British Journal of Mathematical and Statistical Psychology (1980), 33, 247–261 *Printed in Great Britain*

On taking up position in a group: A continuous-time Markov model for biased random movement

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A model is developed for movement by members of a group when this movement is random but is affected by a preference for a particular region of the space occupied by the group. Asymptotic distributions are derived from a continuous-time Markovian model for the case in which a group member may move to any unoccupied location in one jump, and a method for estimating the degree of attraction of the preferred region is given. Such a group is described as 'fluid'. A 'viscous' group is defined as one in which interchanges take place one step at a time. Movement in such groups was simulated on a computer and some results are given, including the apparently paradoxical finding that an approach tendency will be more strongly evident at asymptote if access to the preferred region is difficult than if it is easy.

1. Introduction

Observing the real world provides data that fall into two classes. We may observe the presence or occurrence of objects or events. Or, at a more abstract level, we may observe relations between these primary observables. Such relations may be clearly evident, for example when they are causal, have no exceptions, and are observable without error, or they may be statistical. In this case a major initial problem may be that of determining whether the apparent relationship is significant, and what its strength is. A particularly difficult instance is the problem of deciding whether apparently random movements by members of a group of animals, or displacements of objects in a collection, which alter their distribution in space over time, are accidental or may express a directional tendency. We offer below an attempt to analyse a simplified situation of this sort, in the hope that it may provide a starting point for modelling real situations.

It is probably rare that members of a group or crowd distribute themselves wholly randomly. Commonly some members will have a preference which influences where they move when the opportunity arises. Clumps may form by coalescence of members who are friends, acquaintances, or are for some reason mutually attractive. Or a goal may be topologically defined. For example, Hamilton (1971) has argued that flocking arises from a tendency to seek the centre of the group, though other explanations have been offered for this phenomenon (Treisman, 1975*a*, *b*). But although rates of movement in a crowd, and sex differences in these rates, have been studied (Henderson, 1971; Palmer & Bailey, 1975), there is little evidence on directional tendencies within a group. Problems arise in studying the structure and dynamics of groups or crowds because of difficulties in making and interpreting the observations and because of the many factors and variables which may be relevant. Recognizing the presence of subgroups or clumps may be difficult because the observer cannot perceive, or does not perceive the significance of, features which may be of significance to members of the group. Prior acquaintanceship may not be evident to him, nor may he understand, for example, that some feature of dress constitutes a covert uniform. And he may be uncertain what weight to give some observed asymmetry in the distribution of individuals of different types.

0007-1102/80/020247-15$02.00/0

Consider a hypothetical example: A mixed flock of sheep consists of animals of breed M (say, Merinoes) with a strong flocking tendency and members of a relatively non-gregarious breed N. If Hamilton's (1971) hypothesis were correct, Merinoes should tend to be found at the centre of the flock. If we describe the flock as containing c central and p peripheral locations the question becomes: Is there evidence of a tendency for M sheep to take the central positions? If an excess of M sheep were found at the centre it would be of interest to define a measure γ of their centripetal tendency. Such a measure could be compared for different breeds or employed as a dependent variable following different experimental treatments. This example assumes that movement occurs by interchange between neighbours. In the simplest case the N type has no directional preference and is amenable to interchanging. However, we might have more than two types, with varying centre-seeking tendencies, or even different topological preferences, varying dominance relationships, and relevant categorizations might overlap. Thus, as well as there being a distinction between the behaviour of M and N sheep, there might be differences between the preferences of males and females.

In this example the density within the group is assumed to be constant. In other cases it may vary. Consider a small audience in a large auditorium. If the distribution of the audience is initially random but some members have a tendency to seek the front row, expression of this preference will be relatively unimpeded since the seats are indifferent to the use made of them. In a crowd in a public place movements of individuals may cause density variations, the crowd becoming thick at one point, thin at another. The distribution, or changes in the distribution, within a group might also be affected by its overall shape, proceeding with different degrees of ease if it is linear or circular, for example. Another relevant factor is facility of movement within the group. We may define a 'fluid' group as one in which a member may move to any alternative location, wherever it may be, at one jump. In a 'viscous' group he may move to nearby locations only; larger movements require a step-by-step progression.

No attempt will be made here to analyse these variants. Instead a simplified situation will be defined and a model developed for it, and the effect of the fluidity or viscosity of the group will be examined for this situation.

2. A Markovian model for random movement within a group

We consider a group of T members distributed over T locations. The locations are of two types, C and P, there being exactly c 'central' locations of type C and p 'peripheral' locations of type P. The members are of two types, M and N, there being m of the former type, n of the latter. Members of type M have a bias to move to region C, members of type N have no directional preference. They are passive in that they do not initiate movement but present no obstacle to an interchange when this is provoked by an M animal. Or, instead of the group containing animals of type N, we may if we wish assume that it contains only sheep of type M together with vacant holes, i.e. unoccupied locations within the group to which an animal may move, and that there are n of the latter. In either case changes in group structure will depend on the M animals, and we have

$$m+n=c+p=T. \tag{1}$$

Most stochastic models for recurrent behaviour of the present sort are Markovian. Such models have the considerable advantage that they guarantee the existence of a unique asymptotic distribution which is independent of the starting configuration. We shall here consider Markov chains with fixed transition probabilities. It is common to employ discrete-time chains in which transitions are supposed to occur at fixed epochs. This has the disadvantage that many changes may take place in any given time interval, and the calculation of the probabilities of the many transitions possible in each time interval may be complex and unwieldy.

An alternative which has tended to be overlooked but which will often be easier and may be more realistic is a Markov chain which is continuous in time (Gray, 1967). One great advantage of this approach is that the probabilistic structure of continuous chains allows the probability that more than one transition takes place in a very short period of time to be disregarded. Thus the number of possible transitions from a given state of the chain may be very small indeed when compared with the possible changes in a unit interval allowed by the discrete-time model. It will also almost certainly be easier to find limiting distributions for the chain after a long length of time.

The present problem may be considered as a special case of a birth and death process, general solutions for which may be found elsewhere (e.g. Bharucha-Reid, 1960; Parzen, 1962). A particularly relevant discussion is given by Feller (1957). A model of this sort will be developed for the present case.

If each of the possible configurations of m sheep among T locations is equally likely then the distribution of the random number X of Merinoes occupying central locations is given by the hypergeometric distribution

$$P(X=r)=P_r=\binom{m}{r}\binom{T-m}{c-r}\bigg/\binom{T}{c}. \tag{2}$$

The m sheep move at random among the locations, multiple occupancy of any point being, of course, prohibited. At any time there are $m(T-m)$ possible transitions from any filled to any unoccupied point: the group is taken to be wholly fluid. This gives four possible transition types, according to whether movements are made between or within C or P regions, and we assume that this process of movement is a Markov chain determined by the probabilities

$$P_{\mathrm{CC}}(t,t+h)=\alpha h+o(h), \tag{3}$$

$$P_{\mathrm{CP}}(t,t+h)=\beta h+o(h), \tag{4}$$

$$P_{\mathrm{PC}}(t,t+h)=\gamma h+o(h), \tag{5}$$

$$P_{\mathrm{PP}}(t,t+h)=\delta h+o(h), \tag{6}$$

where $\alpha, \beta, \gamma, \delta > 0$, $P_{\mathrm{CP}}(t,t+h)$ is the probability that a given animal in a C location will move to a particular unoccupied P location in the time interval $(t,t+h)$, the other probabilities are similarly defined, and $o(h)$ is any function $f(h)$ such that $f(h)/h$ tends to 0 as $h \to 0$. Further, the probability of two or more transitions in $(t,t+h)$ is $o(h)$.

The set of states of this Markov chain is the set of $\binom{T}{m}$ possible configurations of the M sheep. There is no subset of states which is absorbing, and it follows that there exists a unique limiting distribution of the number $X(t)$ of M sheep occupying C points at time t which is independent of the initial configuration.

The infinitesimal transition probabilities of $X(t)$ are given by relations such as

$$P(X(t+h)=r+1\,|\,X(t)=r)=(m-r)(c-r)\gamma h+o(h) \tag{7}$$

and

$$P(X(t+h)=r-1\,|\,X(t)=r)=r(p-m+r)\beta h+o(h) \tag{8}$$

since there are $(m-r)(c-r)$ ways in which an animal may be added to the C region, depending upon which of the $(m-r)$ available occupants of P points chooses to move to which of the $(c-r)$ central holes. Each of these moves may occur with probability $\gamma h+o(h)$ and the probability that more than one change occurs is $o(h)$.

Writing

$$P_r(t)=P[X(t)=r]$$

we have

$$P_r(t+h)=\sum_{i=r-1}^{r+1} P_i(t)P[X(t+h)=r\,|\,X(t)=i]+o(h) \tag{9}$$

which gives

$$P_0(t+h)=(1-mc\gamma h)P_0(t)+(p-m+1)\beta h P_1(t)+o(h), \tag{10}$$

$$\begin{aligned}P_r(t+h)&=(m-r+1)(c-r+1)\gamma h P_{r-1}(t)\\&\quad+[1-(m-r)(c-r)\gamma h-r(p-m+r)\beta h]P_r(t)\\&\quad+(r+1)(p-m+r+1)\beta h P_{r+1}(t)+o(h),\quad 1\leqslant r<R\end{aligned} \tag{11}$$

$$P_R(t+h)=(m-R+1)(c-R+1)\gamma h P_{R-1}(t)+[1-R(p-m+R)\beta h]P_R(t)+o(h), \tag{12}$$

where $R=\min(m,c)$.

Letting h approach zero we obtain the derivatives

$$P_0'(t)=-mc\gamma P_0(t)+(p-m+1)\beta P_1(t), \tag{13}$$

$$\begin{aligned}P_r'(t)&=(m-r+1)(c-r+1)\gamma P_{r-1}(t)-[(m-r)(c-r)\gamma+r(p-m+r)\beta]P_r(t)\\&\quad+(r+1)(p-m+r+1)\beta P_{r+1}(t),\quad 1\leqslant r<R,\end{aligned} \tag{14}$$

$$P_R'(t)=(m-R+1)(c-R+1)\gamma P_{R-1}(t)-R(p-m+R)\beta P_R(t). \tag{15}$$

We wish to calculate the limiting distribution of $X(t)$ given by

$$\pi(r)=\lim_{t\to\infty}P_r(t),\quad 0\leqslant r\leqslant R. \tag{16}$$

Letting $t\to\infty$ the derivatives $P_r'(t)\to 0$ and equations (13) to (15) become

$$0=-mc\gamma\pi(0)+(p-m+1)\beta\pi(1), \tag{17}$$

$$\begin{aligned}0&=(m-r+1)(c-r+1)\gamma\pi(r-1)-[(m-r)(c-r)\gamma+r(p-m+r)\beta]\pi(r)\\&\quad+(r+1)(p-m+r+1)\beta\pi(r+1),\quad 1\leqslant r<R,\end{aligned} \tag{18}$$

$$0=(m-R+1)(c-R+1)\gamma\pi(R-1)-R(p-m+R)\beta\pi(R). \tag{19}$$

From equation (17), for $r=0$, we have

$$\pi(1)/\pi(0)=\frac{mc\gamma}{(p-m+1)\beta}. \tag{20}$$

It follows by mathematical induction that

$$\pi(r+1)/\pi(r)=\frac{(m-r)(c-r)\gamma}{(r+1)(p-m+r+1)\beta},\quad 0\leqslant r<R \tag{21}$$

and we deduce that

$$\pi(r)/\pi(0)=\frac{(\gamma/\beta)^r(m)_r(c)_r}{r!(T-m-c+r)_r}, \tag{22}$$

where $(a)_b=(a)(a-1)\ldots(a-b+1)$. Then

$$\pi(r)=(\pi(0)/P_0)\lambda^r P_r, \tag{23}$$

where $\lambda=\gamma/\beta$, $\{P_r\colon r=0,1,\ldots,R\}$ is the hypergeometric distribution given by equation (2) and

$$\pi(0)=P_0\Big/\sum_{r=0}^{R}\lambda^r P_r. \tag{24}$$

Equation (23) gives the limiting distribution for a fluid group. Note that it depends only upon the ratio of γ and β, and is independent of α and δ. This independence is consistent with the fact that transitions between pairs of C points, and between pairs of P points, do not affect the distribution of $X(t)$. The absolute values of γ and β are similarly irrelevant because a simultaneous change in scale of these parameters preserves the relative time scale of PC and CP transitions and thus does not affect the limiting distribution.

Equation (23) reduces to the ordinary hypergeometric distribution if and only if $\gamma=\beta$, which is to say that the attraction of the centre for members of the group at the periphery is the same as the attraction of the periphery to members of the group when at the centre.

A maximum-likelihood estimate of λ may be obtained from a series of j observations, assumed to be independent, of the quantity $X(t)$ after a long period has elapsed. Let these observations be $X_1, X_2, \ldots, X_j$. Then the estimator is the quantity $\hat{\lambda}$ which maximizes

$$\pi(X_1)\pi(X_2)\ldots\pi(X_j)=\hat{\lambda}^{S_x}P_{X_1}P_{X_2}\ldots P_{X_j}\Big/\left[\sum_{r=0}^{R}\hat{\lambda}^r P_r\right]^j, \tag{25}$$

where $S_x=\sum_{i=1}^{j}X_i$, and it can be shown by elementary calculus that $\hat{\lambda}$ is the unique value of the parameter λ such that the mean of the limiting distribution agrees with the sample mean S_x/j. That is, $\hat{\lambda}$ is the root of the equation

$$E(X\,|\,\lambda)=S_x/j, \tag{26}$$

where $E(X\,|\,\lambda)$, considered as a function of λ, is the mean value of $X(t)$ in its limiting distribution and is given by

$$E(X\,|\,\lambda)=\sum_{r=0}^{R}r\lambda^r P_r\Big/\sum_{r=0}^{R}\lambda^r P_r. \tag{27}$$

Equation (26) has a unique solution $\lambda=\hat{\lambda}$ because $E(X\,|\,\lambda)$ is an increasing function of λ. Thus, for any given sample, simple numerical techniques will yield the estimator $\hat{\lambda}$, which may be considered as a function of the parameters T, m and c and the sample mean S_x/j. Figure 1 shows $E(X\,|\,\lambda)$ plotted against λ for $T=48$, $c=12, 24$, $m=12, 24, 36$.

Suppose $\gamma+\beta=\kappa$, say; then $\gamma=\kappa\gamma'$ where $\gamma'=\lambda/(1+\lambda)$. From the previous observation that the limiting distribution depends only upon the ratio $\lambda=\gamma/\beta$ we might without loss of generality assume that $\kappa=1$. In Fig. 2, $E(X\,|\,\lambda)$ is plotted against γ'. As λ varies between 1 and ∞, γ' varies between 0·5 and 1. The figure shows that for the larger values of m (relative to R) and lower values of γ' the relation is nearly linear.

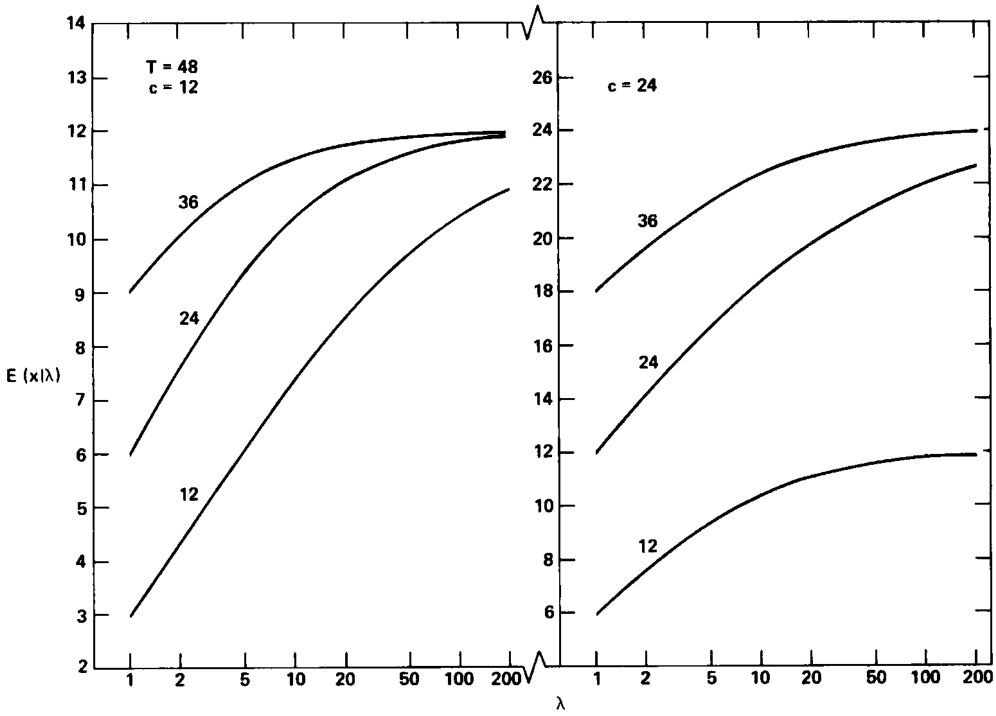

Figure 1. The mean of the limiting distributions, $E(X|\lambda)$, for a space of $T=48$ points, is plotted against λ (on a logarithmic scale). On the left $c=12$, on the right $c=24$. $m=12, 24, 36$ is shown as a parameter on the curves.
Note. The values on the right ordinate are twice those on the left.

As $\gamma' \rightarrow 1$ $(\lambda \rightarrow \infty)$, $E(X|\lambda) \rightarrow R$; this occurs because, as the relative attraction λ of the C points approaches ∞, as many sheep as possible (R) occupy central locations with probability approaching one.

3. Fluid and viscous groups: A computer simulation

Consider a hypothetical experiment to which such a model might apply. It is desired to determine whether subjects from a given population will prefer one colour scheme rather than another. A sample of 24 subjects are distributed randomly in two adjoining rooms, one decorated in purple, the other in cream, and each containing 24 seats. Subjects may move to any unoccupied seat in either room at any time. After a sufficient length of time, measures of the number of subjects in the cream room are taken and an estimate of λ is derived.

In this experiment subjects know the location of all free seats and may move any distance at any time. In most natural groups movement tends to be more restricted. For the most part members of a reasonably dense throng are best aware of the arrangement of the group in their near vicinity, and when they move they do so by interchanging with or shifting among near neighbours; they may progress in some preferred direction, if their direction of movement is not wholly decided by random local factors, by successive interchanges. The model and the hypothetical experiment represent an extreme case in which the group is wholly fluid. Since most natural groups

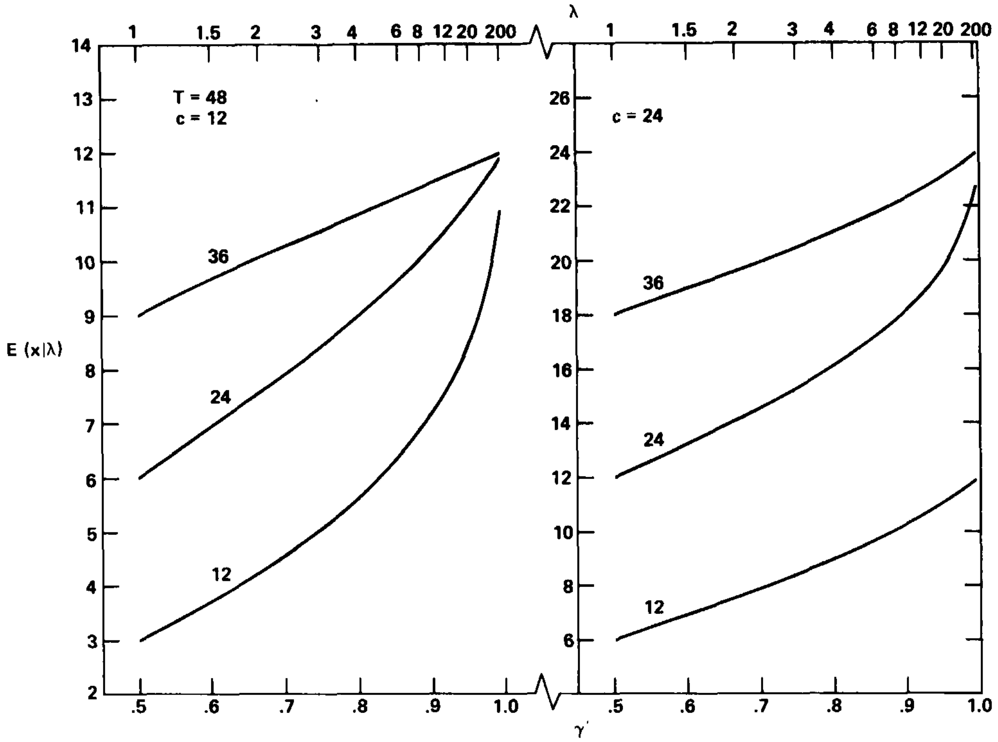

Figure 2. $E(X|\lambda)$ is plotted against $\gamma' = \lambda/(1+\lambda)$. $T=48$ points, $c=12$ on the left, $c=24$ on the right and $m=12$, 24 or 36. m is shown as a parameter on the curves.

are more viscous it would be desirable to be able to modify the model to take this into account.

The following would be an experiment at the opposite extreme. Instead of the seats in the two rooms being randomly distributed they are placed in a single continuous line extending through both rooms, so that 24 C seats are in the cream room and 24 P seats in the purple room. The $m=24$ subjects are seated initially at random positions. They may move at any time, but may only interchange with the neighbour (or vacant seat) to their immediate left or right. There is then only one P point from which subjects may move into the neighbouring set of C points, and vice versa, and their mobility has been reduced to a minimum.

The questions naturally arise what in this case the asymptotic distribution of X will be, and how this will relate to the strength of any C-seeking tendency. But these questions are more difficult to answer. A main reason for the tractability of the previous case was the possibility of assuming, when defining the generator of the chain, that any member could move to any unoccupied point in one jump. A more realistic model which restricts interchanges between the two sets of points to a defined interface between them and limits the range of movement is difficult to provide an analytic solution for.

Although this example is somewhat forced, it is analogous to problems which may arise in real, especially ethological situations. For example, Altmann (1979) has presented an interesting and extensive study of progressions in the yellow baboon, *Papio cynocephalus*. When proceeding from a sleeping site to a foraging area or from one feeding ground to another, these animals often proceed in an elongated formation which may sometimes become a single file. Altmann (1979) finds that most of the progression orders recorded were essentially random, but in the few deviant cases, usually in dangerous situations, juveniles and adult males were overrepresented in the front third, and adult males in the final third.

The study of such phenomena might be facilitated by models which are capable of taking account of the viscosity of the group (the range of locations over which interchanges may take place), the speed at which changes may proceed when danger is indicated, any effects of the elongated as against the linear form, and the length of time that danger is indicated, and which might allow an estimate of the strength of any tendency for certain group members to seek out particular positions to be extracted from the final disposition of the group.

Any model for, say, linear progressions in baboons, would require to take account of special features of the species and of the data, and no such model is offered here. Nevertheless, evidence on the properties of general, simplified situations of this type may be useful as a first step in model-building. We present below some information on the approach to asymptote in both fluid and viscous groups, and on the asymptotic distribution in viscous groups, obtained by simulating the movements of members of such groups on a digital computer. Four types of groups were studied: small and large fluid groups, and small and large viscous groups.

For the small fluid group a space of $T = 48$ locations was defined, consisting of 24 C points and 24 P points, and m members were initially randomly distributed in the space ($m = 12$ or $m = 24$). A series of trials was then run, each trial representing an interval Δt during which one movement will occur, any member moving to any unoccupied location. On each trial a movement was selected at random from the set of movements possible on that trial, the probability of being selected being determined by the simulation parameters α_s, β_s, γ_s and δ_s, defined in accordance with equations (3) to (6). For convenience it was taken that $\gamma_s + \beta_s = 1 = \alpha_s + \delta_s$, with $\alpha_s = \delta_s = 0{\cdot}5$. Then on a trial, immediately prior to which X (the number of members in the C region) equals r, the summed probability density of movements from C to P is given by $r(T-c-m+r)\beta_s$, of movements from one C location to another by $r(c-r)\alpha_s$, of movements from P to C by $(m-r)(c-r)\gamma_s$, and of movements from P to P by $(m-r)(T-c-m+r)\delta_s$. Then the probability that on this trial a P member moves to C is given by

$$(m-r)(c-r)\gamma_s/[(m-r)(c-r)\gamma_s + r(T-c-m+r)\beta_s + r(c-r)\alpha_s + (m-r)(T-c-m+r)\delta_s]$$

and the other probabilities are similarly defined.

The large fluid group differed only in that a space of 288 locations was defined (144 C points and 144 P points) containing $m = 72$ or $m = 144$ members. If we define the density of a group as $D = m/T$, then the small fluid group had two density levels, $12/48 = 0{\cdot}25$, and $24/48 = 0{\cdot}5$, and the large fluid group had the same two densities. In order that the numbers involved should be the same in simulations of the small and large groups, six small groups were run in parallel in each small group simulation: on each trial one of the groups was taken (in order) and a movement randomly selected for it. After each trial, X, the number of members of the six small groups, or the one large group, occupying C points was determined. At intervals of 1000 trials the mean value of X for the past 1000 trials, X_{1000}, was found.

The program ran until asymptote was reached. This was determined in the following arbitrary way. Each value of X_{1000}, after the first, was compared with the preceding value. If it was less than or equal to the preceding value this was noted. Once this had occurred twice the program ran for a further 1000 trials and the final value of X_{1000} was taken as an estimate of X_a, the asymptotic mean value of X. These programs

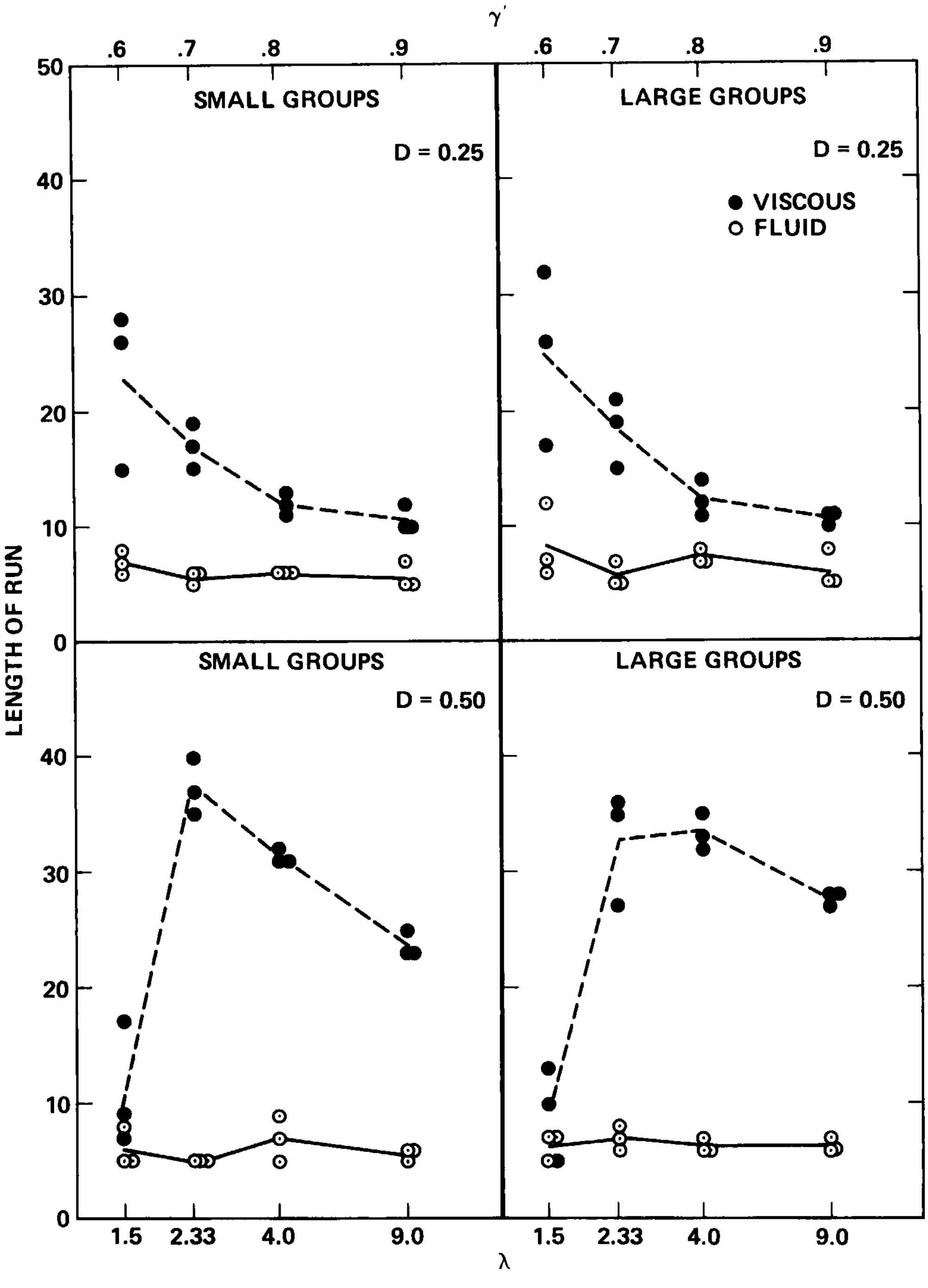

Figure 3. Length of run to asymptote in thousands of trials is plotted against λ (on a logarithmic scale). Small groups are shown on the left and large groups on the right, low density groups ($D = 0{\cdot}25$) above and high density groups ($D = 0{\cdot}5$) below.

can be considered to simulate groups in which members of type M are distributed in a space which also contains vacant locations, or else groups wholly filled with members of two types, M and N, N members being passive but willing to interchange when requested.

Movement in small and large viscous groups was also simulated, using the same parameter values. For the small viscous group the space was defined as a linear array of 48 locations, locations 1–24 being P points and locations 25–48 being C points. For the large group the space consisted of six such linear arrays in parallel, and may be visualized as a matrix with 6 rows and 48 columns, the left 24 columns constituting the P space and the right the C space. As before, six small groups were run in parallel. In a viscous group each member has the same number of possible moves: right, left, up or down. (It was taken that a member could interchange with another member, or with a space, in any neighbouring position on a row or column. An interchange with another member would produce no change in the distribution. A member in an extreme position could select an 'impossible move', i.e. to columns 0 or 49 or to row 0 or 7 for the large group, or any lateral move for the small group, but in this case no change in the distribution resulted.) On each trial a member was selected at random and one of his four possible moves chosen. Thus a P member could choose to move toward C with probability $0{\cdot}5\gamma_s$ or away from C with probability $0{\cdot}5\beta_s$, or up with probability $0{\cdot}5\alpha_s$, or down with probability $0{\cdot}5\delta_s$. Again $\gamma_s + \beta_s = 1$ and $\alpha_s = \delta_s = 0{\cdot}5$. Thus $\gamma' = \gamma_s$. To ensure comparability of run lengths, the same four alternatives were used for the small group although lateral choices produced no change. In these groups, although moves along the rows express directional preferences, only an interchange between columns 24 and 25 involving a hole can alter X.

In the fluid groups the sum of the bias parameters which retained a member in his original group $(\alpha_s + \delta_s)$ was made equal to the sum of the bias parameters expressing a directional tendency $(\beta_s + \gamma_s)$ for both small and large groups. To produce an analogous effect in the simulations of the viscous groups, on half the trials members chose a directional movement, i.e. one towards or away from the other compartment, and on half the trials a 'sideways' movement (up or down) at right angles to the C–P axis. The latter moves changed neither X nor the average distance of members in each compartment from the barrier between the compartments. So that small and large viscous groups should be comparable, the number of sideways moves was identical in each case. In the linear groups the decision to make such a move produced no change, so that effectively members 'marked time' on half their moves. In the large groups a sideways choice moved a member to the corresponding location in the adjacent (higher or lower) row, except that if a member in the highest row selected 'up' or a member in the lowest row selected 'down', no change occurred. Thus the number of moves that altered the distribution along the C–P axis was the same in each case; this should be borne in mind when comparing the simulations. Of course, other assumptions might have been made. Thus if members had only been allowed to choose among 'possible' moves, i.e. the real shifts in position open to them, this would make little difference to the rate of approach to the asymptote for the large viscous groups, but would double it for the linear groups.

The simulations examined the effects of viscosity, group size and group density. Some results are shown in Figs 3 and 4. The length of run to the end-point (in thousands), for three runs with each combination of parameters, is shown in Fig. 3 and the corresponding asymptotic values are plotted in Fig. 4. The estimated asymptotic value of X, X_a, is X_{1000} for the last 1000 trials simulated. The asymptote may also be expressed as the proportion of the members of the group in the C region: $P_a = X_a/6m$

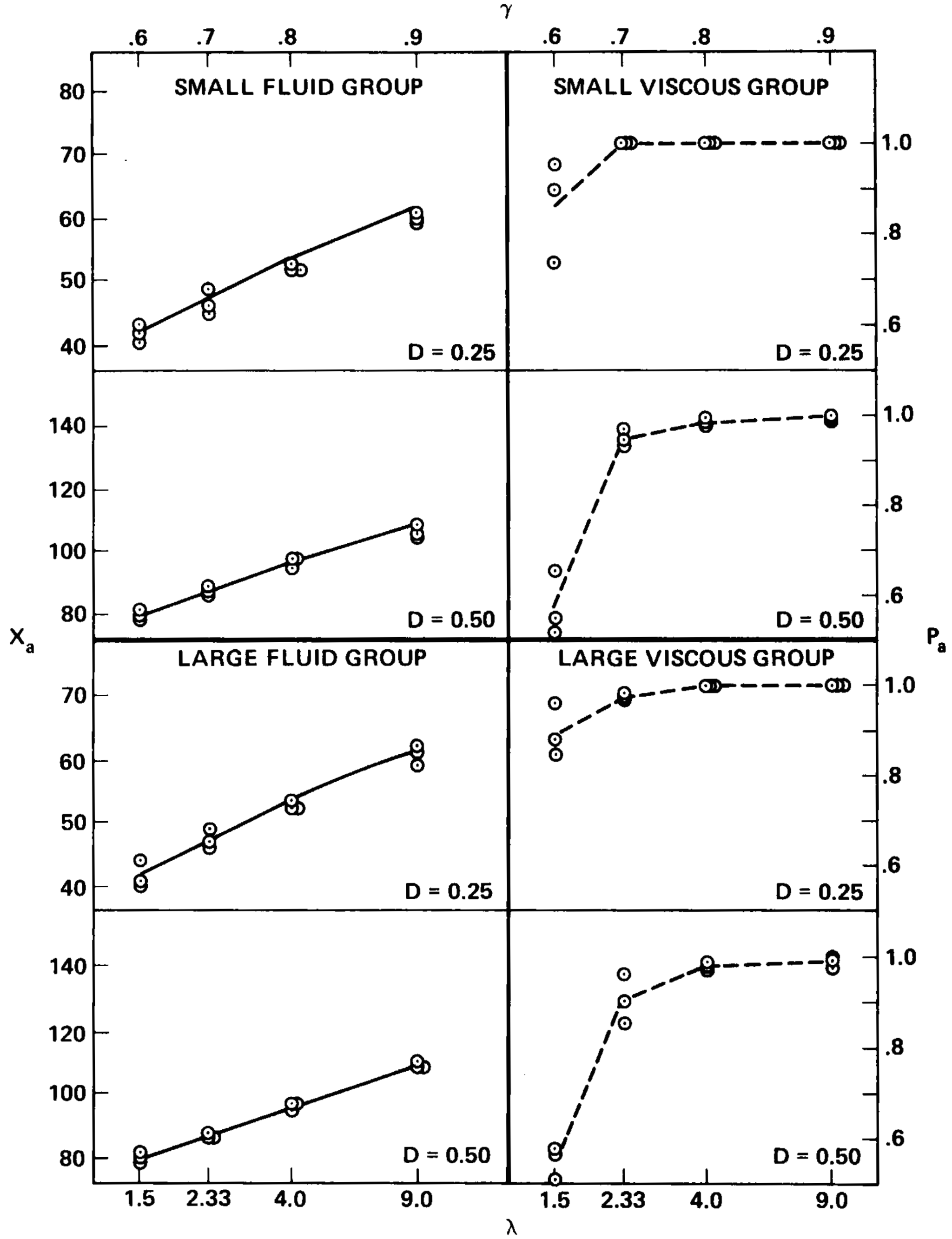

Figure 4. Asymptotic values, X_a, are plotted against λ. The end-points of three runs for each combination of parameters are shown. The curves on the left were calculated from equation (27), as in Fig. 1, for $T=48$, but are here fitted to both the small and large groups. For the viscous groups dashed lines join the mean values. For the small groups, X_a is summed over six small groups. P_a is the proportion of group members in the C region at asymptote.

for the small groups, $P_a = X_a/m$ for the large groups. The curves fitted to the fluid groups show $E(X \mid \lambda)$ plotted against λ (on a logarithmic scale) as in Fig. 1. Corresponding values of $\gamma' = \lambda/(1+\lambda)$ are also shown.

For a given density, the asymptotic proportions, P_a, are very similar for the small and large groups in both the fluid and viscous cases. The curves fitted to the fluid groups were calculated for $T = 48$ but, expressed as P_a, give an equally good fit for $T = 288$. Thus P_a does not appear to be affected by group size, for a fixed density. The good fit of the theoretical curves indicates that the criterion for recognizing asymptote was appropriate.

Group density does affect the asymptotes. At the higher density these are lower not only for the fluid groups, as might be expected from Fig. 1, but also for the viscous groups. This can be understood from the consideration that 12 members can be packed into 24 C locations so as to be well away from the transition locations (24 and 25) but this is not possible with 24 members.

In the viscous simulations there may be a slight tendency for asymptotes to be lower in the large groups than the small. Evidently the opportunities for lateral movement in the former did not facilitate access to the C region, nor did they shorten run length. Of course, in the simulation members were not engaged in strategic decisions when choosing whether to go forwards or sideways. They were concerned with the immediate neighbouring locations only, and could equally readily interchange with another member or a hole.

It is notable that for most combinations of parameters the viscous groups have asymptotes which are higher than those of the fluid groups. They do not show the same approximately linear relation to $\ln \lambda$, over the present range. Instead there is a small range of transition or 'threshold' above which the weaker approach tendencies produce asymptotic concentrations nearly as high as do the strong.

If viscosity does not reduce P_a, it does have a marked effect on run length. For the complete set of fluid simulations the average run was 6333 trials; for the viscous simulations 20 958 trials. The fluid groups approach asymptote at much the same rate for all combinations of parameters. The viscous runs are longer and decrease with λ, with one exception: at the lowest value and higher density the viscous groups never 'get started'. After a short period of oscillating at values just above 0·5 the criterion for end-point is reached. For the same λ and lower density an elevated asymptote is attained but a long run is required.

In considering the similar run lengths shown by the small and large groups in Fig. 3 it should be remembered that six small groups were run in parallel in each small group simulation, each constituent group being sampled once in every six trials. This embodies the assumption that a single trial represents a period Δt in which one in 288 members will choose to move. Thus the findings that members of small and large groups will move to asymptote at about the same rate applies when members have the same probability of moving in a given time interval in each case. One qualification should be noted, in relation to the viscous groups. We chose to compare movement in a (1×48) array with movement in a (6×48) array, rather than with, say, a (1×288) array, in order to test the guess that a member might reach the C region more quickly in a situation which offered more alternative routes. In a linear array, if movement along the straight line is blocked, the member must stay where he is. In a two-dimensional array there is the possibility that a lateral move may take him out of a blocked row into an open row and so facilitate his drift toward the C region.

Figure 3 shows that this hypothesis was false: the occurrence in the large groups of lateral movements, from one row to another, at right angles to the C–P direction, neither speeded nor slowed the rate at which the asymptote was approached. (Of course, in the present model each move is chosen at random without knowledge of the distribution of members in the space or the possible 'routes' open; a model embodying more strategic decisions made with greater knowledge might quite well produce a different result.) But in order to make this comparison, the number of opportunities to move in the C–P axis was made the same for both small and large groups; in each case there was a 50 per cent chance of choosing to move toward C or P, and a 50 per cent chance of a lateral choice. For the small groups, lateral moves were always 'dummy' moves since there were no alternative rows to move to: nothing happened.

For the large groups, one-sixth of the lateral moves were 'dummies': no change was made if a member attempted to move to rows 0 or 7, which did not exist, rather than between two neighbouring rows in the set 1 to 6. Another basis for comparison would have been to restrict members of small and large groups to 'possible' moves. This would have allowed members of the small groups to double the number of moves they made along the C–P axis. It would have allowed members of the large groups to increase C–P moves by a very much smaller proportion. On these assumptions, the speed of approach to asymptote would decrease as the breadth (number of rows) of the space increases.

Density does not affect the time taken to reach asymptote for fluid groups, but with viscous groups much longer runs are required at the higher density except at the lowest λ value.

Conclusion

A model has been developed for movement in a group when this is random but modified by a preference for a particular region. The model gives the asymptotic distribution for fluid groups, and fluid and viscous groups have been compared by computer simulation. A number of results have been obtained, of which perhaps the most striking is the apparently paradoxical finding that a tendency to approach the preferred region will be more evident at asymptote if access is difficult (the group is viscous) than if it is easy (the group is fluid). This can be understood from the consideration that if entry to the C region must be made by a series of interchanges then so must escape from it, and this may be even more difficult.

The model and the simulation results have been discussed in relation to aggregations such as groups of people or flocks of sheep, but they may be applicable to other situations in which small collections of elements interact randomly but with a possible directional basis. The movements of bacteria in a suspension, if some have a tropism determined by the magnetic field and others do not, the distribution of malignant cells in a tissue, or the shifting of pebbles in a pool might be possible applications.

Since the model is highly simplified and has been studied for only a subset of possible parameter values, it is obviously not possible to derive very strong conclusions for it. It is intended more as a starting point for the development of better fitting models for particular sets of data, than as a basis for their analysis as it stands.

Bearing these reservations in mind, two approaches to applying the model might be suggested. First, we may attempt to set up laboratory constraints similar to those envisaged by the model, and study whether the behaviour follows the predicted patterns. Thus, to revert to the rather contrived example discussed earlier, suppose we wish to examine unobtrusively the preferences a certain population might have for cream as against purple furnishings. Then we might set up experiments such as those described earlier, in which the rules of movement produce viscous or fluid groups, the former in linear or rectangular arrangements. If we have an independent measure of the subjects' preferences (for example, ratings obtained on a different occasion), or induce preferences by some prior procedure, we may then test the model (and the ratings) by examining whether subjects distribute themselves in the expected way. Predictions concerning the rate of approach to and the level of the asymptote could then also be tested. Specifically, we might expect to get the highest asymptote levels with viscous low-density groups, given long runs, with run length to asymptote having some relation to the strength of λ. We might also expect that when lateral moves are ruled out, the approach to the asymptote will be quicker. If, however, fluid movement is allowed, we might expect shorter runs to asymptote, and a closer relation between X_a and λ.

An alternative application would be to develop models for real situations, such as the ways animals dispose themselves under different conditions. Thus, for example, the viscosity of an animal progression under given conditions might be assessed by recording the average range of interchanges of position, the movement rates (Henderson, 1971; Palmer & Bailey, 1975), and the general form of the group. We could then examine whether a model on the present lines would provide an account of adjustments in distribution that might occur in response to external factors such as danger (Altmann, 1979). If the model is found to be applicable, one might then attempt to relate the magnitude of the external factor to a measure of λ.

As it stands, the model suggests some tentative predictions. If a group without strong internal organization responds to an external danger by semi-random redistribution, then one response to a rapid build-up of indications of danger, such as sightings of a predator, might be an increase in the fluidity of movement in the group: this would result in more rapid movement towards the more favourable distribution. But if the danger is at a low level but long maintained, as might be the case when traversing an unfavourable terrain during the course of a migration, then high viscosity might be favoured: although the optimal distribution will take longer to reach, it may be maintained at a higher level even if λ is quite low. If the opposite is observed, for example, bunching up, as sheep do at the suspicion of a wolf, i.e. increased viscosity in response to sudden danger, this might have the corollary that assumption of a defensive distribution would not rely on semi-random movements but on directed patterns of behaviour. In such a case, the problem of avoiding detection by the predator, and the nature of defensive reactions would also be relevant (Treisman, 1975*a*, *b*).

The present model cannot be related to any large body of established data: in the absence of appropriate interpretative concepts such data have not been collected. We hope that the existence of a model may facilitate research on movements in a group. The suggestions made above are tentative: they have been put forward to illustrate possible applications, especially but not only in relation to animal ethology.

Acknowledgement

We would like to thank Professor K. Karamcheti for providing research facilities while the second author was a member of the Joint Institute for Aeronautics and Acoustics, Stanford University, and J. Breaux for useful comments.

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Received 2 March 1979; revised version received 17 September 1979

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