The shape theorem for the frog model*

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September 21, 2001

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Abstract

In this work we prove a shape theorem for a growing set of simple random walks on $\mathbb{Z}^d$, known as frog model. The dynamics of this process is described as follows: There are active particles, which perform independent discrete time SRWs, and sleeping particles, which do not move. When a sleeping particle is hit by an active particle, it becomes active too. At time 0 all particles are sleeping, except for that placed at the origin. We prove that the set of the original positions of all active particles, rescaled by the elapsed time, converges to some compact convex set.

Keywords: simple random walk, subadditive ergodic theorem

1 Introduction and results

We study a discrete time particle system on $\mathbb{Z}^d$ named frog model. In this model particles, thought of as frogs, move as independent simple random walks (SRWs) on $\mathbb{Z}^d$. At time zero there is one particle at each site of the lattice and all the particles are sleeping except for the one at the origin. The only active particle starts to perform a discrete time SRW. From then on

*The authors are thankful to CAPES/PICD, CNPq (300226/97–7 and 300676/00–0) and FAPESP (97/12826–6) for financial support.

when an active particle jumps on a sleeping particle, the latter wakes up and starts jumping independently, also performing a SRW. The number of active particles grows to infinity as active particles jump on sites that have not been visited before, awakening the particles that are sitting there. Let us underline that the active particles do not interact with each other and there is no "one-particle-per-site" rule.

This model is a modification of a model for information spreading that the authors have learned from K. Ravishankar. The idea is that every active particle has some information and it shares that information with a sleeping particle at the time the former jumps on the latter. Particles that have the information move freely helping in the process of spreading the information. The model that we deal with in this paper is a discrete-time version of that proposed by R. Durrett (1996, private communication), who also suggested the term "frog model".

To the best of our knowledge, the first published result on this model is due to Telcs, Wormald [10], where it was referred to as the "egg model". They proved that, starting from the initial configuration defined above, the origin will be visited infinitely often a.s. Popov [9] proved that the same is true in dimension $d \geq 3$ for the initial configuration with a sleeping particle (or "egg") at each $x \neq 0$ with probability $\alpha/\|x\|^2$, α being a large positive constant. Recently A. Ramirez and V. Sidoravicius communicated to us that they are working on a continuous-time analog of this model, and that they have proved some results such as shape theorem and convergence to the product of Poissons.

We now define the process in a formal way. Let $\{(S_n^x)_{n\in\mathbb{N}}, x \in \mathbb{Z}^d\}$ be independent SRWs such that $S_0^x = x$ for all $x \in \mathbb{Z}^d$. For the sake of cleanness let $S_n := S_n^0$. These sequences of random variables give the trajectory of the particle placed originally at site x, starting to move at the time it wakes up. Let $t(x,z) = \min\{n : S_n^x = z\}$, remembering that if $d \geq 3$, then $\mathbf{P}[t(x,z) = \infty] > 0$. For technical reasons, besides the process which starts from the initial configuration defined above, we need also to introduce the processes starting from one active particle in x and sleeping particles elsewhere, $x \in \mathbb{Z}^d$. Define by

$$T(x,z) = \inf\Big\{\sum_{i=1}^{k} t(x_{i-1}, x_i) : x = x_0, x_1, \dots, x_k = z \text{ for some } k\Big\}, \qquad (1.1)$$

the passage time from x to z for the frog model. It means that, if the process starts from just one active particle sitting at site x, in the sense that initially that particle is the only active one, then $T(x,z)$ is the time it takes to awaken the particle at site z. Note that the particle which awakens z need not be that from x.

Now, let $Z_y^x(n)$ be the location (at time n) of the particle that started from site y in the process in which the only active particle at time zero was at site x. Formally, we have $Z_x^x(n) = S_n^x$, and

$$Z_y^x(n) = \begin{cases} y, & \text{if } T(x,y) \geq n, \\ S^y_{n-T(x,y)}, & \text{if } T(x,y) < n. \end{cases}$$

Since every random variable of the form $Z_y^x(n)$ is constructed using the same realization of the the random variables $\{(S_n^x)_{n\in\mathbb{N}}, x \in \mathbb{Z}^d\}$, this defines a coupling of processes $\{Z^x, x \in \mathbb{Z}^d\}$, where $Z^x := \{Z_y^x(n) : y \in \mathbb{Z}^d,\ n \in \mathbb{N}\}$. The idea is that as soon as a particle becomes active, it follows the same trajectory in all the processes Z^x.

With the help of these variables we define the sites whose originally sleeping particles have been awakened by time n, provided that initially the only active particle was in x. Namely

$$\xi_n^x = \{y \in \mathbb{Z}^d : T(x,y) \leq n\}.$$

We are mostly concerned with $\xi_n := \xi_n^0$ and its asymptotic behaviour. In order to analyse that behaviour, we define

$$\bar{\xi}_n^x = \{y + (-1/2, 1/2]^d : y \in \xi_n^x\} \subset \mathbb{R}^d,$$

and $\bar{\xi}_n := \bar{\xi}_n^0$.

The main result of this paper is the following

Theorem 1.1 *For any dimension $d \geq 1$ there is a non-empty convex set $\mathsf{A} = \mathsf{A}(d) \subset \mathbb{R}^d$ such that, for any $0 < \varepsilon < 1$*

$$(1-\varepsilon)\mathsf{A} \subset \frac{\bar{\xi}_n}{n} \subset (1+\varepsilon)\mathsf{A}$$

for all n large enough a.s.

Note that, although Theorem 1.1 establishes the existence of the asymptotic shape A, it is difficult to identify this shape exactly. Of course, A is symmetric, the origin belongs to the interior of A, and $\mathsf{A} \subset \mathcal{D}$, where

$$\mathcal{D} = \{x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d : |x^{(1)}| + \cdots + |x^{(d)}| \leq 1\}.$$

Also, note that if the initial configuration is augmented (i.e. some new particles are added), then the asymptotic shape (when it exists) augments as well. We are going to show that if the initial configuration is rich enough, then the limiting shape A may contain some pieces of the boundary of $\mathcal{D}$ (a "flat edge" result).

To formulate that result, we need some additional notation. For $d \geq 2$ and $1 \leq i < j \leq d$ let

$$\Lambda_{ij} = \{x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d : x^{(k)} = 0 \text{ for } k \neq i, j\},$$

and for $0 < \beta < 1/2$ let

$$\Theta_{ij}^{\beta} = \{x = (x^{(1)}, \dots, x^{(d)}) \in \Lambda_{ij} : |x^{(i)}| + |x^{(j)}| = 1, \min\{|x^{(i)}|, |x^{(j)}|\} \geq \beta\}.$$

Define Θ^{β} to be the convex hull of $(\Theta_{ij}^{\beta})_{1 \leq i < j \leq d}$. Denote by A_m the asymptotic shape in the frog model when the initial configuration is such that any site $x \in \mathbb{Z}^d$ contains exactly m particles. The existence of A_m for arbitrary m can be derived in just the same way as in the case $m = 1$ (Theorem 1.1).

Theorem 1.2 *For each $d \geq 2$ there exists $m_0 = m_0(d)$ such that for all $m \geq m_0$ there exists $0 < \beta < 1/2$ such that $\Theta^{\beta} \subset \mathsf{A}_m$.*

The paper is organized in the following way. Section 2 contains some well known results about large deviations and SRW on $\mathbb{Z}^d$. We need these results later in the course of the proof of Theorem 1.1. In Section 3 we apply the subadditive ergodic theorem to our model, and the rest of the proof of the shape theorem is given in Section 4. Besides that, in Section 4 we prove the "flat edge" result.

2 Basic facts

Along this section we state some facts about large deviations and random walks which we need in order to prove our results. As usual, $C, C_1, C_2, \dots$ stand for positive finite constants. For what follows we use these constants freely. Also, $\lfloor x \rfloor$ stands for the largest integer which is less than or equal to x, while $\lceil x \rceil$ is the smallest integer which is greater than or equal to x.

The following large deviation result is an immediate consequence of Theorem 1.1, p. 748 of [8].

Lemma 2.1 *Let $\{X_i, i \geq 1\}$ be i.i.d. positive random variables such that there are $C_1 > 0$ and $0 < \alpha < 1$ such that for all n*

$$\mathbf{P}[X_i \geq n] \leq C_1 \exp\{-n^\alpha\}. \tag{2.1}$$

Then there exist $a > 0,\ 0 < \beta < 1$ and $C_2 > 0$ such that for all n

$$\mathbf{P}\Big[\sum_{i=1}^n X_i \geq an\Big] \leq C_2 \exp\{-n^\beta\}.$$

Let

$$\mathsf{R}_n^{\mathsf{B}} = \{S_i^{\mathsf{B}} : 0 \leq i \leq n\} = \{y \in \mathbb{Z}^d : t(x,y) \leq n \text{ for some } x \in \mathsf{B}\}$$

be the set of distinct sites visited by the family of SRWs starting from the set of sites B, up to time n. Some authors refer to R_n^0 as the range of SRW. As usual, $|\mathsf{R}_n^{\mathsf{B}}|$ stands for the cardinality of $\mathsf{R}_n^{\mathsf{B}}$. A useful basic fact is that $|\mathsf{R}_n^{\mathsf{B}}| \leq (n+1)|\mathsf{B}|$.

Theorem 2.1 (see e.g. [4], p. 333, 338.)

- *If $d = 2$ then there is $a_2 > 0$ such that*

$$\lim_{n\to\infty} \frac{\mathbf{E}|\mathsf{R}_n^0|}{n/\log n} = a_2. \tag{2.2}$$

- *If $d \geq 3$ then there is $a_3 := a_3(d) > 0$ such that*

$$\lim_{n\to\infty} \frac{\mathbf{E}|\mathsf{R}_n^0|}{n} = a_3. \tag{2.3}$$

Let $\mathsf{p}_n(x) = \mathbf{P}[S_n = x]$ and $\|x\|$ be the Euclidean norm. From Section 3 onwards we also work with the norm $\|x\|_1 = \|(x^{(1)}, \ldots, x^{(d)})\|_1 = \sum_{i=1}^d |x^{(i)}|$. Let $G_n(x) := \sum_{j=0}^n \mathsf{p}_j(x)$ be the mean number of visits to site x up to time n and $G(x) := G_\infty(x)$. These are the well known *Green's functions.* Let $\mathsf{q}(n,x) = \mathbf{P}[t(0,x) \leq n]$.

Theorem 2.2

- *If $d = 2, x \neq 0$ and $n \geq \|x\|^2$, then there exists $C_2 > 0$ such that*

$$\mathsf{q}(n,x) \geq \frac{C_2}{\log\|x\|}. \tag{2.4}$$

- *Suppose that $d \geq 3, x \neq 0$ and $n \geq \|x\|^2$. Then there exists $C_3 = C_3(d) > 0$ such that*

$$\mathsf{q}(n,x) \geq \frac{C_3}{\|x\|^{d-2}}. \tag{2.5}$$

Proof. Suppose without loss of generality that $\|x\|^2 \leq n \leq \|x\|^2 + 1$. Observe that

$$\begin{aligned} G_n(x) &= \sum_{j=0}^{n} \mathsf{p}_j(x) = \sum_{j=0}^{n}\sum_{k=0}^{j} \mathsf{p}_k(0)\mathbf{P}[t(0,x) = j-k] \\ &= \sum_{k=0}^{n} \mathsf{p}_k(0)\mathsf{q}(n-k,x) \leq \mathsf{q}(n,x)G_n(0). \end{aligned}$$

So

$$\mathsf{q}(n,x) \geq \frac{G_n(x)}{G_n(0)} \geq \begin{cases} \dfrac{\sum_{j=\lfloor n/2\rfloor}^{n} \mathsf{p}_j(x)}{\sum_{j=0}^{n} \mathsf{p}_j(0)}, & d = 2, \\ (G(0))^{-1}\sum_{j=\lfloor n/2\rfloor}^{n} \mathsf{p}_j(x), & d \geq 3. \end{cases}$$

Using Theorem 1.2.1 of [6], after some elementary computations we finish the proof. □

3 Subadditive ergodic theorem

The basic tools for proving shape theorems are the subadditive ergodic theorems. Next we state a result of Liggett (1985) (cf. [7]), which is an improved version of Kingman's subadditive ergodic theorem (cf. [5]).

Theorem 3.1 *Suppose that $\{Y(m,n)\}$ is a collection of positive random variables indexed by integers satisfying $0 \leq m < n$ such that*

- *$Y(0,n) \leq Y(0,m) + Y(m,n)$ for all $0 \leq m < n$ (subadditivity);*
- *The joint distribution of $\{Y(m+1, m+k+1), k \geq 1\}$ is the same as that of $\{Y(m,m+k), k \geq 1\}$ for each $m \geq 0$;*
- *For each $k \geq 1$ the sequence of random variables $\{Y(nk,(n+1)k), n \geq 1\}$ is a stationary ergodic process;*
- *$\mathbf{E}Y(0,1) < \infty$.*

Then

$$\lim_{n\to\infty} \frac{Y(0,n)}{n} \to \gamma \qquad a.s.,$$

where

$$\gamma = \inf_{n\geq 0} \frac{\mathbf{E}Y(0,n)}{n}.$$

In the sequel we show that the hypotheses of Theorem 3.1 hold for $Y(m,n) := T(mx,nx)$, for each fixed $x \in \mathbb{Z}^d$.

First of all observe that the set of variables $\{T(x,y) : x,y \in \mathbb{Z}^d\}$ defined in Section 1 is *subadditive* in the sense that

$$T(x,z) \leq T(x,y) + T(y,z) \tag{3.1}$$

for all $x,y,z \in \mathbb{Z}^d$ and *all* the realizations of the random variables S_n^x. Indeed, if site z is reached before site y, then (3.1) is evident. If that does not happen, recall that the random variables $T(y,z)$, $y,z \in \mathbb{Z}^d$ are constructed using the same collection of the random variables S_n^x, i.e., each particle follows the same trajectory as soon as it wakes up. So the process departuring from only site y awakened (the one which gives the passage time $T(y,z)$) is coupled with the original process (i.e., that started from x), and for the latter one may have other particles awakened at time $T(x,y)$ besides that from y. Consequently, from (1.1) it follows that $T(x,z) - T(x,y)$, which is the remaining time to reach site z for the original process, is less than or equal to $T(y,z)$, thus proving (3.1).

The second hypothesis, as well as the fact that for fixed $x \in \mathbb{Z}^d$ and $k \in \mathbb{N}$ the sequence $\{T((n-1)kx, nkx), n \geq 1\}$ is *stationary*, immediately follow from the definition. *Ergodicity* holds because the sequence of random variables $\{T((n-1)kx, nkx), n \geq 1\}$ is strongly mixing. That can be checked easily because the events $\{T(n_1kx, (n_1+1)kx) = a\}$ and $\{T(n_2kx, (n_2+1)kx) = b\}$ are independent provided that $a + b < \|(n_1 - n_2)kx\|_1$.

It is simple to see that the fourth hypothesis holds when $d = 1$. To see that remember that for $\tau =$ *the first return to the origin of a SRW*, we can assure that $\mathbf{P}[\tau > t] \leq Ct^{-1/2}$. Besides that, in a random time with exponential tail we will have at least three awakened particles jumping independently in the frog model. Combining these two facts we have that $\mathbf{E}T(0,1) < \infty$. So, for $d = 1$ one gets $T(0,n)/n \to \gamma$ a.s., and consequently we have the proof of Theorem 1.1 with $\mathsf{A} = [-\gamma^{-1}, \gamma^{-1}]$ in dimension 1. Thus, from now on we assume that $d \geq 2$.

To take care of the fourth hypothesis in dimension $d \geq 2$, we need the following result.

Theorem 3.2 *For all $d \geq 2$ and $x_0 \in \mathbb{Z}^d$ there exist positive finite constants $C = C(x_0, d)$ and $\beta = \beta(d)$ such that*

$$\mathbf{P}[T(0, x_0) \geq n] \leq C \exp\{-n^\beta\}$$

for all n.

Proof. We begin by considering the case $d \geq 3$. Pick $n \geq \|x_0\|^2$. Fix some $0 < \varepsilon < 1$ (to be chosen later). Denote for $1 \leq i \leq \lfloor d/2 \rfloor$

$$\mathsf{D}_{i,\varepsilon} := \{x \in \mathbb{Z}^d : \|x\| \leq i n^{1/2+\varepsilon}\},$$

and define the event

$$A_1 := A_1(n, \varepsilon) := \{|\mathsf{R}_n^0 \cap \mathsf{D}_{1,\varepsilon}| \geq r_1 n^{1-\varepsilon}\}, \tag{3.2}$$

where $r_1 = r_1(d)$ is a positive constant to be chosen later.

Lemma 3.1 *For any $d \geq 3$ the number r_1 can be chosen in such a way that for some positive constants α_1, α_1' and all n we have*

$$\mathbf{P}[A_1] \geq 1 - \alpha_1 \exp\{-\alpha_1' n^\varepsilon\}. \tag{3.3}$$

Proof. By Theorem 2.1 and the fact that for any random variable X with $0 \leq X \leq a$ a.s. and $\mathbf{E}X \geq b$ it is true that $\mathbf{P}[X \geq b/2] \geq b/(2a)$, it follows that for some $r_1, C_1 > 0$

$$\mathbf{P}[|\mathsf{R}_k^0| \geq r_1 k] \geq C_1. \tag{3.4}$$

Let

$$A_1' := A_1'(n, \varepsilon) := \{|\mathsf{R}_n^0| \geq r_1 n^{1-\varepsilon}\}.$$

Divide the time interval $[0, n]$ into (roughly speaking) n^ε disjoint intervals of size $n^{1-\varepsilon}$. For each subinterval of size $n^{1-\varepsilon}$, the cardinality of the corresponding subrange does not depend on the cardinalities of other subranges, so using (3.4) with $k = n^{1-\varepsilon}$ one gets that

$$\mathbf{P}[A_1'] \geq 1 - (1 - C_1)^{n^\varepsilon}. \tag{3.5}$$

Consider the event

$$B = B(n, \varepsilon) = \{ \sup_{0 \le i \le n} \|S_i^0\| < n^{1/2+\varepsilon} \}$$

and observe that by Lemma 1.5.1 of [6] there is C_2 such that

$$\mathbf{P}[B^c] \le C_2 \exp\{-n^{\varepsilon}\}. \tag{3.6}$$

Combining (3.5) and (3.6), we get (3.3). □

Now consider the finite sequence of times

$$n_1 := n, n_2 := n + 9n^{1+2\varepsilon}, \dots, n_{\lfloor d/2 \rfloor} := n + n^{1+2\varepsilon} \sum_{j=2}^{\lfloor d/2 \rfloor} (2j-1)^2.$$

Define the random sets

$$\tilde{G}_1 = \{x \in \mathsf{D}_{1,\varepsilon} : t(0, x) \le n_1\}$$

and, for $k = 2, \dots, \lfloor d/2 \rfloor$,

$$\tilde{G}_k = \{x \in \mathsf{D}_{k,\varepsilon} \backslash \mathsf{D}_{k-1,\varepsilon} : \text{there exists } y \in \tilde{G}_{k-1} \text{ such that } t(y, x) \le n_k - n_{k-1}\}.$$

For $k = 2, \dots, \lfloor d/2 \rfloor$ define the events

$$A_k = A_k(n, \varepsilon) = \{|\tilde{G}_k| \ge r_k n^k\},$$

where the numbers $r_k = r_k(d)$, $k = 2, \dots, \lfloor d/2 \rfloor$, will be chosen later. Define also

$$\varepsilon(k) := \begin{cases} \varepsilon/2, & \text{if } k = 1, \\ \varepsilon, & \text{if } k \ge 2. \end{cases}$$

Lemma 3.2 *Let $d \ge 4$. One can choose the numbers r_i, $i = 2, \dots, \lfloor d/2 \rfloor$, in such a way that for $1 \le k \le \lfloor d/2 \rfloor - 1$ we have (with the event A_1 defined by (3.2))*

$$\mathbf{P}[A_{k+1} \mid A_k] \ge 1 - \alpha_k \exp\{-\alpha'_k n^{2\varepsilon(k)}\} \tag{3.7}$$

for some positive constants α_k, α'_k and all n. Moreover, for any fixed $d \ge 3$ there exist positive constants $\hat{\alpha}_0, \hat{\alpha}_1, \gamma_1$ such that

$$\mathbf{P}[A_{\lfloor d/2 \rfloor}] \ge 1 - \hat{\alpha}_0 \exp\{-\hat{\alpha}_1 n^{\gamma_1}\} \tag{3.8}$$

for all n.

Proof. Consider the set $\tilde{G}_k$ and pick from this set $r_k n^{2\varepsilon(k)}$ disjoint groups with $n^{k-2\varepsilon}$ particles in each group (note that, when A_k happens, there are enough particles in $\tilde{G}_k$ to do it). Name these groups $G_k^1, \dots, G_k^{r_k n^{2\varepsilon(k)}}$. Name their union $G_k \subset \tilde{G}_k$. Note that, by definition, the particles from G_k start to move until the moment n_k. Fix $i \le r_k n^{2\varepsilon(k)}$; for each y in the ring $\mathsf{D}_{k+1,\varepsilon} \setminus \mathsf{D}_{k,\varepsilon}$ let $\zeta_i^{(k+1)}(y)$ be the indicator function of the event

$$\{\text{there exists } x \in G_k^i \text{ such that } t(x,y) \le n_{k+1} - n_k\}.$$

Note that the quantities n_k were defined in such a way that if $x \in \mathsf{D}_{k,\varepsilon}$, $y \in \mathsf{D}_{k+1,\varepsilon}$, then $\|x-y\|^2 \le n_{k+1} - n_k = (2k+1)^2 n^{1+2\varepsilon}$. So, using the independence of random walks starting from G_k^i and Theorem 2.2, we have

$$\begin{aligned} \mathbf{E}(\zeta_i^{(k+1)}(y) \mid A_k) &= \mathbf{P}[\zeta_i^{(k+1)}(y) = 1 \mid A_k] \\ &\ge 1 - \prod_{x \in G_k^i} (1 - \mathsf{q}((2k+1)^2 n^{1+2\varepsilon}, y - x)) \\ &\ge 1 - \Big(1 - \frac{C_3}{(2k+1)^{d-2} n^{(1/2+\varepsilon)(d-2)}}\Big)^{n^{k-2\varepsilon}} \\ &\ge \frac{C_4}{n^{d/2+d\varepsilon-(k+1)}} \end{aligned} \tag{3.9}$$

(note that $d/2 + d\varepsilon > k+1$ for $k \le \lfloor d/2 \rfloor - 1$). Let

$$\zeta_i^{(k+1)} = \sum_{y \in \mathsf{D}_{k+1,\varepsilon} \setminus \mathsf{D}_{k,\varepsilon}} \zeta_i^{k+1}(y).$$

Since $|\mathsf{D}_{k+1,\varepsilon} \setminus \mathsf{D}_{k,\varepsilon}|$ is of order $n^{d/2+d\varepsilon}$, it follows that there exists $r_{k+1} > 0$ such that

$$\mathbf{E}(\zeta_i^{(k+1)} \mid A_k) \ge 2 r_{k+1} n^{k+1}$$

and, clearly,

$$\zeta_i^{(k+1)} \le n^{k-2\varepsilon} \times (2k+1)^2 n^{1+2\varepsilon} = (2k+1)^2 n^{k+1}.$$

So, again using the fact that for any random variable X with $0 \le X \le a$ a.s. and $\mathbf{E}X \ge b$ it is true that $\mathbf{P}[X \ge b/2] \ge b/(2a)$, one gets that there is C_5 such that

$$\mathbf{P}[\zeta_i^{(k+1)} \ge r_{k+1} n^{k+1} \mid A_k] \ge C_5 > 0.$$

Considering now all the $r_k n^{2\varepsilon(k)}$ groups and using the fact that the random walks starting from there are independent, one gets

$$\mathbf{P}[A_{k+1}^c \mid A_k] \leq (1 - C_5)^{r_k n^{2\varepsilon(k)}},$$

which by its turn is equivalent to (3.7).

Now, by Lemma 3.1 and using (3.7) together with the following inequality

$$\mathbf{P}[A_{\lfloor d/2 \rfloor}] \geq \mathbf{P}[A_{\lfloor d/2 \rfloor} \mid A_{\lfloor d/2 \rfloor - 1}] \cdots \mathbf{P}[A_2 \mid A_1]\mathbf{P}[A_1],$$

it follows that (3.8) holds, which concludes the proof of Lemma 3.2. $\square$

Now, suppose first that $d \geq 4$. The idea is to consider the particles in $\tilde{G}_{\lfloor d/2 \rfloor}$ (which start moving until the moment $n_{\lfloor d/2 \rfloor}$) and wait until the moment $n_{\lfloor d/2 \rfloor} + (\lfloor d/2 \rfloor + 1)^2 n^{1+2\varepsilon}$ in order to have an overwhelming probability for them to reach the site x_0.

Let

$$H := \{\text{no particle from } \tilde{G}_{\lfloor d/2 \rfloor} \text{ hits } x_0 \text{ until the time } n_{\lfloor d/2 \rfloor} + (\lfloor d/2 \rfloor + 1)^2 n^{1+2\varepsilon}\}.$$

When the event $A_{\lfloor d/2 \rfloor}$ happens, the number of particles in $\tilde{G}_{\lfloor d/2 \rfloor}$ is at least $r_{\lfloor d/2 \rfloor} n^{\lfloor d/2 \rfloor}$ and they are all at the distance at most $(\lfloor d/2 \rfloor + 1) n^{1/2+\varepsilon}$ from x_0. So by using Theorem 2.2 together with the fact that the random walks starting from $\tilde{G}_{\lfloor d/2 \rfloor}$ are independent, we obtain

$$\begin{aligned}
\mathbf{P}[T(0, x_0) > n_{\lfloor d/2 \rfloor} + (\lfloor d/2 \rfloor + 1)^2 n^{1+2\varepsilon} \mid A_{\lfloor d/2 \rfloor}] \\
\leq \ & \mathbf{P}[H \mid A_{\lfloor d/2 \rfloor}] \\
\leq \ & \Big(1 - \frac{C_8}{n^{(1/2+\varepsilon)(d-2)}}\Big)^{r_{\lfloor d/2 \rfloor} n^{\lfloor d/2 \rfloor}}.
\end{aligned}$$

Now, choosing $\varepsilon < \frac{1}{2(d-2)}$, and using the fact that

$$\begin{aligned}
&\mathbf{P}[T(0, x_0) > n_{\lfloor d/2 \rfloor} + (\lfloor d/2 \rfloor + 1)^2 n^{1+2\varepsilon}] \\
&\quad \leq \ \mathbf{P}[T(0, x_0) > n_{\lfloor d/2 \rfloor} + (\lfloor d/2 \rfloor + 1)^2 n^{1+2\varepsilon} \mid A_{\lfloor d/2 \rfloor}] + \mathbf{P}[A_{\lfloor d/2 \rfloor}^c]
\end{aligned}$$

together with (3.8), we are finished for $d \geq 4$.

Analogously, for the case $d = 3$, with the event H defined as above, we have

$$\mathbf{P}[H \mid A_1] \leq \Big(1 - \frac{C_9}{n^{1/2+\varepsilon}}\Big)^{r_1 n^{1-\varepsilon}}.$$

By choosing $\varepsilon < 1/4$ and using Lemma 3.1, the result follows for $d = 3$.

The case $d = 2$ is treated quite analogously to the case $d = 3$. That is, first, dividing the time interval $[0, n]$ into $n^{1/2}$ subintervals of size $n^{1/2}$ (i.e., taking $\varepsilon = 1/2$) and using (2.2), we prove that with large probability the original particle will awaken $const \times n^{1/2}/\log n$ sleeping particles in the ball of radius n until the moment n, where $n \geq \|x_0\|^2$. Considering the independent random walks of those particles until the time $n + 4n^2$ and using Theorem 2.2, we get the result. Thus, the proof of Theorem 3.2 is finished. $\square$

4 Asymptotic shape

In the previous section it was proved that for all $x \in \mathbb{Z}^d$, the collection $T(nx, mx)$, $m > n \geq 0$, satisfies the hypotheses of the subadditive ergodic theorem. Therefore, defining $T(x) := T(0, x)$ for all $x \in \mathbb{Z}^d$, it holds that there exists $\mu(x) \geq 0$ such that

$$\frac{T(nx)}{n} \to \mu(x) \qquad \text{a.s., } n \to \infty. \tag{4.1}$$

From the fact $T(nx) \geq n\|x\|_1$ it follows that $\mu(x) \geq \|x\|_1$ for all $x \in \mathbb{Z}^d$.

Let us extend the definition of $T(x, y)$ to the whole $\mathbb{R}^d \times \mathbb{R}^d$ by defining

$$T(x, y) = \min\{n : y \in \bar{\xi}_n^{x_0}\},$$

where $x_0 \in \mathbb{Z}^d$ is such that $x \in (-1/2, 1/2]^d + x_0$. Note that the subadditive property (3.1) is preserved. The next goal is to show that μ can be extended to $\mathbb{R}^d$ in such a way that (4.1) holds for all $x \in \mathbb{R}^d$.

Lemma 4.1 *For all $x \in \mathbb{Q}^d$*

$$\frac{T(nx)}{n} \to \frac{\mu(mx)}{m} =: \mu(x),$$

where m is the smallest positive integer such that $mx \in \mathbb{Z}^d$.

Proof. Let $n = km + r$, where $k, r \in \mathbb{N}$ and $0 \leq r < m$. Since $T(nx) \leq T(kmx) + T(kmx, kmx + rx)$, it is true that, a.s.,

$$\limsup_{n\to\infty} \frac{T(nx)}{n} \leq \frac{\mu(mx)}{m}. \tag{4.2}$$

Analogously, writing $n = (k+1)m - l$, one gets $T((k+1)mx) - T(nx, (k+1)mx) \le T(nx)$, which implies that, a.s.,

$$\liminf_{n\to\infty} \frac{T(nx)}{n} \ge \frac{\mu(mx)}{m}. \tag{4.3}$$

Combining (4.2) and (4.3), we finish the proof of Lemma 4.1. $\square$

From the definition of μ it can be easily checked that for any $x, y \in \mathbb{Q}^d, \alpha \in \mathbb{Q}$ it holds that $\mu(x+y) \le \mu(x) + \mu(y)$ and $\mu(\alpha x) = \alpha\mu(x)$. Note that μ is uniformly continuous in $\mathbb{Q}^d$ (as all the norms in a finite dimensional space are equivalent), and therefore it can be continuously extended to $\mathbb{R}^d$ in such a way that for all $x \in \mathbb{R}^d$

$$\lim_{n\to\infty} \frac{T(nx)}{n} = \mu(x) \tag{4.4}$$

(the fact that (4.4) still holds can be proved by approximating x by rationals and using Lemma 4.3 below). So, it follows that μ is a norm in $\mathbb{R}^d$.

Now, the next step is to assure that ξ_n grows *at least* linearly.

Lemma 4.2 *For all $x \in \mathbb{Z}^d$, $d \ge 2$, there are constants $0 < \delta_0 < 1$, $C > 0$ and $0 < \gamma < 1$, which depend only on the dimension, such that*

$$\mathbf{P}\Big[T(x) \ge \frac{\|x\|_1}{\delta_0}\Big] \le C\exp\{-\|x\|_1^\gamma\}.$$

Proof. Let $n := \|x\|_1$ and $0 = x_0, x_1, \dots, x_n = x$ be a path connecting the origin to site x such that for all i, $\|x_i - x_{i-1}\|_1 = 1$; note that $\|x_k\|_1 = k$, $k = 0, \dots, n$. Let $Y_i := T(x_{i-1}, x_i)$. Due to the subadditivity, it is enough to prove that

$$\mathbf{P}\Big[\sum_{i=1}^n Y_i \ge \frac{\|x\|_1}{\delta_0}\Big] \le C\exp\{-\|x\|_1^\gamma\}. \tag{4.5}$$

Let

$$B := \Big\{Y_i < \frac{\sqrt{n}}{2}, i = 1, \dots, n\Big\}.$$

Clearly, by Theorem 3.2 we have

$$\mathbf{P}[B] \ge 1 - C_1 n\exp\{-n^{\gamma'}\} \tag{4.6}$$

for some $\gamma' > 0$. For $i = 1, \ldots, \lceil\sqrt{n}\rceil$ let

$$\sigma_i := \sum_{j=0}^{M_i} Y_{i+j\lceil\sqrt{n}\rceil},$$

where

$$M_i := \max\{j \in \mathbb{N} : i + j\lceil\sqrt{n}\rceil \leq n\}.$$

Observe that, if the event B happens, then each σ_i is as a sum of independent identically distributed random variables, since in this situation the variables $\{Y_{i+j\lceil\sqrt{n}\rceil} : j = 1, \ldots, M_i\}$ depend on disjoint sets of random walks.

We point out that we cannot guarantee the existence of the moment generating function of Y_i. All we have is a sub exponential estimate as in (2.1) (see Theorem 3.2). Lemma 2.1 takes care of the situation and allows us to obtain (for $\delta_0 = 1/a$)

$$\begin{aligned}
\mathbf{P}\Big[\sum_{i=1}^{n} Y_i > \frac{n}{\delta_0}\Big| B\Big] &\leq \mathbf{P}\Big[\Big\{\sigma_1 \leq \frac{M_1}{\delta_0}, \ldots, \sigma_{\lceil\sqrt{n}\rceil} \leq \frac{M_{\lceil\sqrt{n}\rceil}}{\delta_0}\Big\}^c \Big| B\Big] \\
&\leq \sum_{i=1}^{\lceil\sqrt{n}\rceil} \mathbf{P}\Big[\sigma_i \geq \frac{M_i}{\delta_0}\Big| B\Big] \\
&\leq C_2 \sum_{i=1}^{\lceil\sqrt{n}\rceil} \exp\{-M_i^{\beta}\}. \qquad (4.7)
\end{aligned}$$

Note that if n is large then for all $i \leq \lceil\sqrt{n}\rceil$ it holds that $M_i = \mathcal{O}(\sqrt{n})$. The result follows from (4.6) and (4.7). $\square$

For $y \in \mathbb{R}^d$ and $a > 0$ denote $D(y, a) = \{x \in \mathbb{R}^d : \|x - y\|_1 \leq a\}$.

Lemma 4.3 *For all $d \geq 2$ there exist constants $0 < \delta < 1$, $C > 0$, $\gamma_0 > 0$, which depend only on the dimension, such that*

$$\mathbf{P}[D(0, n\delta) \subset \bar{\xi}_n] \geq 1 - C\exp\{-n^{\gamma_0}\}$$

for all n.

Proof. By Lemma 4.2, if x is such that $\|x\|_1 = n$ then we have

$$\mathbf{P}\Big[T(x) \geq \frac{n}{\delta_0}\Big] \leq C\exp\{-n^{\gamma}\}.$$

Now, let $y \in \mathbb{Z}^d$ be such that $0 < \|y\|_1 < n$. Then there exists $x \in \mathbb{Z}^d$ such that $\|x\|_1 = n$ and $\|x-y\|_1$ equals n or $n+1$. Suppose that $\|x-y\|_1 = n$; the other case is treated similarly. Since $T(y) \leq T(x) + T(x,y)$ and by Lemma 4.2 we have

$$\begin{aligned} \mathbf{P}\Big[T(y) \geq \frac{2n}{\delta_0}\Big] &\leq \mathbf{P}\Big[T(x) \geq \frac{n}{\delta_0}\Big] + \mathbf{P}\Big[T(x-y) \geq \frac{n}{\delta_0}\Big] \\ &\leq 2C\exp\{-n^\gamma\}. \end{aligned} \tag{4.8}$$

As the number of points in $D(0, n\delta)$ is of order n^d, by using (4.8) we finish the proof of Lemma 4.3. □

Now we are able to finish the proof of the shape theorem for the frog model.

Proof of Theorem 1.1. Let $\mathsf{A} := \{x \in \mathbb{R}^d : \mu(x) \leq 1\}$. The following argument is rather standard (see e.g. [1, 2]), we keep it to preserve the self-containedness of the paper.

Let $\varepsilon' = (1-\varepsilon)^{-1} - 1$, and $\varepsilon'' = 1-(1+\varepsilon)^{-1}$. Clearly, to prove Theorem 1.1, it is enough to prove that $n\mathsf{A} \subset \bar{\xi}_{(1+\varepsilon')n}$ and $\bar{\xi}_{(1-\varepsilon'')n} \subset n\mathsf{A}$ for all n large enough, a.s.

Since A is compact, there exists a finite set $\mathsf{F} := \{x_1, \ldots, x_k\} \subset \mathsf{A}$ such that $\mu(x_i) < 1$ for $i = 1, \ldots, k$, and (with δ from Lemma 4.3) $\mathsf{A} \subset \cup_{i=1}^k D(x_i, \varepsilon'\delta)$. Note that (4.4) implies that $n\mathsf{F} \subset \bar{\xi}_n$ for all n large enough a.s. Now, Lemma 4.3 and Borel-Cantelli imply that a.s. for all n large enough we have $D(nx_i, n\varepsilon'\delta) \subset \bar{\xi}_{n\varepsilon'}^{nx_i}$, for all $i = 1, 2, \ldots, k$. So $n\mathsf{A} \subset \bar{\xi}_{(1+\varepsilon')n}$ and this part of the proof is done.

Now, choose $\mathsf{G} := \{y_1, \ldots, y_k\} \subset 2\mathsf{A}\setminus\mathsf{A}$ such that $2\mathsf{A}\setminus\mathsf{A} \subset \cup_{i=1}^k D(y_i, \varepsilon''\delta)$. Notice that $\mu(y_i) > 1$ for $i = 1, \ldots, k$. Again, (4.4) implies that $n\mathsf{G} \cap \bar{\xi}_n = \emptyset$ for all n large enough a.s. Analogously, by Lemma 4.3 and Borel-Cantelli we get that for all n large enough, if $\bar{\xi}_{(1-\varepsilon'')n} \cap n(2\mathsf{A}\setminus\mathsf{A}) \neq \emptyset$, then $\bar{\xi}_n \cap n\mathsf{G} \neq \emptyset$. This shows that $\bar{\xi}_{(1-\varepsilon'')n} \subset n\mathsf{A}$ for all n large enough, a.s., and so concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. To prove the theorem, it is enough to prove the following fact: for fixed i, j, there exists $\beta \in (0, 1/2)$ such that

$$\Theta_{ij}^\beta \subset \mathsf{A}_m. \tag{4.9}$$

Indeed, in this case (4.9) holds for all i, j with the same β by symmetry, hence $\Theta^\beta \subset \mathsf{A}_m$ by virtue of convexity of A_m.

Now, the proof of (4.9) is just a straightforward adaptation of the proof of the "flat edge" result of [3]. To keep the paper self-contained, let us outline the ideas of the proof. Suppose, without loss of generality, that $i = 1, j = 2$, and m is even. We are going to prove that the frog model observed only on $\Lambda_{12} \cap \mathbb{Z}_+^d$ dominates the oriented percolation process in $\mathbb{Z}_+^2$ with parameter $\theta = 1 - (1 - (2d)^{-1})^{m/2}$. To show this, first suppose that initially for any x all the particles in x are labeled "$x{\rightarrow}$" or "$x{\uparrow}$" in such a way that x contains exactly $m/2$ particles of each label. Define e_1, e_2 to be the first two coordinate vectors. The oriented percolation is then defined in the following way: For $x \in \Lambda_{12} \cap \mathbb{Z}_+^d$

- the bond from x to $x + e_1$ is open if for the frog model at the moment next to that of activation of the site x at least one particle labeled "$x{\rightarrow}$" goes to $x + e_1$;

- the bond from x to $x + e_2$ is open if at that moment at least one particle labeled "$x{\uparrow}$" goes to $x + e_2$.

Clearly, the two above events are independent, and their probabilities are exactly θ. So the frog model indeed dominates the oriented percolation in the following sense: if a site $x = (x^{(1)}, x^{(2)})$ (for the sake of brevity forget the zero coordinates from 3 to d) belongs to cluster of 0 in the oriented percolation, then in the frog model the corresponding site is awakened *exactly* at time $x^{(1)} + x^{(2)}$. Now it rests only to choose m as large as necessary to make the oriented percolation supercritical ($\theta \to 1$ as $m \to \infty$) and use the result that (conditioned on the event that the cluster of 0 is infinite) the intersection of the cluster of 0 with the line $\{(x^{(1)}, x^{(2)}) : x^{(1)} + x^{(2)} = n\}$ grows linearly in n (cf. [3]). $\square$

Acknowledgements

This work was done when O.S.M. Alves was visiting the Statistics Department of the Institute of Mathematics and Statistics of the University of São Paulo. He is thankful to the probability group of IME-USP for hospitality. The authors have benefited on useful comments and suggestions from K. Ravishankar and L. Fontes and wish to thank them. E. Kira should be thanked for a careful reading of the first draft of this paper. F.P.M. is also indebted to R. Durrett and M. Bramson for discussions about a continuous-time version of the model. Special thanks are due to the anonymous referee whose comments permitted us to improve the presentation substantially.

References

[1] M. Bramson, D. Griffeath (1980) On the Williams-Bjerknes tumor growth model II. *Math. Proc. Cambridge Philos. Soc.* **88**, 339–357.

[2] R. Durrett, D. Griffeath (1982) Contact processes in several dimensions. *Z. Wahrsch. Verw. Gebiete* **59** (4), 535–552.

[3] R. Durrett, T.M. Liggett (1981) The shape of the limit set in Richardson's growth model. *Ann. Probab.* **9** (2), 186–193.

[4] B.D. Hughes (1995) *Random Walks and Random Environments, vol. 1.* Clarendon press, Oxford.

[5] J.F.C. Kingman (1973) Subadditive ergodic theory. *Ann. Probab.* **1**, 883–909.

[6] G.F. Lawler (1991) *Intersections of Random Walks.* Birkhäuser Boston.

[7] T.M. Liggett (1985) An improved subadditive ergodic theorem. *Ann. Probab.* **13**, 1279–1285.

[8] S.V. Nagaev (1979) Large deviations of sums of independent random variables. *Ann. Probab.* **7** (5), 745–789.

[9] S.Yu. Popov (2001) Frogs in random environment. *J. Statist. Phys.* **102** (1/2), 191–201.

[10] A. Telcs, N.C. Wormald (1999) Branching and tree indexed random walks on fractals. *J. Appl. Probab.* **36** (4), 999–1011.