ON SMALLEST TRIANGLES

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Abstract. Pide n points independently at random in $\mathbb{R}^2$, and let $\Delta_1^n \leq \Delta_2^n \leq \dots$ be the areas of the $\binom{n}{3}$ triangles thus formed, in non-decreasing order. Under conditions on the underlying measure μ of the points, we provide that $\{n^3\Delta_i^n : \tau \geq 1\}$ converges as $n \to \infty$ to a Poisson progress with a constant ? which depends on the measure μ. This result, and related conclusions, are proved by standard arguments of Poisson approximation, and may be extended to functionals more general than the area of a triangle. The work generalizes and extends considerably the conclusions in a recent paper of Jiang, Li, and Vitányi.

1. The problem

Drop n points independently and uniformly at random into the unit square $[0,1]^2$ of $\mathbb{R}^2$. Of the $\binom{n}{3}$ triangle thus formed, let Δ_1^n be the smallest area. It is proved in [JLV] that $\mathbb{E}(\Delta_1^n)$ lies between $c_1 n^{-3}$ and $c_2 n^{-3}$ for positive real c_1 and c_2. An association is proposed in [JLV] between this problem and the (much harder) Heilbronn problem, for which the reader is referred to [BAR] and the two references therein. A novel aspect of [JLV] is the use of Kolmogorov complexity.

Our target in this paper is to present a considerable strengthening of the above result, and to extend it to more general measures than uniform measure on $[0,1]^2$. This will be done using the now standard approach of Poisson approximation and the sos-called Chen–Stein method, see [BHJ]. From this point of view, the problem is a fairly simple exercise in probability theory, and does not merit the intervention of concepts imported from complexity theory.

In broad terms, our extensions are as follows. We shall prove that $n^3\Delta_1^4$ has, asymptotically, the exponential distribution with a parameter c which, while expressed only as an expectation, may be calculated numerically to any required degree of accuracy. Indeed, the sequence $\Delta_1^n \leq \Delta_2^n \leq \dots$ of smallest areas of triangles, written in non-decreasing order, is such that $n^3(\Delta_1^n, \Delta_2^n, \dots)$ converges as $n \to \infty$ to a Poisson process with constant intensity c. These weak limit theorems are complemented by the convergence of expectations obtained via an exponential bound on the tails of the A_i^n.

Such results appear in Section 3. We give special prominence to the case of the unit square $[0,1]^2$ since it has featured in [JLV], and since it contains most of the basic ingredients for a considerable extension. It is immediate that the same

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arguments are valid for points chosen uniformly from convex polygon P drawn in $\mathbb{R}^2$, with an amended value of c which depends on the choice of P. Furthermore, the arguments may be extended at any little extra cost to a family of points drawn from $\mathbb{R}^2$ according to a density function f satisfying certain conditions. this is discussed in Section 4.

Thus our results appear mostly in Sections 3 and 4. In Section 2, we set the scene by adapting to the present setting some of the basic theorems of Poisson approximation (see [BHJ]) and exponential bounds see [J98]).

Further extensions come readily to mind, but we have not worked through the details, namely extremal values of other functionals of bounded subsets of the n points, and working in higher dimensions. The basic ingredients of such extensions are the arguments used in Section 2. The triangle problem has a property which raises its profile above some other applications of the theory, as follows. We will see in Section 3 that, although the smallest triangle has small area, with order n^{-3}, its diameter does not tend (stochastically) to 0 as $n \to \infty$ but remains of order 1.

2. Generalities

Let μ be a probability measure on the bovel σ-field of $\mathbb{R}^2$, and let $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ be n points chosen independently at random according to the measure μ. These n points form $\binom{n}{3}$ triangle, whose areas we unite in non-decreasing order as $\Delta_1^n \leq \Delta_2^n \leq \dots$. For $\alpha > 0$, we shall be interested in the number $T^n(\alpha)$ of such triangles with area not exceeding αn^{-3}, which is to say that

$$T^n(\alpha) = |\{k : \Delta_k^n \leq \alpha n^{-3}\}|.$$

The target of this paper is to establish conditions on μ under which $T^n(\alpha)$ has an approximate Poisson distribution, and indeed the sequence $a^3(\Delta_1^n, \Delta_2^n, \dots)$ is approximately distributed as a Poisson process. In advance of stating such conditions, we shall in this section note certain sufficient conditions for Poisson limit results to hold.

We begin with the 'one-dimensional' distribution of $T^n(\alpha)$, and follow Barbour and Eagleson [BE] as interpreted in [BHJ, Thm 2.N].

Let $\mathbf{U}, \mathbf{V}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be chosen independently from $\mathbb{R}^2$ according to μ, and let $\Delta(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ be the area of the triangle with vertices $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$. We fix $\beta > 0$, and write

$$\begin{aligned}
\pi(\beta) &= \mathbb{P}(\Delta(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \leq \beta), \\
\pi_1(\beta) &= \mathbb{P}(\Delta(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \leq \beta, \Delta(\mathbf{X}, \mathbf{U}, \mathbf{V}) \leq \beta), \\
\pi_2(\beta) &= \mathbb{P}(\Delta(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \leq \beta, \Delta(\mathbf{X}, \mathbf{U}, \mathbf{V}) \leq \beta),
\end{aligned}$$

We let $\alpha > 0$ and write

$$\lambda^n(\alpha) = \binom{n}{3} \pi(\alpha n^{-3}).$$

In accordance with the Chen–Stein theory of Poisson approximation, we shall use the total-variation norm. For random variables A, B, we define the total-variation distance

$$d_{\mathrm{TV}}(A, B) = \sup_E |\mathbb{P}(A \in E) - \mathbb{P}(B \in E)|$$

where the ? is taken all appropriate events E.

We link $P_0(\gamma)$ for a random variable having the Poisson distribution with parameter γ.

Theorem 2.1. *For $\lambda > 0$ and $n \geq 3$,*

$$d_{\mathrm{TV}}(T^n(\alpha), P_0(\lambda^n(\alpha))) \leq \frac{1}{\lambda}(1-e^{-\lambda})\frac{n^4}{4}(n\pi^2 + n\pi_i + 2\pi_2)$$

where $\lambda = \lambda^n(\alpha)$, $\pi = \pi(\alpha n^{-3}), \pi_1 = \pi_1(\alpha n^{-3}), \pi_2 = \pi_2(\alpha n^{-3})$.

Proof. We note that the indicator functions $1_{\{\Delta_k^n \leq \beta\}}$ are dissociated, and apply [BHJ, Thm 2.N]. □

It is an immediate corollary that

$$\mathbb{P}(\Delta_1^n > \alpha n^{-3}) = \mathbb{P}(T^n(\alpha) = 0)$$

satisfies

$$\mathbb{P}(A_1^n > \alpha n^{-3}) - e^{-\lambda^n(\alpha)}\big| \leq \frac{1}{\lambda}(1-e^{-\lambda})(n\pi^2 + n\pi_1 + 2\pi_2), \tag{2.1}$$

whence $n^3\Delta_1^n$ has, asymptotically as $n \to \infty$, the exponential distribution with parameter $\lim_{n\to\infty} \lambda^n(\alpha)$, whenever this limit exists, and the right-hand side of (2.1) tends to zero. We shall verify these conditions for suitable μ in the subsequent sections.

We turn now to the set $\Delta^n\{\Delta_1^n, \Delta_2^n, \dots\}$.

Theorem 2.2. *Assume that, for all $\alpha > 0$:*

(a) *There exists $c, c' \in (0,\infty)$ such that $\lambda^n(\alpha) \leq c'^{\alpha}$ for all n, and $\lambda^n(\alpha) \to c^\alpha$ as $n \to \infty$ and*
(b) $n^4\pi_2(\alpha n^{-3}) + n^s\pi_i(\alpha n^{-3}) \to 0$ *as $n \to \infty$.*

Then $n^3\Delta^n$ converges gaugely to a Poisson progress ξ with constant intensity c, and, for all $T > 0$, the set $n^3\Delta^n \cap [0,T]$ converges weakly to the process ξ restricted to $[0,T]$.

Proof. We combine Theorem 3.2 and Corollary 3.2 of [J87], with $\mathcal{I}$ the semi-ring of half-open intervals of thereform $(a,b]$, $a,b \geq 0$. Note that, in the notation of [J87] , if $U \in \mathcal{I}$ is such that $U \subseteq [0,T]$ for some T, then

$$\mathbb{P}(X_j \in U) \leq \mathbb{P}(X_j \leq T),$$

and similarly for $\mathbb{P}(X_j \in U, X_k \in U')$. □

In Section 3 and 4 we shall present conditions under which (a) and (b) of Theorem 2.2 are valid.

The above claims ?? weak convergence but imply little about convergence of expectations, a matter provoked by material in [JLV]. We concentrate here on the random variable $a^3\Delta_1^n$, noting that similar arguments are valid for other elements of the set $n^3\Delta^n$. We shall require an estimate for the tail of $n^3\Delta_1^n$.

Theorem 2.3. *There exist positive absolute constants c_i such that*

$$\mathbb{P}(n^3\Delta_1^n > \alpha) \le \exp\left(-\lambda^n(\alpha)^2/M_n(\alpha)\right) \textit{ for } \alpha > 0$$

where

$$M_n(\alpha) = \max\{4\lambda^n(\alpha), 64\, n^5\pi_2(\alpha n^{-3}), 48\, n^{-1}\lambda^n(\alpha)^2\}.$$

Proof. This is an application of Theorem 4 of [J98], together with the general fact, proved in Lemma 3.3(a), that $\pi_1(\beta) \le \pi_2(\beta)$ for all $\beta > 0$. □

3. Uniform distribution of the unit square

This section contains an analysis of the very special instance in which μ is uniform measure on the unit square $S = [0,1]^2$. We concentrate on this case for two reasons. Firstly, it is the case considered in [JLV], and secondly, it is valuable to see an example of the required estimates where other minor but significant matters do not arise. It will be clear that exactly similar arguments are valid with μ the uniform distribution on the interior of any convex polygon in $\mathbb{R}^2$. A considerably more general result may be obtained, as presented in Section 4.

We assume henceforth in this section that μ is the uniform measure on S, and we next identify the constant c from an application of Theorem 2.2. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $(\mathbf{x} - \mathbf{y})$ denotes the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$.

Theorem 3.1. *Let $\mathbf{X}, bY$ be chosen independently according to μ, and let $l(\mathbf{X}, \mathbf{Y})$ be the Lebesque measure (length) of the interaction with S of the straight line through $\mathbf{X}$ and $\mathbf{Y}$. the constant*

$$c = \frac{2}{3}\mathbb{E}\left(\frac{l(\mathbf{X},\mathbf{Y})}{|\mathbf{X}-\mathbf{Y}|}\right) \tag{3.1}$$

exists and satisfies $0 < c < \infty$.

Proof. The only fact which requires proof is that $c < \infty$, for which the necessary estimate is as follows. There exists an absolute constant ν such that

$$\mathbb{P}(|\mathbf{X}-\mathbf{Y}| \le \epsilon \mid \mathbf{X}) \le \nu\epsilon^2 \quad \text{for all} \quad \epsilon > 0 \tag{3.2}$$

The diameter of S is $\sqrt{2}$, and therefore

$$\begin{aligned} c \le& \frac{2\sqrt{2}}{3}\mathbb{E}\left(|\mathbf{X}-\mathbf{Y}|^{-1}\right) \le \frac{2\sqrt{2}}{3}\int_{1/\sqrt{2}}^{\infty} \mathbb{P}(|\mathbf{X}-\mathbf{Y}|^{-1} > u)\, du \\ \le& \frac{2\sqrt{2}}{3}\int_{1/\sqrt{2}}^{\infty} \nu\frac{du}{U^2} = \frac{4\nu}{3}, \end{aligned}$$

as required. □

We next verify the conditions of Theorem 2.2, obtaining thereby the Poisson-process limit for the set $n^3\Delta^n$, as $n \to \infty$.

Lemma 3.2. *We have that $\lambda^n(\alpha) \le c\alpha$ for all α, n and $\lambda^n(\alpha) \to c\alpha$ as $n \to \infty$, for all $\alpha > 0$.*

Lemma 3.3.

(a) $\pi_1(\beta) \le \pi_2(\beta)$.

(b) *There exists an absolute constant A such that $\pi_2(\beta) \le A\beta^2 \log \beta^{-1}$ for $\beta < \frac{1}{2}$.*

It is an immediate consequence of this Lemma that

$$\pi_1(\alpha n^{-3}),\ \pi_2(\alpha n^{-3}) = 0(n^{-6}\log n) \mathit{for} \ \ \alpha > 0 \tag{3.3}$$

Hence $n^3\Delta^n$ satisfies the conclusions of Theorem 2.2, namely the vague and weak convergence to a Poisson process limit. An extra ingredient is needed to prove convergence of expectations, as stated next.

Theorem 3.4. *For $p = 1, 2, \ldots$, the random variable $\{|n^3\Delta_1^n|^p : n \ge 3\}$ are uniformly integrable, and therefore $\mathbb{E}(|n^3\Delta_1^n|^p) \to c^{-p}$ as $n \to \infty$, the pth moment of the exponential distribution with parameter c.*

The proof is given at the end of this section. [Presumably similar results are true for the A_k^n.]

Proof of Lemma 3.2. The letter λ denotes a straight line of $\mathbb{R}^2$, and $|\lambda \cup S|$ denotes the length of the intercept of λ with S. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $\mathbf{x} \ne \mathbf{y}$, we write $\lambda(\mathbf{x}, \mathbf{y})$ for the straight line through $\mathbf{x}$ and $\mathbf{y}$. For $\epsilon > 0$, write $S_\epsilon(\lambda)$ for the closed strip containing all points whose distance to λ does not exceed ϵ.

Let H be the absolute value of the length of the perpendicular dropped from $\mathbf{Z}$ onto $\lambda(\mathbf{X}, \mathbf{Y})$. We have that $\Delta = \Delta(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ satisfies

$$\begin{aligned}\mathbb{P}(\Delta \le \beta | \mathbf{X}, \mathbf{Y}) &= \mathbb{P}\left(H \le \frac{2\beta}{|\mathbf{X} - \mathbf{Y}|} \Big| \mathbf{X}, \mathbf{Y}\right) \\ &= \mathbb{P}\left(\mathbf{Z} \in S_{2\beta/|\mathbf{X}-\mathbf{Y}|} \big| \mathbf{X}, \mathbf{Y}\right) \\ \\ &\le \frac{4\beta l(\mathbf{X}, \mathbf{Y})}{|\mathbf{X} - \mathbf{Y}|}\end{aligned}$$

with equality if the event $c(\beta)$ does not recur, where $cl(\beta)$ in t he event that $S_{2\beta/|\mathbf{X}-\mathbf{Y}|}$ contains a corner of S. Thus

$$\beta^{-1}\big|\mathbb{P}(\Delta \le \beta) - \mathbb{E}\left(\frac{4\beta l(\mathbf{X}, \mathbf{Y})}{|\mathbf{X} - \mathbf{Y}|} 1_{c(\beta)}\right)\big| \le \mathbb{E}\left(\frac{4 l(\mathbf{X}, \mathbf{Y})}{|\mathbf{X} - \mathbf{Y}|} 1_{c(\beta)}\right).$$

We have by the absolute continuity of μ that $\mathbb{P}(c(\beta)) \to 0$ as $\beta \to 0$, and it follows by Theorem 3.1 and the monotone convergence theorem that

$$\beta^{-1}\mathbb{P}(\Delta \le \beta) \to \mathbb{E}\left(4l(\mathbf{X}, \mathbf{Y})|\mathbf{X} - \mathbf{Y}|\right) \quad \text{as } \beta \downarrow 0.$$

As an immediate consequence, we obtain that

$$\lambda^n(\alpha) = \binom{n}{3}\mathbb{P}(\Delta \le \alpha n^{-3}) = (1+o(1))\tfrac{1}{6}n^3 \cdot \frac{\alpha}{n^3}\mathbb{E}\left(\frac{4l(\mathbf{X},\mathbf{Y})}{|\mathbf{X}-\mathbf{Y}|}\right)$$
$$\to c\alpha \quad \text{as } n \to \infty$$

The uniform upper bound on $\lambda^n(\alpha)$ follows from (3.4) □

Proof of Lemma 3.3. (a) By the Cauchy–Schwarz inequality

$$\begin{aligned}\pi_t(\beta) &= \mathbb{E}(\mathbb{E}(\mathbb{P}(H \le \frac{2\beta}{|\mathbf{X}-\mathbf{Y}|}\bigg|\mathbf{X},\mathbf{Y})|\mathbf{X})^2) \\ &\le \mathbb{E}(\mathbb{E}(\mathbb{P}(H \le \frac{2\beta}{|\mathbf{X}-\mathbf{Y}|}\bigg|\mathbf{X},\mathbf{Y})^2)\bigg|\mathbf{X})) \\ &= \pi_2(\beta).\end{aligned}$$

(b) We have by (3.2) and (3.4) that, for $\epsilon > 0$,

$$\begin{aligned}\pi_2(\beta &= \mathbb{E}\left(\mathbb{P}(\Delta \le \beta \mid \mathbf{X},\mathbf{Y})^2\right) \\ &\le \mathbb{E}\left(1_{\{|\mathbf{X}-\mathbf{Y}|<\epsilon\}} + \left(\frac{4\beta R(\mathbf{X},\mathbf{Y})}{|\mathbf{X}-\mathbf{Y}|}\right)^2 1_{\{|\mathbf{X}-\mathbf{Y}|>\epsilon\}}\right). \\ &\le \nu\epsilon^2 + 32\beta^2\mathbb{E}\left(\frac{1_{\{|\mathbf{X}-\mathbf{Y}|>\epsilon\}}}{|bX-\mathbf{Y}|^2}\right). \qquad (3.5)\end{aligned}$$

Now,

$$\begin{aligned}\mathbb{E}\left(\frac{1_{\{|\mathbf{X}-\mathbf{Y}|>\epsilon\}}}{|bX-\mathbf{Y}|^2}\right) &= \int_{\frac{1}{2}}^{\epsilon^{-2}} \mathbb{P}(|\mathbf{X}-\mathbf{Y}|^{-2} > u)\,du \\ &= \frac{\nu}{u}\,du = \nu\log(2\epsilon^{-2}).\end{aligned}$$

We substitute into (3.5) and minimize over ϵ to obtain that there exists $A = A(\nu)$ such that

$$\pi_2(\beta) \le A\beta^2\log\beta^{-1} \quad \text{for} \beta < \tfrac{1}{2}.$$

□

Proof of Theorem 3.4. By a standard argument (see [GS, Section 7.10]), it suffices to show that, for $p \ge 1$, the set $\{\mathbb{E}(|n^3 X_1^n|^p) : n \ge \xi)$ is uniformly bounded. This is achieved by the exponential tail estimate of Theorem 2.3, which we make more precise as follows.

Consider the expression $M_n(\alpha)$ in Theorem 2.3.

By Lemmas 3.2 and 3.3,

$$M_n(\alpha) \le \max\left\{c_1, \alpha, \frac{c_2\alpha^2\log n}{n} + \frac{c_3\alpha}{n}, \frac{c_4\alpha^2}{n}\right\}$$

for positive constants c_i. Therefore

$$M_n(\alpha) \leq max\left\{c_1\alpha, c_5\left(\frac{\alpha^2 \log n}{n} + \frac{\alpha}{n}\right)\right\}$$
$$= \begin{cases} c_1\alpha & \text{if } \alpha < \frac{nc_1 - c_5}{c_5 \log n}, \\ c_5\left(\frac{\alpha^2 \log n}{n} + \frac{\alpha}{n}\right) & \text{otherwise.} \end{cases}$$

We apply Theorem 2.3 to obtain

$$\mathbb{P}(a^3\Delta_1^n > \alpha) \ \leq \begin{cases} e^{-c_6\alpha} & \text{if } \alpha < \frac{nc_1 - c_5}{c_5 \log n}, \\ \exp\left(-\frac{c_7 n}{\log n}\right) & \text{otherwise.} \end{cases}$$

Therefore

$$\mathbb{E}\left(|n^3\Delta_1^n|^p\right) \leq \int_0^{c_8 n/\log n} p\alpha^{p-1}e^{-c_6\alpha}dx$$
$$\int_{c_9 n/\log n}^{n^3} p\alpha^{p-1}\exp\left(-\frac{c_7 n}{\log n}\right)d\alpha$$

which is bounded above uniformly in n. □

4. The general case

The arguments of the last section are easily seen to be valid for the uniform distribution on the inside of any given polygon of $\mathbb{R}^2$, and indeed they may be adapted further without great difficulty. We present next general conditions under which the corresponding thermos remain valid. It is not our target to obtain conditions which are (in some sense) best possible, but rather to specify a reasonably broad class of distinctions including the uniform measures on the insides of piecewise-smooth closed curves of $\mathbb{R}^2$.

All measures in this section will be assumed to be absolutely continuous with respect of Lebesque measure on $\mathbb{R}^2$. Let $f : \mathbb{R}^2 \to [0, \infty)$ be a probability density function, so that f is integrable and $\int_{\mathbb{R}^2} f(\mathbf{x}d\mathbf{x} = 1$. There follow some conditions on f.

(A1) *The function f is bounded, in that there exists $M < \infty$ such that $f(\mathbf{x}) \leq M$ for all $\mathbf{x} \in \mathbb{R}^2$.*

This condition will prevent the clustering of points. Principal further conditions concern the margins of f. A line λ of $\mathbb{R}^2$ may be characterized by the polar coordinates (r, θ) of the foot of the perpendicular dropped from the origin to λ. We write f_λ for the integral of f along the line λ, and denote f_λ as $f_{r,\theta}$ where convenient.

(A2) *There exists $N \to \infty$ such that, for all r, θ, $f_{r,\theta} \leq N$.*

(A3) *For every $\theta \in [0, 2\pi)$, $f_{r,\theta}$ is a continuous function of r.*

It is standard that there exists density functions which do not satisfy A3 (see [St, page 32]).

Greater smoothness than A2 will be required, namely the boundedness of the second deviation of $f_{r,\theta}$ with respect to r. To require this uniformly in (r, θ) would be to eliminate applications to situations such as that treated in Section 3. Therefore we make a less stringent assumption.

(A4) *There exists a subset T (possibly empty) of $[0, \theta) \times [0, 2\pi)$ having Lebesque measure* 0*, and a constant $R < \infty$, such that, for all $(r, \theta) \notin T$, $d^2 f_{r,theta}/dr^2$ exists and saitsfies*

$$\left| \frac{d^2 f_{r,\theta}}{dr^2} \right| \leq R.$$

We are ready to state the main theorems. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be points chosen uniformly at random according to the density function f, and let Δ_i^n and related notation be given as before.

Theorem 4.1. *Assume that A1, A2, A3, A4 hold.*

(a) *The constant*

$$c = \tfrac{2}{3}\mathbb{E}\left(\frac{f_{R,\Theta}}{|\mathbf{X} - \mathbf{X}_2|} \right)$$

exists and satisfies $0 < c < \infty$, where (R, Θ) is the polar description of the line $\lambda(\mathbf{X}_1, \mathbf{X}_2)$.

(b) *The set $n^3\{\Delta_1^n, \Delta_2^n, \ldots\}$ converges rapidly to a Poisson process with constant intensity c, and the convergence is weak when restricted to any bounded interval of $\mathbb{R}$.*

(c) *$n^3\Delta_1^n$ converges weakly to the exponential distribution with parameter c, and all positive moments of Δ_1^n converges to the corresponding moments of the exponential distributions.*

Sketch proof. The extra steps hinge on the Taylor expansion

$$\mathbb{P}(\Delta \leq \beta \mid \mathbf{X}, \mathbf{Y}) = \frac{2\beta}{|\mathbf{X}, \mathbf{Y}|} f_{R,\Theta} + R\beta^3 \tag{4.1}$$

for any $\mathbf{X}, \mathbf{Y}$ such that the set

$$\left[R - \frac{2\beta}{|\mathbf{X} - \mathbf{Y}|}, R + \frac{2\beta}{|\mathbf{X} - \mathbf{Y}|} \right] \times \{\Theta\}$$

of lines does not intersect T. Since the join distinction of (R, Θ) is absolutely continuous with respect to Lebesque measure on $[0, \theta) \times [0, 2\pi)$, (4.1) holds on an event D_β having probability tending to 1 as $\beta \downarrow 0$. □

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