{p(1-p)}\mu_p(A)(1-\mu_p(A))\log[1/(2m_{A,p})], \tag{5.4}$$

where $m_{A,p} = \max_{s\in S} J_{A,p}(s)$ and $\xi_p = \min_{s\in S}\big[\mu_p(1_s)(1-\mu_p(1_s))\big]$.

Corollary 5.5. *In the notation of Theorem 5.3,*

$$\mu_{p_1}(A)[1-\mu_{p_2}(A)] \leq \kappa^{B(p_2-p_1)}, \qquad 0 < p_1 \leq p_2 < 1,$$

where

$$B = \inf_{p\in(p_1,p_2)}\left\{\frac{c\xi_p}{p(1-p)}\right\}, \qquad \kappa = 2\sup_{\substack{p\in(p_1,p_2),\\ s\in S}} J_{A,p}(s).$$

The corresponding inequality for product measures may be found in [37, Cor. 1.2]. Throughout this note, the letter c shall refer only to the constant of Theorem 5.3.

Proof of Theorem 5.3. It is proved in [8, 16] that

$$\frac{d}{dp}\mu_p(A) = \frac{1}{p(1-p)}\sum_{s\in S}\mu_p(1_s)(1-\mu_p(1_s))J_{A,p}(s). \tag{5.6}$$

Let $K = [0,1]^S$ be the 'continuous' cube, endowed with Lebesgue measure λ, and let B be an increasing subset of K. The influence $I_B(s)$ of an element s is given in [11] as

$$I_B(s) = \lambda(1_B(\psi^s) \neq 1_B(\psi_s))$$

where ψ^s (respectively, ψ_s) is the member of K obtained from $\psi \in K$ by setting $\psi(s) = 1$ (respectively, $\psi(s) = 0$). The conclusion of [11] may be expressed as follows. There exists a constant $c > 0$, independent of all other quantities, such that: for any increasing event $B \subseteq K$,

$$\sum_{s\in S} I_B(s) \geq c\lambda(B)(1-\lambda(B))\log[1/(2m_B)] \tag{5.7}$$

where $m_B = \max_{s\in S} I_B(s)$. The main result of [11] is a lower bound on m_B that is easily seen to follow from (5.7).

Equation (5.7) does not in fact appear explicitly in [11], but it may be derived from the arguments presented there, very much as observed in the case of the discrete cube from the arguments of [25]. See [15, Thm 3.4]. The factor of 2 on the right side of (5.7) is of little material consequence, since the inequality is important only when m_B is small, and, when $m_B < \frac{1}{3}$ say, the 2 may be removed with an amended value of the constant c. The literature on influence and sharp-threshold can seem a little disordered, and a coherent account may be found in [20]. The method used there introduces the factor 2 in a natural way, and for this reason we have included it in the above.

It is shown in [16] (see the proof of Theorem 2.10) that there exists an increasing subset B of K such that $\mu_p(A) = \lambda(B)$, and $J_{A,p}(s) \geq I_B(s)$ for all $s \in S$. Inequality (5.4) follows by (5.6)–(5.7). □

Proof of Corollary 5.5. By (5.4),

$$\left(\frac{1}{\mu_p(A)} + \frac{1}{1-\mu_p(A)}\right)\mu_p'(A) \geq B\log(\kappa^{-1}), \qquad p_1 < p < p_2,$$

whence, on integrating over (p_1, p_2),

$$\frac{\mu_{p_2}(A)}{1-\mu_{p_2}(A)} \Big/ \frac{\mu_{p_1}(A)}{1-\mu_{p_1}(A)} \geq \kappa^{-B(p_2-p_1)}.$$

The claim follows. □

6. Proofs of Theorems 3.4 and 3.10

Note first that a random-cluster measure has the form of (5.1) with $S = E$ and $\mu(\omega) = q^{k(\omega)}$, and it is known and easily checked that μ satisfies the FKG lattice condition when $q \geq 1$ (see [19, Sect. 3.2], for example). We shall apply Theorem 5.3 to a random-cluster $\phi_{p,q}$ measure with $q \geq 1$. It is standard (see [19, Thm 4.17(b)]) that

$$\frac{p}{q} \leq \frac{p}{p+q(1-p)} \leq \phi_{p,q}(1_e) \leq p, \tag{6.1}$$

whence

$$\phi_{p,q}(1_e)[1-\phi_{p,q}(1_e)] \geq \frac{p(1-p)}{q}.$$

We may thus take

$$B = \frac{c}{q} \tag{6.2}$$

in Corollary 5.5.

Let $q \geq 1$, $1 \leq k, m < n$, and consider the random-cluster measures $\phi^b_{n,p} = \phi^b_{\Lambda_n,p,q}$ on the box Λ_n. For $e \in \mathbb{E}^2$, write $J^b_{k,m,n}(e)$ for the (conditional) influence of e on the event $H_{k,m}$ under the measure $\phi^b_{n,p}$. We set $J^b_{k,m,n}(e) = 0$ for $e \notin \mathbb{E}_{\Lambda_n}$.

Lemma 6.3. *Let $q \geq 1$. We have that*

$$\sup_{e\in\mathbb{E}^2} J^0_{k,m,n}(e) \leq \frac{q}{p}\eta_k, \qquad 0 < p \leq p_{\mathrm{sd}}(q), \quad 1 \leq k, m < n, \tag{6.4}$$

$$\sup_{e\in\mathbb{E}^2} J^1_{k,m,n}(e) \leq \frac{q}{p_{\mathrm{d}}}\eta_{m+1}, \qquad p_{\mathrm{sd}}(q) \leq p < 1, \quad 1 \leq k, m < n, \tag{6.5}$$

where p_{d} satisfies (2.3) *and*

$$\eta_k = \phi^0_{p_{\mathrm{sd}}(q),q}(0 \leftrightarrow \partial\Lambda_{k/2}) \to 0 \qquad \text{as } k \to \infty.$$

Proof. For any configuration $\omega \in \Omega$ and vertex z, let $C_z(\omega)$ be the open cluster at z, that is, the set of all vertices joined to z by open paths.

Suppose first that $0 < p \leq p_{\mathrm{sd}}(q)$, and let $e = \langle x, y\rangle$ be an edge of Λ_n. We couple the two conditional measures $\phi^0_{n,p}(\cdot \mid \omega(e) = b)$, $b = 0, 1$, in the following manner. Let Ω_n be the configuration space of the edges in Λ_n, and let $T = \{(\pi, \omega) \in \Omega_n^2 : \pi \leq \omega\}$ be the set of all ordered pairs of configurations. There exists a measure μ^e on T such that:

(a) the first marginal of μ^e is $\phi^0_{n,p}(\cdot \mid 1_e = 0)$,
(b) the second marginal of μ^e is $\phi^0_{n,p}(\cdot \mid 1_e = 1)$,
(c) for any subset γ of Λ_n, conditional on the event $\{(\pi, \omega) : C_x(\omega) = \gamma\}$, the configurations π and ω are μ^e-almost-surely equal on all edges having no endvertex in γ.

The details of this coupling are omitted. The idea is to build the paired configuration (π, ω) edge by edge, beginning at the edge e, in such a way that $\pi(f) \leq \omega(f)$ for each edge f examined. The (closed) edge-boundary of the cluster $C_x(\omega)$ is closed in π also. Once this boundary has been uncovered, the configurations π, ω on the rest of space are governed by the same (conditional) measure, and may be taken equal. Such an argument has been used in [2] and [19, Thm 5.33(a)], and has been carried further in [4].

We claim that

$$J^0_{k,m,n}(e) \leq \phi^0_{n,p}(D_x \mid 1_e = 1), \tag{6.6}$$

where D_x is the event that C_x intersects both the left and right sides of $B_{k,m}$. This is proved as follows. By (5.2),

$$\begin{aligned} J^0_{k,m,n}(e) &= \mu^e(\omega \in H_{k,m},\, \pi \notin H_{k,m}) \\ &\leq \mu^e(\omega \in H_{k,m} \cap D_x) \\ &\leq \mu^e(\omega \in D_x) = \phi^0_{n,p}(D_x \mid 1_e = 1), \end{aligned}$$

since, when $\omega \notin D_x$, either both or neither of ω, π belong to $H_{k,m}$. By (6.6),

$$J^0_{k,m,n}(e) \le \frac{\phi^0_{n,p}(D_x)}{\phi^0_{n,p}(1_e)}. \tag{6.7}$$

On D_x, the radius of the open cluster at x is at least $\frac{1}{2}k$. Since $\phi^0_{n,p} \le_{\mathrm{st}} \phi_{p,q}$ and $\phi_{p,q}$ is translation-invariant,

$$\phi^0_{n,p}(D_x) \le \phi_{p,q}(x \leftrightarrow x + \partial\Lambda_{k/2}) = \phi_{p,q}(0 \leftrightarrow \partial\Lambda_{k/2}).$$

By (3.9),

$$\phi_{p,q}(0 \leftrightarrow \partial\Lambda_{k/2}) \le \phi^0_{p_{\mathrm{sd}}(q),q}(0 \leftrightarrow \partial\Lambda_{k/2}) \to 0 \qquad \text{as } k \to \infty,$$

and, by (6.1) and (6.7), the conclusion of the lemma is proved when $p \le p_{\mathrm{sd}}(q)$.

Suppose next that $p_{\mathrm{sd}}(q) \le p < 1$. Instead of working with the open paths, we work with the dual open paths. Each edge $e_{\mathrm{d}} = \langle u, v\rangle$ of the dual lattice traverses some edge $e = \langle x, y\rangle$ of the primal, and, for each configuration ω, we define the dual configuration ω_{d} by $\omega_{\mathrm{d}}(e_{\mathrm{d}}) = 1 - \omega(e)$. Thus, the dual edge e_{d} is open if and only if e is closed. It is well known (see [19, eqn (6.12)], for example) that, with ω distributed according to $\phi^1_{n,p}$, ω_{d} has as law the random-cluster measure, denoted $\phi_{n,p_{\mathrm{d}},\mathrm{d}}$, on the dual of Λ_n with free boundary condition. The event $H_{k,m}$ occurs if and only if there is no dual open path traversing the dual of $B_{k,m}$ from top to bottom. We may therefore apply the above argument to the dual process, obtaining thus that

$$J^1_{k,m,n}(e) \le \frac{\phi_{n,p_{\mathrm{d}},\mathrm{d}}(V_u)}{\phi_{n,p_{\mathrm{d}},\mathrm{d}}(1_e)}, \tag{6.8}$$

where V_u is the event that C_u intersects both the top and bottom sides of the dual of $B_{k,m}$.

On the event V_u, the radius of the open cluster at u is at least $\frac{1}{2}(m+1)$. Since $\phi_{n,p_{\mathrm{d}},\mathrm{d}} \le_{\mathrm{st}} \phi_{p_{\mathrm{d}},q}$,

$$\phi_{n,p_{\mathrm{d}},\mathrm{d}}(V_u) \le \phi_{p_{\mathrm{d}},q}(u \leftrightarrow u + \partial\Lambda_{(m+1)/2}) = \phi_{p_{\mathrm{d}},q}(0 \leftrightarrow \partial\Lambda_{(m+1)/2}).$$

As above, by (2.4),

$$\phi_{p_{\mathrm{d}},q}(0 \leftrightarrow \partial\Lambda_{(m+1)/2}) \le \phi^0_{p_{\mathrm{sd}}(q),q}(0 \leftrightarrow \partial\Lambda_{(m+1)/2}) = \eta_{m+1},$$

and this completes the proof when $p \ge p_{\mathrm{sd}}(q)$. □

Proof of Theorem 3.10. This follows immediately from Corollary 5.5 by (6.2) and Lemma 6.3. □

Proof of Theorem 3.4. By planar duality,

$$\phi^0_{p,q}(H_k) = 1 - \phi^1_{p_{\mathrm{d}},q}(H_k),$$

where p, p_{d} are related by (2.3), see [19, Thms 6.13, 6.14]. Since $\phi^0_{p_{\mathrm{sd}}(q),q} \le_{\mathrm{st}} \phi^1_{p_{\mathrm{sd}}(q),q}$,

$$\phi^0_{p_{\mathrm{sd}}(q),q}(H_k) \le \tfrac{1}{2} \le \phi^1_{p_{\mathrm{sd}}(q),q}(H_k),$$

and Theorem 3.4 follows from Theorem 3.10. □

7. Proof of Theorems 4.1 and 4.3

Only an outline of the proof of Theorem 4.1 is included here, since it follows the 'usual' route (see [18, Sect. 11.3] or [19, Sect. 6.2], for examples of the argument). The measure $\pi_{\beta,h}$ is automorphism-invariant, ergodic, and has the finite-energy property. By the main result of [12], the number N^+ (respectively, N^-) of infinite + clusters (respectively, infinite − $*$-connected clusters) satisfies

$$\text{either} \quad \pi_{\beta,h}(N^\pm = 0) = 1 \quad \text{or} \quad \pi_{\beta,h}(N^\pm = 1) = 1.$$

Assume that $\theta^+(\beta,h)\theta^-(\beta,h) > 0$, which is to say that $\pi_{\beta,h}(N^+ = N^- = 1) = 1$. One may find a box Λ sufficiently large that, with $\pi_{\beta,h}$-probability at least $\frac{1}{2}$: the top and bottom of its boundary $\partial\Lambda$ are + connected to infinity off Λ, and the left and right sides are − $*$-connected to infinity off Λ. Since $N^+ = 1$ almost surely, there is a + path connecting the two infinite + paths above, and this contradicts the fact that $N^- = 1$ almost surely.

We turn to the proof of Theorem 4.3. For the moment, let $\pi_{\beta,h}$ be the Ising measure on a finite graph $G = (V,E)$ with parameters $\beta \ge 0$ and $h \ge 0$. It is well known than $\pi_{\beta,0}$ satisfies the FKG lattice condition (3.2) on the partially ordered set $\Sigma_V = \{-1,+1\}^V$. We identify Σ_V with $\{0,1\}^V$ via the mapping $\sigma_x \mapsto \omega_x = \frac{1}{2}(\sigma_x + 1)$, and we choose p by

$$\frac{p}{1-p} = e^{2h}. \tag{7.1}$$

Then $\pi_{\beta,h}$ may be expressed in the form (5.1), and we may thus apply the results of Section 5. By conditioning on the states of the neighbours of x,

$$\frac{e^{2h-\Delta\beta}}{e^{\Delta\beta} + e^{2h-\Delta\beta}} \le \pi_{\beta,h}(1_x) \le \frac{e^{2h+\Delta\beta}}{e^{-\Delta\beta} + e^{2h+\Delta\beta}}, \tag{7.2}$$

where Δ is the degree of the vertex x, and 1_x is the indicator function that $\sigma_x = +1$. Therefore,

$$\begin{aligned}\pi_{\beta,h}(1_x)[1 - \pi_{\beta,h}(1_x)] &\ge \min\left\{\frac{e^{2h}}{(e^{\Delta\beta} + e^{2h-\Delta\beta})^2}, \frac{e^{2h}}{(e^{-\Delta\beta} + e^{2h+\Delta\beta})^2}\right\} \\ &= \frac{e^{2h+2\Delta\beta}}{(1 + e^{2h+2\Delta\beta})^2}. \end{aligned} \tag{7.3}$$

This bound will be useful with $\Delta = 4$, and we write

$$\xi_{\beta,h} = \frac{e^{2h+8\beta}}{(1+e^{2h+8\beta})^2}. \tag{7.4}$$

Note that $\xi_{\beta,h}$ is decreasing in h.

We follow the argument of the proof of Theorem 5.3. Let $\beta \in [0, \beta_c)$, $h > 0$, and $1 \le k, m \le r < n$, and consider the Ising measure $\pi_{n,h} = \pi_{\Lambda_n,\beta,h}$ on the box $\Lambda_n = [-n, n]^2$. For $x \in \mathbb{Z}^2$, write $J_{k,m,n}(x)$ for the (conditional) influence of x on the event $H_{k,m}$ under the measure $\pi_{n,h}$. We set $J_{k,m,n}(x) = 0$ for $x \notin \Lambda_n$.

Lemma 7.5. *Uniformly in $x \in \mathbb{Z}^2$,*

$$J_{k,m,n}(x) \le (1 + e^{8\beta-2h}) \left[\pi_{n,h}(B_{k,m} \overset{+}{\leftrightarrow} \partial\Lambda_r) + \sup_{x\in\Lambda_r} \pi_{n,h}(x \overset{+}{\leftrightarrow} x + \partial\Lambda_{k/2})\right], \tag{7.6}$$

$$J_{k,m,n}(x) \le (1 + e^{8\beta+2h}) \left[\pi_{n,h}(B_{k,m} \overset{-}{\leftrightarrow}_* \partial\Lambda_r) + \sup_{x\in\Lambda_r} \pi_{n,h}(x \overset{-}{\leftrightarrow}_* x + \partial\Lambda_{m/2})\right]. \tag{7.7}$$

Proof. Let $h > 0$. Let C_x^+ be the set of all vertices joined to x by a path of vertices all of whose states are $+1$ (thus, $C_x^+ = \varnothing$ if $\sigma_x = -1$). We may couple the conditioned measures $\pi_{n,h}(\cdot \mid \sigma_x = b)$, $b = \pm 1$, such that the Ising equivalents of (a)–(c) hold as in Section 6. As in (6.7),

$$J_{k,m,n}(x) \le \frac{\pi_{n,h}(D_x^+)}{\pi_{n,h}(1_x)}, \tag{7.8}$$

where D_x^+ is the event that C_x^+ intersects both the left and right sides of $B_{k,m}$. On D_x^+, the radius of C_x^+ is at least $\frac{1}{2}k$.

For $x \notin \Lambda_r$,

$$\pi_{n,h}(D_x^+) \le \pi_{n,h}(B_{k,m} \overset{+}{\leftrightarrow} \partial\Lambda_r).$$

For $x \in \Lambda_r$, we shall use the bound

$$\pi_{n,h}(D_x^+) \le \pi_{n,h}(x \overset{+}{\leftrightarrow} x + \partial\Lambda_{k/2}).$$

Combining the above inequalities with (7.2), we obtain (7.6).

Let C_x^- be the set of all vertices joined to x by a $*$-connected path of vertices all of whose states are -1. The event $H_{k,m}$ occurs if and only if there is no $-$ $*$-connected path from the top to the bottom of $B_{k,m}$. Therefore, the conditional influence of x on $H_{k,m}$ equals that of x on this new event. As in (7.8),

$$J_{k,m,n}(x) \le \frac{\pi_{n,h}(V_x^-)}{\pi_{n,h}(1 - 1_x)}, \tag{7.9}$$

where V_x^- is the event that C_x^- intersects both the top and bottom of $B_{k,m}$. The above argument leads now to (7.7). $\square$

Proof of Theorem 4.3. Let $R > h_c$ and $\delta > 0$, and let $k, m \le r < n$. We set

$$\kappa^{\delta}_{n,r,+} = 2(1+e^{8\beta})\left[\pi_{n,h_c-\delta}(B_{k,m} \xleftrightarrow{+} \partial\Lambda_r) + \sup_{x\in\Lambda_r} \pi_{n,h_c-\delta}(x \xleftrightarrow{+} x + \partial\Lambda_{k/2})\right],$$

$$\kappa^{\delta}_{n,r,-} = 2(1+e^{8\beta+2R})\left[\pi_{n,h_c+\delta}(B_{k,m} \xleftrightarrow{-}_{*} \partial\Lambda_r) + \sup_{x\in\Lambda_r} \pi_{n,h_c+\delta}(x \xleftrightarrow{-}_{*} x + \partial\Lambda_{m/2})\right].$$

Let $0 < h_1 < h_c < h_2 \le R$, and choose $\delta < \min\{h_c - h_1, h_2 - h_c\}$. By (7.1), (7.3), Lemma 7.5, and Theorem 5.3, $f_n(h) = \pi_{n,h}(H_{k,m})$ satisfies

$$\frac{1}{f_n(h)(1-f_n(h))}\cdot\frac{df_n}{dh} \ge B_+ \log(1/\kappa^{\delta}_{n,r,+}), \qquad h_1 \le h \le h_c - \delta, \tag{7.10}$$

where $B_+ = 2c\xi_{\beta,h_c}$, see (7.4). The corresponding inequality for $h_c+\delta \le h \le R$ holds with $\kappa^{\delta}_{n,r,+}$ replaced by $\kappa^{\delta}_{n,r,-}$, and B_+ replaced by $B_- = 2c\xi_{\beta,R}$.

We integrate (7.10) over the intervals $(h_1, h_c - \delta)$ and $(h_c + \delta, h_2)$, add the results, and use the fact that $f_n(h)$ is non-decreasing in h, to obtain that

$$\log \frac{f_n(h)}{1-f_n(h)}\bigg|_{h_1}^{h_2} \ge (h_c-\delta-h_1)B_+\log(1/\kappa^{\delta}_{n,r,+}) + (h_2-h_c-\delta)B_-\log(1/\kappa^{\delta}_{n,r,-}).$$

Take the limits as $n\to\infty$, $r\to\infty$, and $\delta\to 0$ in that order, and use the monotonicity in h of $\pi_{\beta,h}$, to obtain the theorem. $\square$

8. The coloured random-cluster model

There is a well known coupling of the random-cluster and Potts models that provides a transparent explanation of how the analysis of the former aids that of the latter. Formulated as in [14] (see also the historical account of [19]), this is as follows. Let $p \in (0,1)$ and $q \in \{2,3,\dots\}$. Let ω be sampled from the random-cluster measure $\phi_{p,q}$ on the finite graph $G = (V,E)$. To each open cluster of ω we assign a uniformly chosen element of $\{1,2,\dots,q\}$, these random spins being independent between clusters. The ensuing spin-configuration σ on G is governed by a Potts measure, and pair-spin correlations in σ are coupled to open connections in ω. This coupling has inspired a construction that we describe next.

Let $p \in (0,1)$, $q \in (0,\infty)$, and $\alpha \in (0,1)$. Let ω have law $\phi_{p,q}$. To the vertices of each open cluster of ω we assign a random spin chosen according to the Bernoulli measure on $\{0,1\}$ with parameter α. These spins are constant within clusters, and independent between clusters. We call this the *coloured random-cluster model* (CRCM). With σ the ensuing spin-configuration, we write $\kappa_{p,q,\alpha}$ for the measure governing the pair (ω,σ), and $\pi_{p,q,\alpha}$ for the marginal law of σ. When $q \in \{2,3,\dots\}$ and $q\alpha$ and $q(1-\alpha)$ are integers, the CRCM is a vertex-wise contraction of the Potts model from the spin-space $\{1,2,\dots,q\}^V$ to $\Sigma = \{0,1\}^V$.

The CRCM has been studied in [26] under the name 'fractional fuzzy Potts model', and it is inspired in part by the earlier work of [13, 21, 22], as well as the study of the so-called 'divide-and-colour model' of [5].

The following seems to be known, see [13, 21, 22, 26], but the short proof given below may be of value.

Theorem 8.1. *The measure $\pi_{p,q,\alpha}$ is monotone for all finite graphs G and all $p \in (0,1)$ if and only if $q\alpha, q(1-\alpha) \ge 1$.*

We identify the spin-vector $\sigma \in \Sigma$ with the set $A = \{v \in V : \sigma_v = 1\}$. Let $\pi_h = \pi_{p,q,\alpha,h}$ be the probability measure obtained from $\pi_{p,q,\alpha}$ by including an external field with strength $h \in \mathbb{R}$,

$$\pi_h(A) \propto e^{h|A|}\pi_{p,q,\alpha}(A), \qquad A \subseteq V. \tag{8.2}$$

It is an elementary consequence of Theorem 8.1 and (8.2) that, when $q\alpha, q(1-\alpha) \ge 1$, π_h is a monotone measure, and π_h is increasing in h. When $q = 2$ and $\alpha = \frac{1}{2}$, π_h is the Ising measure with external field. The purpose of this section is to extend the arguments of Section 4 to the CRCM with external field.

There is a special case of the CRCM with an interesting interpretation. Let ω be sampled from $\phi_{p,q}$ as above, and let $\sigma = (\sigma_v : v \in V)$ be a vector of independent Bernoulli (γ) variables. Let B be the event that σ is constant on each open cluster of ω. The pair (ω, σ), conditional on B, is termed the *massively coloured random-cluster measure* (MCRCM). The law of σ is simply $\pi_{p,2q,\frac{1}{2},h}$ where $h = \log[\gamma/(1-\gamma)]$.

Just as $\pi_{p,q,\alpha}$ and $\phi_{p,q}$ may be coupled via $\kappa_{p,q,\alpha}$, so we can couple π_h with an 'edge-measure' $\phi_h = \phi_{p,q,\alpha,h}$ via the following process. With B given as above, and $(\omega,\sigma) \in B$, denote by $\sigma(C)$ the common spin-value of σ on an open cluster C of ω. Let $\kappa_h = \kappa_{p,q,\alpha,h}$ be the probability measure on $\Omega \times \Sigma$ given by

$$\kappa_h(\omega,\sigma) \propto \phi_{p,q}(\omega) 1_B(\omega,\sigma) \prod_C \big[(\alpha e^{h|C|})^{\sigma(C)}(1-\alpha)^{1-\sigma(C)}\big], \tag{8.3}$$

where the product is over the open clusters C of ω, and $|C|$ is the number of vertices of C. The marginal and conditional measures of κ_h are easily calculated. The marginal on Σ is π_h, and the marginal on Ω is $\phi_h = \phi_{p,q,\alpha,h}$ given by

$$\phi_h(\omega) \propto \phi_{p,q}(\omega) \prod_C \big[\alpha e^{h|C|} + 1 - \alpha\big], \qquad \omega \in \Omega. \tag{8.4}$$

Note that $\phi_0 = \phi_{p,q}$. Given ω, we obtain σ by labelling the open clusters with independent Bernoulli spins in such a way that the odds of cluster C receiving spin 1 are $\alpha e^{h|C|}$ to $1-\alpha$.

By (8.2), or alternatively by summing $\kappa_h(\omega,\sigma)$ over ω, we find that

$$\pi_h(A) \propto e^{h|A|}(1-p)^{|\Delta A|} Z_{A,q\alpha} Z_{\overline{A},q(1-\alpha)}, \qquad A \subseteq V, \tag{8.5}$$

where ΔA is the set of edges of G with exactly one endvertex in A, and $Z_{B,q}$ is the partition function of the random-cluster measure on the subgraph induced by $B \subseteq V$ with edge-parameter p and cluster-weight q. It may be checked as in the proof of Theorem 8.1 that, for given p, q, α, the measure π_h is bounded above (respectively, below) by a product measure with parameter $a(h)$ (respectively, $b(h)$) where

$$a(-h) \to 0, \quad b(h) \to 1, \qquad \text{as } h \to \infty. \tag{8.6}$$

The measure ϕ_h has a number of useful properties, following.

Proposition 8.7. *Let $q\alpha, q(1-\alpha) \geq 1$.*

(i) *The probability measure ϕ_h is monotone.*

(ii) *The marginal measure of κ_h on Ω, conditional on $\sigma_x = b$, satisfies*

$$\begin{aligned} \kappa_h(\cdot \mid \sigma_x = 1) &\geq_{\mathrm{st}} \kappa_h(\cdot \mid \sigma_x = 0), \qquad h \geq 0,\\ \kappa_h(\cdot \mid \sigma_x = 1) &\leq_{\mathrm{st}} \kappa_h(\cdot \mid \sigma_x = 0), \qquad h \leq 0. \end{aligned}$$

(iii) *If $p_1 \leq p_2$ and the ordered three-item sequence $(0, h_1, h_2)$ is monotonic, then $\phi_{p_1,q,\alpha,h_1} \leq_{\mathrm{st}} \phi_{p_2,q,\alpha,h_2}$.*

(iv) *We have that $\phi_{p,q,\alpha,h} \leq_{\mathrm{st}} \phi_{p,Q}$, where $Q = Q(h)$ is defined by*

$$Q(h) = \begin{cases} q\alpha, & h > 0,\\ q, & h = 0,\\ q(1-\alpha), & h < 0. \end{cases}$$

We assume henceforth that $q\alpha, q(1-\alpha) \geq 1$, and we consider next the infinite-volume limits of the above measures. Let G be a subgraph of the square lattice $\mathbb{Z}^2$ induced by the vertex-set V, and label the above measures with the subscript V. By standard arguments (see [19, Chap. 4]), the limit measure

$$\phi_h = \lim_{V \uparrow \mathbb{Z}^2} \phi_{V,h}$$

exists, is independent of the choice of the V, and is translation-invariant and ergodic. By an argument similar to that of [19, Thm 4.91], the measures $\pi_{V,h}$ have a well-defined infinite-volume limit π_h as $V \uparrow \mathbb{Z}^2$. Furthermore, the pair (ϕ_h, π_h) may be coupled in the same manner as on a finite graph. That is, a *finite* cluster C of ω receives spin 1 with probability $\alpha e^{h|C|}/[\alpha e^{h|C|} + 1 - \alpha]$. An *infinite* cluster receives spin 1 (respectively, 0) if $h > 0$ (respectively, $h < 0$). When $h = 0$, the spin of an infinite cluster has the Bernoulli distribution with parameter α.

Since ϕ_h is translation-invariant, so is π_h. As in [19, Thm 4.10], π_h is positively associated, and the proof of [19, Thm 4.91] may be adapted to obtain that π_h is ergodic. By a simple calculation, the $\pi_{V,h}$ have the finite-energy property, with bounds that are uniform in V (see [19, eqn (3.4)]), and therefore so does π_h. Adapting the notation used in Section 4 for the Ising model, let

$$\theta^1(p,q,\alpha,h) = \pi_h(0 \overset{1}{\leftrightarrow} \infty),$$
$$\theta^0(p,q,\alpha,h) = \pi_h(0 \overset{0}{\leftrightarrow}_* \infty).$$

As in Theorem 4.1, and with an essentially identical proof,

$$\theta^1(p,q,\alpha,h)\theta^0(p,q,\alpha,h) = 0. \tag{8.8}$$

By the remark after (8.2) and [19, Thm 4.10], π_h is stochastically increasing in h, whence there exists $h_c = h_c(p,q,\alpha) \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$\theta^1(p,q,\alpha,h) \begin{cases} = 0 & \text{if } h < h_c, \\ > 0 & \text{if } h > h_c. \end{cases}$$

By comparisons with product measures (see the remark prior to (8.6)), we have that $|h_c| < \infty$.

We call a probability measure μ on Σ *subcritical* (respectively, *supercritical*) if the μ-probability of an infinite 1-cluster is 0 (respectively, strictly greater than 0); we shall use the corresponding terminology for measures on Ω. There is a second type of phase transition, namely the onset of percolation in the measure ϕ_h. An infinite edge-cluster under ϕ_h forms part of an infinite vertex-cluster under π_h. Let $p_c(q)$ be the critical point of the random-cluster measure $\phi_{p,q}$ on $\mathbb{Z}^2$, as usual. By Proposition 8.7(iv), ϕ_h is subcritical for all h when $p < p_c(q\min\{\alpha, 1-\alpha\})$; in particular, for such p, ϕ_h is subcritical for h lying in some open neighbourhood of h_c. On the other hand, suppose that $\phi_0 = \phi_{p,q}$ is supercritical. By the remarks above, $\theta^1 > 0$ for $h > 0$, and $\theta^0 > 0$ for $h < 0$. By (8.8), θ^1 is discontinuous at $h = h_c = 0$. By Proposition 8.7(iii), $\phi_h \geq_{\text{st}} \phi_0$, whence θ^1 is discontinuous at $h = h_c = 0$ whenever $p > p_c(q)$.

With $k, m \in \mathbb{N}$, let $H_{k,m}$ be the event that there exists a left–right 1-crossing of the box $B_{k,m}$. A result corresponding to Theorem 4.3 holds, subject to a condition on ϕ_h with h near h_c. This condition has not, to our knowledge, been verified for the Ising model, although it is expected to hold. In this sense, the next theorem does not quite generalize Theorem 4.3.

Theorem 8.9. *Let $R \geq 0$. When $h_c \neq 0$ we require in addition that $R \leq |h_c|$. Suppose that ϕ_h is subcritical for $h \in [h_c - R, h_c + R]$. There exist $\rho_{i,1} = \rho_{i,1}(p,q,\alpha,R)$ and $\rho_{i,0} = \rho_{i,0}(p,q,\alpha,R)$ satisfying*

$$\rho_{i,1}\rho_{i,0} \to 0 \qquad \textit{as } i \to \infty,$$

such that: for $h_1 \in [h_c - R, h_c]$, $h_2 \in [h_c, h_c + R]$,

$$\pi_{h_1}(H_{k,m})[1 - \pi_{h_2}(H_{k,m})] \leq \rho_{k,1}^{h_c - h_1}\rho_{m,0}^{h_2 - h_c}, \qquad k, m \geq 1.$$

As in the proof of Theorem 4.3, the first step is to establish bounds on the one-point marginals of π_h. This may be strengthened to a finite-energy property, but this will not be required here. The proof is deferred to the end of the section.

Lemma 8.10. *Let $G = (V, E)$ be a finite graph with maximum vertex-degree Δ. Then*

$$\frac{\alpha e^h}{\alpha e^h + 1 - \alpha}(1-p)^\Delta \le \pi_h(\sigma_x = 1) \le 1 - \frac{1-\alpha}{\alpha e^h + 1 - \alpha}(1-p)^\Delta.$$

Consider the subgraph of $\mathbb{Z}^2$ induced by $\Lambda_n = [-n, n]^d$, and let $x \in \Lambda_n$. Objects associated with the finite domain Λ_n are labelled with the subscript n. For $b = 0, 1$, let $\pi^b_{n,h}$ (respectively, $\phi^b_{n,h}$) be the marginal measure on Σ_n (respectively, Ω_n) of the coupling $\kappa_{n,h}$ conditioned on $\sigma_x = b$.

By Proposition 8.7, $\phi^1_{n,h} \ge_{\rm st} \phi^0_{n,h}$ when $h \ge 0$, and $\phi^1_{n,h} \le_{\rm st} \phi^0_{n,h}$ when $h \le 0$. It is convenient to work with a certain coupling of the pairs $(\phi^0_{n,h}, \pi^0_{n,h})$ and $(\phi^1_{n,h}, \pi^1_{n,h})$. Recall that $C_x(\omega)$ denotes the open cluster at x in the edge-configuration $\omega \in \Omega$.

Lemma 8.11. *Let $h \in \mathbb{R}$. There exists a probability measure $\kappa^{01}_{n,h}$ on $(\Omega_n \times \Sigma_n)^2$ with the following properties. Let $(\omega^0, \sigma^0, \omega^1, \sigma^1)$ be sampled from $(\Omega_n \times \Sigma_n)^2$ according to $\kappa^{01}_{n,h}$.*

(i) *For $b = 0, 1$, ω^b has law $\phi^b_{n,h}$.*
(ii) *For $b = 0, 1$, σ^b has law $\pi^b_{n,h}$.*
(iii) *If $h \le 0$, $\omega^0 \ge \omega^1$. If $h \ge 0$, $\omega^1 \ge \omega^0$.*
(iv) *The spin configurations σ^0 and σ^1 agree at all vertices $y \notin C_x(\omega^0) \cup C_x(\omega^1)$.*

Proof. Assume first that $h \ge 0$. There exists a probability measure $\overline{\phi}_n$ on Ω_n^2, with support $D_1 = \{(\omega^0, \omega^1) \in \Omega_n^2 : \omega^0 \le \omega^1\}$, whose first (respectively, second) marginal is $\phi^0_{n,h}$ (respectively, $\phi^1_{n,h}$). By sampling from $\overline{\phi}_n$ in a sequential manner beginning at x, and proceeding via the open connections of the upper configuration, we may assume in addition that $(\omega^0, \omega^1) \in D_2$, where D_2 is the set of pairs such that $\omega^0(e) = \omega^1(e)$ for any edge e having at most one endpoint in $C_x(\omega^1)$. Let $(\omega^0, \omega^1) \in D = D_1 \cap D_2$.

The spin vectors σ^b may be constructed as follows:

(a) attach spin b to the cluster $C_x(\omega^b)$,
(b) attach independent Bernoulli spins to the other ω^b-open clusters in such a way that the odds of cluster C receiving spin 1 are $\alpha e^{h|C|}$ to $1 - \alpha$.

We may assign spins σ^b to the open clusters of the ω^b in such a way that: σ^b has law $\pi^b_{n,h}$, and $\sigma^0_y = \sigma^1_y$ for $y \notin C_x(\omega^1)$. Write $\kappa^{01}_{n,h}$ for the joint law of the ensuing pairs (ω^0, σ^0), (ω^1, σ^1).

When $h \le 0$, let $\kappa^{01}_{n,h}$ be the coupling as above, with the differences that: $\omega^0 \ge \omega^1$, and $\sigma^0_y = \sigma^1_y$ for $y \notin C_x(\omega^0)$. $\square$

We seek next a substitute for Lemma 7.5 in the current setting. Let $J_{k,m,n}(x)$ be the conditional influence of vertex x on the event $H_{k,m}$, with reference measure $\pi_{n,h}$ on Λ_n.

Let $(\omega^0, \sigma^0, \omega^1, \sigma^1)$ be sampled according to the measure $\kappa^{01}_{n,h}$ of Lemma 8.11. Define random clusters $C^H_x, C^V_x \subseteq \mathbb{Z}^2$ as follows,

$$C^H_x(\omega^0, \sigma^0, \omega^1, \sigma^1) := \{z \in \mathbb{Z}^2 : \exists y \in C_x(\omega^0),\ y \overset{1}{\leftrightarrow} z \text{ in } \sigma^1\}$$
$$C^V_x(\omega^0, \sigma^0, \omega^1, \sigma^1) := \{z \in \mathbb{Z}^2 : \exists y \in C_x(\omega^1),\ y \overset{0}{\leftrightarrow}_* z \text{ in } \sigma^0\}$$

Notice that, if $h \ge 0$ (respectively, $h \le 0$), C^H_x (respectively, C^V_x) is the spin-1 cluster (respectively, spin-0 $*$-cluster) at x under σ^1 (respectively, σ^0). It may be checked as before that:

$$J_{k,m,n}(x) \le \kappa^{01}_{n,h}\big(C^H_x \text{ contains a horizontal crossing of } B_{k,m}\big), \tag{8.12}$$
$$J_{k,m,n}(x) \le \kappa^{01}_{n,h}\big(C^V_x \text{ contains a vertical } *\text{-crossing of } B_{k,m}\big). \tag{8.13}$$

The notation C^H_x, C^V_x is introduced in order to treat the cases $h > 0$ and $h < 0$ simultaneously.

Lemma 8.14. *Let R be as in Theorem 8.9.*

(i) *If $\theta^1(p, q, \alpha, h_c) = 0$, and ϕ_h is subcritical for $h \in [h_c - R, h_c]$, there exists $\nu_{k,1}$ satisfying $\nu_{k,1} \to 0$ as $k \to \infty$ such that*
$$\limsup_{n\to\infty} \sup_{h\in[h_c-R,h_c]} \sup_{x\in\Lambda_n} J_{k,m,n}(x) \le \nu_{k,1}.$$

(ii) *If $\theta^0(p, q, \alpha, h_c) = 0$, and ϕ_h is subcritical for $h \in [h_c, h_c + R]$, there exists $\nu_{m,0}$ satisfying $\nu_{m,0} \to 0$ as $m \to \infty$ such that*
$$\limsup_{n\to\infty} \sup_{h\in[h_c,h_c+R]} \sup_{x\in\Lambda_n} J_{k,m,n}(x) \le \nu_{m,0}.$$

Proof. We prove part (i) only, the proof of (ii) being similar. If $[h_c - R, h_c] \subseteq [0, \infty)$, let $\phi = \phi_{h_c}$; if $[h_c - R, h_c] \subseteq (-\infty, 0]$, let $\phi = \phi_{h_c-R}$. By Proposition 8.7, and the assumptions of (i),

(a) $\phi_{n,h} \le_{\mathrm{st}} \phi$ for $n \ge 1$ and $h \in [h_c - R, h_c]$,
(b) ϕ is subcritical,
(c) π_{h_c} is subcritical, and $\pi_{n,h} \le_{\mathrm{st}} \pi_{n,h_c}$ for $h \in [h_c - R, h_c]$.

By Lemma 8.10, there exists $L > 0$ such that
$$\pi_{n,h}(\sigma_x = 1)\pi_{n,h}(\sigma_x = 0) \ge L \tag{8.15}$$
for all $n \ge 1$, $x \in \Lambda_n$, and $h \in [h_c - R, h_c + R]$. Let
$$A_x(\omega) = \sup\{r \ge 0 : x \leftrightarrow x + \partial\Lambda_r\}$$

denote the radius $\mathrm{rad}(C_x)$ of the edge cluster $C_x = C_x(\omega)$ at x, and note that $\phi(A_x < \infty) = 1$.

Let $r \geq \max\{k, m\}$ and $x \in \Lambda_r$. By (8.12) and the positive association of $\pi^1_{n,h}$, and as in (6.7),

$$\begin{aligned} J_{k,m,n}(x) &\leq \kappa^{01}_{n,h}\big(\mathrm{rad}(C^H_x) \geq \tfrac{1}{2}k\big) \\ &\leq \sum_{a=0}^{\infty} \phi^0_{n,h}(A_x = a)\alpha^1_{n,h}(x, a, \tfrac{1}{2}k) \\ &\leq \frac{1}{L}\sum_{a=0}^{\infty} \phi_{n,h}(A_x = a)\alpha_{n,h}(x, a, \tfrac{1}{2}k), \end{aligned}$$

where

$$\alpha^\xi_{n,h}(x, a, b) = \pi^\xi_{n,h}\big(x + \Lambda_a \overset{1}{\leftrightarrow} x + \partial\Lambda_b \,\big|\, \sigma_y = 1 \text{ for } y \in x + \Lambda_a\big).$$

Since $\alpha_{n,h}(x, a, b)$ is non-decreasing in a, and furthermore $\phi_{n,h} \leq_{\mathrm{st}} \phi$ and ϕ is translation-invariant,

$$\sup_{x\in\Lambda_r} J_{k,m,n}(x) \leq \frac{1}{L}\sum_{a=0}^{\infty} \phi(A_0 = a) \sup_{x\in\Lambda_r}\big\{\alpha_{n,h}(x, a, \tfrac{1}{2}k)\big\}. \tag{8.16}$$

By (8.15) and the fact that $\pi_{n,h} \leq_{\mathrm{st}} \pi_{n,h_c}$,

$$\alpha_{n,h}(x, a, \tfrac{1}{2}k) \leq \min\left\{1, \frac{1}{L^{|\Lambda_r|}}\pi_{n,h_c}(x + \Lambda_a \overset{1}{\leftrightarrow} x + \partial\Lambda_{k/2})\right\}. \tag{8.17}$$

Suppose now that $x \in \Lambda_n \setminus \Lambda_r$. Then

$$\begin{aligned} J_{k,m,n}(x) &\leq \kappa^{01}_{n,h}(C^H_x \cap B_{k,m} \neq \varnothing) \\ &\leq \sum_{a=0}^{\infty} \phi^0_{n,h}(A_x = a)\beta^1_{n,h}(x, a) \\ &\leq \frac{1}{L}\sum_{a=0}^{\infty} \phi_{n,h}(A_x = a)\beta_{n,h}(x, a), \end{aligned}$$

where

$$\beta^\xi_{n,h}(x, a) = \pi^\xi_{n,h}\big(x + \Lambda_a \overset{1}{\leftrightarrow} B_{k,m} \,\big|\, \sigma_y = 1 \text{ for } y \in x + \Lambda_a\big)$$

is a non-decreasing function of a. Since $\phi_{n,h} \leq_{\mathrm{st}} \phi$, and ϕ is translation-invariant,

$$J_{k,m,n}(x) \leq \frac{1}{L}\sum_{a=0}^{\infty} \phi(A_0 = a)\beta_{n,h}(x, a).$$

As above,

$$\begin{aligned}\beta_{n,h}(x,a) &\le \frac{1}{L^{|\Lambda_a|}}\pi_{n,h}(x+\Lambda_a \overset{1}{\leftrightarrow} B_{k,m})\\ &\le \frac{1}{L^{|\Lambda_a|}}\pi_{n,h}(B_{k,m} \overset{1}{\leftrightarrow} \partial\Lambda_{r-a}) \qquad \text{if } a \le r,\end{aligned}$$

whence

$$J_{k,m,n}(x) \le \frac{1}{L}\sum_{a=0}^{\infty}\phi(A_0=a)\min\left\{1, \frac{1}{L^{|\Lambda_a|}}\pi_{n,h_c}(B_{k,m} \overset{1}{\leftrightarrow} \partial\Lambda_{r-a})\right\}, \tag{8.18}$$

where the minimum is interpreted as 1 when $a > r$.

We add (8.16)–(8.17) and (8.18), and take the limit $n \to \infty$, to obtain by the bounded convergence theorem that

$$\begin{aligned}&\limsup_{n\to\infty}\sup_{x\in\Lambda_n} J_{k,m,n}(x)\\ &\quad\le \frac{1}{L}\left[\sum_{a=0}^{\infty}\phi(A_0=a)\min\left\{1, \frac{1}{L^{|\Lambda_a|}}\pi_{h_c}(x+\Lambda_a \overset{1}{\leftrightarrow} \partial\Lambda_{k/2})\right\}\right.\\ &\qquad\qquad \left.+\sum_{a=0}^{\infty}\phi(A_0=a)\min\left\{1, \frac{1}{L^{|\Lambda_a|}}\pi_{h_c}(B_{k,m} \overset{1}{\leftrightarrow} \partial\Lambda_{r-a})\right\}\right].\end{aligned}$$

We now send $r \to \infty$. Since $\theta^1(p,q,\alpha,h_c) = 0$ by assumption, the last summand tends to 0. By the bounded convergence theorem,

$$\limsup_{n\to\infty}\sup_{x\in\Lambda_n} J_{k,m,n}(x) \le \nu_{k,1}, \tag{8.19}$$

where

$$\nu_{k,1} = \frac{1}{L}\sum_{a=0}^{\infty}\phi(A_0=a)\min\left\{1, \frac{1}{L^{|\Lambda_a|}}\pi_{h_c}(x+\Lambda_a \overset{1}{\leftrightarrow} \partial\Lambda_{k/2})\right\}.$$

By the bounded convergence theorem again, $\nu_{k,1} \to 0$ as $k \to \infty$. Since (8.16)–(8.17) and (8.18) are uniform in $h \in [h_c - R, h_c]$, one may include the supremum over h in (8.19), as required for the lemma. □

Proof of Theorem 8.9. Let $f_n(h) = \pi_{n,h}(H_{k.m})$. By (5.4) and Lemma 8.10,

$$\frac{1}{f_n(h)[1-f_n(h)]}\frac{d}{dh}f_n(h) \ge cL\log\left[\frac{1}{2\max_x J_{k,m,n}(x)}\right], \tag{8.20}$$

with L as in the proof of Lemma 8.14. Let

$$\xi_{n,k,1} = \sup_{h\in[h_c-R,h_c]}\sup_{x\in\Lambda_n} 2J_{k,m,n}(x), \quad \xi_{n,m,0} = \sup_{h\in[h_c,h_c+R]}\sup_{x\in\Lambda_n} 2J_{k,m,n}(x).$$

By (8.20),

$$\log \frac{f_n(h)}{1-f_n(h)}\bigg|_{h_1}^{h_2} \ge (h_\mathrm{c}-h_1)cL\log(\xi_{n,k,1}^{-1}) + (h_2-h_\mathrm{c})cL\log(\xi_{n,m,0}^{-1}),$$

whence

$$f_n(h_1)[1-f_n(h_2)] \le \xi_{n,k,1}^{cL(h_\mathrm{c}-h_1)}\xi_{n,m,0}^{cL(h_2-h_\mathrm{c})}.$$

Take the limit as $n\to\infty$ and use Lemma 8.14. □

Proof of Proposition 8.7. A strictly positive measure μ on $\Omega = \{0,1\}^E$ is monotone if and only if: for all $\omega\in\Omega$ with $\omega(e)=\omega(f)=0$, $e\ne f$,

$$\mu(\omega^{e,f})\mu(\omega) \ge \mu(\omega^e)\mu(\omega^f), \tag{8.21}$$

see, for example, [19, Thm 2.19]. Given two strictly positive measures μ_1 and μ_2, at least one of which is monotone, it is sufficient for $\mu_1 \le_{\mathrm{st}} \mu_2$ that:

$$\frac{\mu_1(\omega^e)}{\mu_1(\omega)} \le \frac{\mu_2(\omega^e)}{\mu_2(\omega)} \qquad \omega\in\Omega,\ e\in E. \tag{8.22}$$

This is proved in [19, Thm 2.6]. Condition (8.22) is non-trivial only when $\omega(e)=0$.

We shall prove (i) by checking that ϕ_h satisfies (8.21). Write $\mathcal{C}(\omega)$ for the set of open clusters under ω, and let $f_h(k)=\alpha e^{hk}+1-\alpha$. Substituting (8.4) into (8.21), we must check

$$\phi_{p,q}(\omega^{e,f})\phi_{p,q}(\omega)\prod_{C\in\mathcal{C}(\omega^{e,f})} f_h(|C|)\prod_{C\in\mathcal{C}(\omega)} f_h(|C|)$$
$$\ge \phi_{p,q}(\omega^{e})\phi_{p,q}(\omega^f)\prod_{C\in\mathcal{C}(\omega^{e})} f_h(|C|)\prod_{C\in\mathcal{C}(\omega^f)} f_h(|C|). \tag{8.23}$$

On using the monotonicity of $\phi_{p,q}$, and on cancelling the factors $f_h(|C|)$ for $C\in\mathcal{C}(\omega)\cap\mathcal{C}(\omega^{e,f})$, we arrive at the following three cases.

(i) There are clusters $C_1, C_2\in\mathcal{C}(\omega)$, such that $C_1\cup C_2\in\mathcal{C}(\omega^e)=\mathcal{C}(\omega^f)$. It suffices that

$$qf_h(a)f_h(b)\ge f_h(a+b), \quad a=|C_1|,\ b=|C_2|,$$

and this is easily checked for $a,b\ge 0$ since $q\alpha, q(1-\alpha)\ge 1$.

(ii) There are clusters $C_1,C_2,C_3\in\mathcal{C}(\omega)$, such that $C_1\cup C_2\in\mathcal{C}(\omega^e)$ and $C_2\cup C_3\in\mathcal{C}(\omega^f)$. It suffices that

$$f_h(a+b+c)f_h(b)\ge f_h(a+b)f_h(b+c), \quad a=|C_1|,\ b=|C_2|,\ c=|C_3|,$$

and this is immediate.

(iii) There are clusters $C_1,C_2,C_3,C_4\in\mathcal{C}(\omega)$ such that $C_1\cup C_2\in\mathcal{C}(\omega^e)$ and $C_3\cup C_4\in\mathcal{C}(\omega^f)$. In this case, inequality (8.23) simplifies to a triviality.

It may be checked similarly that the marginal measure of $\kappa_h(\,\cdot \mid \sigma_x = b)$ on Ω is monotone if either $h \geq 0$, $b = 1$ or $h \leq 0$, $b = 0$. One uses the expression

$$\kappa_h(\omega \mid \sigma_x = b) \propto \phi_{p,q}(\omega) e^{hb|C_x(\omega)|} \prod_{C \in \mathcal{C}(\omega) \setminus \{C_x(\omega)\}} f_h(|C|), \qquad \omega \in \Omega.$$

Parts (ii) and (iii) then follow by checking (8.22) with appropriate μ_i. Part (iv) follows from part (iii) by taking the limit as $|h| \to \infty$. Many of the required calculations are rather similar to part (i), and we omit further details. $\square$

Proof of Theorem 8.1. We identify the spin-vector $\sigma \in \Sigma$ with the set $A = \{v \in V : \sigma_v = 1\}$. In order that $\pi = \pi_{p,q,\alpha}$ be monotone it is necessary and sufficient (see inequality (8.21)) that

$$\pi(A^{xy})\pi(A) \geq \pi(A^x)\pi(A^y), \qquad A \subseteq V,\ x, y \in V \setminus A,\ x \neq y. \tag{8.24}$$

Let $A \subseteq V$, $x, y \in V \setminus A$, $x \neq y$. Let a be the number of edges of the form $\langle x, z\rangle$ with $z \in A$, let b be the number of edges of the form $\langle x, z\rangle$ with $z \notin A$ and $z \neq x, y$, and let e be the number of edges joining x and y.

We write $A^x = A \cup \{x\}$, etc. By (8.5) with $h = 0$,

$$\frac{\pi(A^x)}{\pi(A)} = (1-p)^{b+e-a} \frac{Z_{A^x,q\alpha} Z_{\overline{A^x},q(1-\alpha)}}{Z_{A,q} Z_{\overline{A},q(1-\alpha)}} = \frac{\alpha}{1-\alpha} \cdot \frac{\phi_{\overline{A},q(1-\alpha)}(I_x)}{\phi_{A^x,q\alpha}(I_x)},$$

where I_x is the event that x is isolated, and $\phi_{A,q}$ is the random-cluster measure on the subgraph induced by vertices of A with edge-parameter p and cluster-weight q. Similarly,

$$\frac{\pi(A^{xy})}{\pi(A^y)} = \frac{\alpha}{1-\alpha} \cdot \frac{\phi_{\overline{A^y},q(1-\alpha)}(I_x)}{\phi_{A^{xy},q\alpha}(I_x)}.$$

The ratio of the left to the right sides of (8.24) is

$$\frac{\phi_{A^x}(I_x)}{\phi_{A^{xy}}(I_x)} \cdot \frac{\phi_{\overline{A^y}}(I_x)}{\phi_{\overline{A}}(I_x)} = \frac{\phi_{A^{xy},q\alpha}(I_x \mid I_y)}{\phi_{A^{xy},q\alpha}(I_x)} \cdot \frac{\phi_{\overline{A},q(1-\alpha)}(I_x \mid I_y)}{\phi_{\overline{A},q(1-\alpha)}(I_x)}. \tag{8.25}$$

Inequality (8.24) holds by the positive association of random-cluster measures with cluster-weights at least 1.

That the conditions are necessary for monotonicity follows by an example. Suppose $0 < q\alpha < 1$ and $q(1-\alpha) \geq 1$. Let G be a cycle of length four, with vertices (in order, going around the cycle) u, x, v, y. Take $A = \{u, v\}$ above, so that $e = 0$. The final ratio in (8.25) equals 1, and the penultimate is strictly less than 1. $\square$

Proof of Lemma 8.10. By Proposition 8.7(iv) and inequality (6.1),

$$\phi_h(I_x) \geq \phi_{p,Q}(I_x) \geq (1-p)^{\Delta},$$

where I_x is the event that x is isolated. Conditional on I_x, the spin of x under the coupling κ_h has the Bernoulli distribution with parameter $\alpha e^h/[\alpha e^h + 1 - \alpha]$. □

Acknowledgements

The second author acknowledges the hospitality of the Mathematics Department at the University of British Columbia, the Institut Henri Poincaré–Centre Emile Borel, Paris, and the Section de Mathématiques at the University of Geneva, where this work was largely done. Rob van den Berg proposed the elimination of equation (4.2) from the proofs.

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