A GENERALIZATION OF A THEOREM OF KLEITMAN AND MILNER

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Kleitman and Milner [1] present two proofs of an attractive result about the average cardinal of the sets of a Sperner family. One of their proofs uses ideas from linear programming; the other is based on Sperner's original proof of his lemma [2]. In this note, we give a simple proof of a more general theorem about lattices which contains their result as a special case.

Let L be a finite modular lattice with point set P and height function h. Associated with L is the partially ordered set $(P, \leqslant)$ which has zero element ϕ and unit element I. A *maximal chain* of L is a totally ordered set $\{a_0, a_1, \ldots, a_n\}$ where

$$\phi = a_0 < a_1 < \ldots < a_n = I, \quad a_i \in P \text{ and } n = h(I).$$

We write $\mu(a)$ for the number of maximal chains which pass through the point a, and w_i for the number of points of L with height i.

We call the finite lattice L *regular* if L is modular and satisfies the following conditions:

(1) $\mu(a) = \mu(b)$ whenever $h(a) = h(b)$,

(2) $\{w_i^{-1} : i = 0, 1, \ldots, n\}$ is a convex sequence; that is, $w_i^{-1} - w_{i-1}^{-1} (i = 1, 2, \ldots, n)$ increases with i.

Condition (2) is weaker than the condition that $\{w_i\}$ be log-concave (that is,

$$w_i^2 \geqslant w_{i-1} w_{i+1}, \qquad i = 1, 2, \ldots, n-1),$$

but is stronger than the condition that $\{w_i\}$ be unimodal. Two well known examples of regular lattices are the set of subsets of a finite set and the set of subspaces of a projective geometry, with inclusion as the partial order.

In the rest of this paper we shall suppose that L is a regular lattice of height n. By (1), we may write $\mu(h(a))$ for $\mu(a)$. Thus $\mu(n) = \mu(o)$ is the total number of maximal chains of L. We define $m(L)$ to be the greatest integer such that $\{w_i : i = 0, 1, \ldots, m(L)\}$ is a strictly increasing sequence. By (2), $w_i \leqslant w_{i-1}$ whenever $i+1 \geqslant m(L)$.

THEOREM. *Let $\mathscr{F}$ be a set of incomparable points of L such that $|\mathscr{F}| \geqslant w_k$, where $k \leqslant m(L)$. Then*

$$|\mathscr{F}|^{-1} \sum_{a \in \mathscr{F}} h(a) \geqslant k. \tag{3}$$

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If $\{w_i^{-1}\}$ *is a strictly convex sequence then equality holds in* (3) *if and only if* $\mathscr{F}$ *is the set of all points with height* k.

This reduces to the result of [**1**] when L is the lattice of subsets of the set $\{1, 2, \ldots, n\}$ ordered by inclusion. Other applications are to the set of subspaces of a projective or affine geometry.

Proof of the theorem

Let

$$p = |\mathscr{F}|^{-1} \sum_{a \in \mathscr{F}} h(a) \tag{4}$$

be the average height of the points in $\mathscr{F}$, and suppose that $p < k$.

It is immediate by counting maximal chains and using (1) that

$$\mu(n) = \mu(i)\, w_i, \; i = 0, 1, \ldots, n, \tag{5}$$

and

$$\sum_{a \in \mathscr{F}} \mu(h(a)) \leqslant \mu(n). \tag{6}$$

Let $\gamma : [0, n] \to \mathbb{R}$ be defined by $\gamma(x) = \mu(x)$ if x is an integer and by linear interpolation elsewhere. Then, by (5), $\gamma(x)$ is strictly decreasing for $0 \leqslant x \leqslant m(L)$, and is convex for $0 \leqslant x \leqslant n$. To see the latter, let i be an integer such that $1 \leqslant i \leqslant n-1$. Then the second central difference of γ satisfies

$$\begin{aligned} \Delta^2 \gamma(i) &= \gamma(i+1) - 2\gamma(i) + \gamma(i-1) \\ &= \mu(n)(w_{i+1}^{-1} - 2w_i^{-1} + w_{i-1}^{-1}) \\ &\geqslant 0 \end{aligned} \tag{7}$$

by (2). Thus

$$|\mathscr{F}|^{-1} \sum_{a \in \mathscr{F}} \gamma(h(a)) \geqslant \gamma(p) > \gamma(k), \tag{8}$$

and we deduce that

$$\sum_{a \in \mathscr{F}} \mu(h(a)) > |\mathscr{F}| \mu(k) \geqslant \mu(n), \tag{9}$$

which contradicts (6). Hence $p \geqslant k$.

If $\{w_i^{-1} : i = 0, 1, \ldots, n\}$ is a strictly convex sequence, then it is easy to see from (8) that $p = k$ if and only if $\mathscr{F}$ is the set of all points of L with height k.

References

1. D. Kleitman and E. C. Milner, " On the average size of the sets in a Sperner family ", to appear.
2. E. Sperner, " Eine Satz über Untermengen einer endlichen Menge ", *Math. Zeit.*, 27 (1928), 544–548.

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