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CRITICAL SPONGE DIMENSIONS IN PERCOLATION THEORY

G. R. GRIMMETT,* *University of Bristol*

Abstract

In the bond percolation process on the square lattice, with $p<\frac{1}{2}$, let $S(k)$ be the probability that some open path joins the longer sides of a sponge with dimensions k by $a \log k$. There exists a positive constant $\alpha = \alpha_p$ such that

$$S(k) \to \begin{cases} 0 & \text{if} \quad a\alpha > 1 \\ 1 & \text{if} \quad a\alpha < 1, \quad \text{as} \quad k \to \infty. \end{cases}$$

Consequently, the subset of the square lattice $\{(x, y) : 0 \leqq y \leqq f(x)\}$ which lies between the curve $y = f(x)$ and the x-axis has the same critical probability as the square lattice itself if and only if $f(x)/\log x \to \infty$ as $x \to \infty$.

PERCOLATION

1. Introduction

Considerable progress has been made recently in percolation theory. Kesten (1980b) has verified the truth of the conjecture that the critical probability of bond percolation on the square lattice is $\frac{1}{2}$; Wierman (1981) has adapted Kesten's techniques to deal with bond percolation on the triangular and honeycomb lattices. Of crucial importance in their proofs is the notion of sponge percolation, which was introduced independently by Seymour and Welsh (1978) and Russo (1978).

Several authors have derived estimates for the tail of the cluster size distribution (see Kunz and Souillard (1978), Kesten (1980a), Aizenmann, Delyon and Souillard (1980), Grimmett (1981)). In this paper, we obtain similar exponential estimates for the probabilities of the occurrences of certain types of open path. These estimates are asymptotically exact; we use them to find the critical dimensions of a large rectangular sponge with the property that wider sponges are almost surely traversible by open paths but narrower sponges are almost surely not traversible. More precisely, suppose $0<p<\frac{1}{2}$ and let $S_p(a \log k, k)$ denote the probability that a sponge of size $a \log k$ by k is

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* Postal address: School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, U.K.

traversible by an open path between its opposite sides of length k. Then there exists a positive constant α_p, depending on p alone, such that, as $k \to \infty$,

$$S_p(a \log k, k) \to \begin{cases} 0 & \text{if} \quad a\alpha_p > 1 \\ 1 & \text{if} \quad a\alpha_p < 1. \end{cases}$$

For $p = 1 - q > \frac{1}{2}$, the dual lattice technique allows us to deduce that, as $m \to \infty$,

$$S_p(\exp(bm), m) \to \begin{cases} 0 & \text{if} \quad b > \alpha_q \\ 1 & \text{if} \quad b < \alpha_q. \end{cases}$$

Similar arguments provide an exact solution to a problem discussed by Wierman (1978) and McDiarmid (1980). Let $f: [0, \infty) \to [0, \infty)$ be an increasing function with $f(0) = 0$. The percolative properties of the region of the square lattice bounded by the x-axis and the curve $y = f(x)$ depend on how fast f grows with x. We show here that the critical case is when $f(x) = O(\log x)$. That is to say, if $p > \frac{1}{2}$ and $f(x) = a \log(x+1)$, then the probability $P_\infty(a)$ that the region in question contains an infinite open cluster satisfies

$$P_\infty(a) = \begin{cases} 0 & \text{if} \quad a\alpha_q < 1 \\ 1 & \text{if} \quad a\alpha_q > 1, \end{cases}$$

where $q = 1 - p < \frac{1}{2}$.

The methods used here do not rely directly upon the recent sophisticated arguments of Seymour, Welsh, Russo and Kesten; they do however use the notion of sponge percolation, and the previously unknown constants p_H, p_T appear here in their numerical value of $\frac{1}{2}$.

For a recent review of percolation on the square lattice, see the monograph by Smythe and Wierman (1978).

2. Notation

Let Z_2 denote the square lattice, with origin 0. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in Z_2, write $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ for the *distance* between x and y. The vertices of Z_2 are called *sites* and the line segments joining sites x and y with $d(x, y) = 1$ are called *bonds*. Each bond is declared *open* with probability p or *closed* with probability $q = 1 - p$ independently of all other bonds, where $p \in (0, 1)$ is a prescribed number.

A *path* from x to y is an alternating sequence of distinct sites and bonds of the form $x_0, b_1, x_1, b_2, \cdots, b_n, x_n$ where $x_0 = x$, $x_n = y$ and b_{i+1} joins x_i to x_{i+1}. A *circuit* is a path $x_0, b_1, \cdots, b_n, x_n$ together with a further bond b_{n+1} joining x_n to x_0. A path or circuit is *open* (or *closed*) if all its bonds are open (or closed). For any set A of sites, the *internal boundary* ΔA of A is the set of all sites in A which are adjacent to some site which is not in A. For any sets A and B of sites, we write $\{A \sim B\}$ (or $\{A \overset{c}{\sim} B\}$) for the event that there exists an open (or closed) path joining some site of A to some site of B. The number of sites in A is denoted by $|A|$. The *open cluster* W_x containing x is the set of sites y such that $x \sim y$. The *radius* $r(W_x)$ is $\max\{d(x, y) : y \in W_x\}$.

The m by n *sponge* $T(m, n)$ is a rectangle of sites and bonds with $m+1$ sites on each of the left- and right-hand edges, $n+1$ sites on each of the bottom and top edges, and with the property that $T(m, n)$ is isomorphic to the rectangle $\{(x, y) : 0 \leqq x \leqq n, 0 \leqq y \leqq m\}$. $T(m, n)$ is *traversible horizontally* (or *vertically*) if there exists an open path lying entirely in T which joins two sites, one taken from each of the left and right (or bottom and top) edges of T. Write $\{\rightarrow T\}$ and $\{\uparrow T\}$ for these events. Let $S_p(m, n) = P(\uparrow T(m, n))$. We shall sometimes use the dual lattice technique applied to percolation across sponges; see Seymour and Welsh (1978) and Kesten (1980b) for information about this.

The integer part of a real number x is denoted by $[x]$. To avoid unnecessary notation, the square brackets will sometimes be omitted when it is clear that an integer-valued quantity is required. For example, we write $S_p(\log k, k)$ in place of $S_p([\log k], k)$.

3. Statement of main results

Let L_k denote the vertical line $x = k$, and let $\alpha(k) = \alpha_p(k) = P(0 \sim L_k)$ be the probability that the origin is joined to some site of L_k by an open path. We shall see in the next section that, whenever $p < \frac{1}{2}$, there exists a constant $\alpha = \alpha_p \in (0, \infty)$ such that

$$k^{-1} \log \alpha_p(k) \rightarrow -\alpha_p \quad \text{as} \quad k \rightarrow \infty. \tag{3.1}$$

Consider the sponge $T(m, k)$ where $m = m(k)$ is a specified positive increasing function of k. The probability $S_p(m, k) = P(\uparrow T(m, k))$ is a non-increasing function of m; the asymptotic form of $S_p(m, k)$ as $k \rightarrow \infty$ depends critically upon the way in which $m(k)$ grows with k. For example, if $p < \frac{1}{2}$, then

$$S_p(m, k) \rightarrow \begin{cases} 0 & \text{if} \quad m(k) = k \qquad \text{(see Kesten (1980b))} \\ 1 & \text{if} \quad m(k) = O(1). \end{cases}$$

The critical value for m at which the transition of $\lim_{k\rightarrow\infty} S_p(m(k), k)$ occurs is given by the following theorem.

Theorem 1. *If* $p < \frac{1}{2}$ *then as* $k \rightarrow \infty$,

$$S_p(m(k), k) \rightarrow \begin{cases} 0 & \textit{if} \quad k\alpha(m) \rightarrow 0 \\ 1 & \textit{if} \quad k\alpha(m)/m \rightarrow \infty. \end{cases}$$

Combining this with (3.1) yields the next theorem.

Theorem 2. *Let* $p<\frac{1}{2}$ *and* $m(k)=a\log k$. *Then, as* $k\to\infty$,

$$S_p(m(k),k)\to\begin{cases}0 & \textit{if} \quad a\alpha_p>1\\ 1 & \textit{if} \quad a\alpha_p<1,\end{cases}$$

where α_p *is given by* (3.1).

Let $f:[0,\infty)\to[0,\infty)$ be an increasing function with $f(0)=0$, and let $Z_2(f)$ denote the collection of sites and bonds of Z_2 which lie in the closed region bounded by the x axis and the curve $y=f(x)$. Let $I(f,\infty)$ denote the event that $Z_2(f)$ contains an infinite open connected cluster. It is easy to see that $I(f,\infty)$ is the tail event of an infinite collection of independent Bernoulli random variables; thus $P(I(f,\infty))$ is 0 or 1 for any value of p. Furthermore $P(I(f,\infty))=0$ whenever $p\leqq\frac{1}{2}$, since any infinite open cluster in $Z_2(f)$ is a subgraph of Z_2. Wierman (1978) and McDiarmid (1980) have found weak sufficient conditions on f which ensure that $Z_2(f)$ contains an infinite open cluster if and only if Z_2 contains such a cluster. The next theorem provides a necessary and sufficient condition for this to hold.

Theorem 3. *Suppose* $p>\frac{1}{2}$ *and let* $f(x)=a\log(x+1)$. *Then*

$$P(I(f,\infty))=\begin{cases}0 & \textit{if} \quad a\alpha_q<1\\ 1 & \textit{if} \quad a\alpha_q>1.\end{cases}$$

Thus, if $a\alpha_q>1$ then there is strictly positive probability that there exists an infinite path in $Z_2(f)$ containing the origin. Let $P(I(\infty,\infty))$ denote the probability that Z_2 contains an infinite cluster. It follows from the theorem that $P(I(f,\infty))=P(I(\infty,\infty))$ for all p if and only if $f(x)/\log x\to\infty$ as $x\to\infty$.

It is clear from its definition that α_p is a decreasing function of p on the domain $(0,\frac{1}{2})$, and $\alpha_{\frac{1}{2}}=0$. We shall see in the proof of (4.6) that α_p is a right-continuous function of p.

The choice of functions $f(x)=O(\log x)$ occurs elsewhere in analytical estimates of crossing probabilities; see for example Cox and Grimmett (1981).

4. Some inequalities and limits

Firstly, we consider the probabilities of the occurrences of certain types of open path, and we explore the relationships between them. We assume throughout this section that $p<\frac{1}{2}$; thus, all open clusters are almost surely finite.

Let $\pi(k)=P(r(W_0)\geqq k)$, $\alpha(k)=P(0\sim L_k)$, and let $\beta(k)$ be the probability that 0 is joined to L_k by an open path which, apart from 0, uses only sites with strictly positive x-coordinate. $\pi(k)$ is the probability that 0 is joined by an open path to some site distance at least k from 0; sometimes we write $\pi_p(k)$ to emphasize its dependence on p. The sequences $\pi(k)$, $\alpha(k)$, $\beta(k)$ are non-increasing.

Lemma 1. *For $k, r, s \geqq 1$, we have that*

$$\beta(k) \leqq \alpha(k) \leqq \pi(k) \tag{4.1}$$

$$\pi(2k) \leqq 4\alpha(k) \tag{4.2}$$

$$\pi(r+s) \leqq 4r\pi(r)\pi(s) \tag{4.3}$$

$$\alpha(r+s) \leqq (2k+1)\alpha(r)\beta(s) + \pi(k+r). \tag{4.4}$$

Consequently, the rates of decay of π, α and β are given by the next lemma.

Lemma 2. *There exist positive finite constants $\pi = \pi_p$, $\alpha = \alpha_p$ depending on p such that, as $k \to \infty$,*

$$k^{-1} \log \pi(k) \to -\pi_p \tag{4.5}$$

$$k^{-1} \log \alpha(k) \to -\alpha_p \tag{4.6}$$

$$k^{-1} \log \beta(k) \to -\alpha_p. \tag{4.7}$$

Furthermore

$$\pi_p \leqq \alpha_p \leqq 2\pi_p.$$

Equation (3.1) appears in this lemma as Equation (4.6).

Proof of Lemma 1.

Equation (4.1). Obvious.

Equation (4.2). Let D_r denote the collection of sites of Z_2 which are at most distance r from 0. D_r is a diamond with corners at $(\pm r, 0)$ and $(0, \pm r)$, and $\pi(r) = P(0 \sim \Delta D_r)$. Let $L_k^i (1 \leqq i \leqq 4)$ be the four lines $x = k$, $y = -k$, $x = -k$ and $y = k$ respectively. Clearly $\{0 \sim \Delta D_{2k}\} \subseteq \bigcup_{i=1}^4 \{0 \sim L_k^i\}$, and (4.2) follows immediately.

Equation (4.3). Clearly

$$\{0 \sim \Delta D_{r+s}\} \subseteq \{0 \sim \Delta D_r\} \cap \{\Delta D_r \sim \Delta D_{r+s}\}.$$

But these two last events are independent since they are defined in terms of disjoint bond sets, and so

$$\begin{aligned} \pi(r+s) &\leqq \pi(r)\, |\Delta D_r|\, \pi(s) \\ &= 4r\pi(r)\pi(s). \end{aligned}$$

Equation (4.4). Let L_r and L_{r+s} be given as before, and write L_r^k for the segment $\{(r, y) : -k \leqq y \leqq k\}$ of L_r. Let A be the event that some site in L_r^k is joined to 0 by an open path which intersects L_r only at its first endpoint, and let B be the event that some site in L_r^k is joined to L_{r+s} by an open path which intersects L_r only at its first endpoint. Then

$$\{0 \sim L_{r+s}\} \subseteq \{r(W_0) \geqq k+r\} \cup (A \cap B),$$

since if $\{0 \sim L_{r+s}\}$ occurs without $A \cap B$ occurring then 0 is joined by an open path to a site on L_r which is not in L_r^k; thus by the independence of A and B,

$$\begin{aligned}\alpha(r+s) &\leqq \pi(k+r) + \alpha(r)\,|L_r^k|\,\beta(s)\\ &= \pi(k+r) + (2k+1)\alpha(r)\beta(s).\end{aligned}$$

Proof of Lemma 2.

Equation (4.5). Take logarithms of (4.3) and apply the limit theorem for the generalized subadditive inequality (see Hammersley (1962)) to obtain

$$k^{-1}\log \pi(k) \to -\pi \quad \text{as} \quad k \to \infty \tag{4.8}$$

where $-\infty < \pi \leqq \infty$. Trivially $0 \leqq \pi < \infty$, since $p^k \leqq \pi(k) \leqq 1$. It remains to show that $\pi > 0$; there are at least two ways of doing this, depending upon how much of the current knowledge of percolation theory we choose to use. For example, it follows immediately from Kesten (1980a) (see also the discussion in Kesten (1980b)) that $\pi(k) \leqq c_1 \exp(-kc_2)$ for some $c_1, c_2 > 0$, showing that $\pi \geqq c_2 > 0$. There is another way which is more basic, and which provides an alternative proof of Kesten's observation. It is a consequence of Hammersley's theorem (Hammersley (1962), Equation (2)) that

$$\begin{aligned}\pi &\geqq -k^{-1}\log(\pi(k)/(4k)) - 4\sum_{r=2k}^{\infty}\frac{\log(4r)}{r(r+1)}\\ &\geqq -(k-\tfrac{1}{2})^{-1}\log(16e^2k\pi(k)) \quad \text{for all } k.\end{aligned} \tag{4.9}$$

But, for $p < p_T = \frac{1}{2}$,

$$k\pi(k) \leqq \sum_{i=k}^{\infty} iP(|W_0| = i) \to 0 \quad \text{as} \quad k \to \infty,$$

because this sum is the tail of the series expansion of $E\,|W_0|$. Thus the right-hand side of (4.9) is positive for large k. When $p = \frac{1}{2}$, $\pi = 0$, and thus $\pi(k) \geqq (16e^2k)^{-1}$; a better lower bound is given by $\pi(k) \geqq \alpha(k) \geqq (k+1)^{-1}S_{\frac{1}{2}}(k, k) \geqq (2(k+1))^{-1}$ (see Smythe and Wierman (1978), p. 61).

It is not difficult to see that inequality (4.9) implies that $\pi = \pi_p$ is a lower semicontinuous function of p, since the right-hand side of (4.9) is continuous in p and converges to π as $k \to \infty$. Also π_p is non-increasing, and so it is a right-continuous function of p.

Equation (4.6). We rearrange (4.4) and use the subadditive limit theorem. Let $k = Ms + (M-1)r$ where M is a positive integer to be chosen shortly. Let

$g=g(M)=2M(1+\sqrt{3})$ be the greater root of the quadratic equation

(4.10) $$g^2=(2M+g)4M.$$

We claim that the sequence $\alpha'(r)$ given by $\alpha'(r)=2M\alpha(r)+g\pi(Mr)$ satisfies

(4.11) $$\alpha'(r+s)\leqq(r+s)\alpha'(r)\alpha'(s).$$

For, by (4.1) and (4.4),

$$\alpha(r+s)\leqq 2M(r+s)\alpha(r)\alpha(s)+\pi(Mr+Ms),$$

and so, by (4.3) and (4.10),

$$\begin{aligned}\alpha'(r+s)&\leqq(r+s)(4M^2\alpha(r)\alpha(s)+(2M+g)4M\pi(Mr)\pi(Ms))\\&\leqq(r+s)\alpha'(r)\alpha'(s)\end{aligned}$$

which is Equation (4.11). Take logarithms of (4.11) and proceed as in the proof of (4.5) to obtain $k^{-1}\log\alpha'(k)\to-\alpha'_M$, as $k\to\infty$, where $\alpha'_M\in(-\infty,\infty)$.

By (4.5) there exist positive quantities c_1, c_2, c_3, c_4, depending on p alone, such that

(4.12) $$c_3\exp(-kc_4)\leqq\pi(k)\leqq c_1\exp(-kc_2)\quad\text{for all } k.$$

By (4.1) and (4.2)

(4.13) $$\tfrac{1}{4}c_3\exp(-2kc_4)\leqq\alpha(k)\leqq c_1\exp(-kc_2)\quad\text{for all } k.$$

Thus, by the definition of $\alpha'(k)$,

$$0\leqq\log\alpha'(k)-\log\alpha(k)\leqq\log(2M+(24Mc_1/c_3)\exp((2c_4-Mc_2)k)).$$

Pick M large enough so that $2c_4-Mc_2<0$, divide by k and let $k\to\infty$ to see that $k^{-1}\log\alpha'(k)$ and $k^{-1}\log\alpha(k)$ have the same limit $-\alpha=-\alpha'_M$. It is easy to see from (4.1) and (4.2) that $\pi\leqq\alpha\leqq2\pi$. This proves (3.1) also. As in the proof of (4.5), $\alpha=\alpha_p$ is a right-continuous function of p.

Equation (4.7). Rearrange (4.4) with $r=s$ and $k=(M-1)r$, with M to be chosen shortly, to obtain $\beta(r)\geqq(\alpha(2r)-\pi(Mr))/(2Mr\alpha(r))$. Thus

(4.14) $$\log\beta(r)\geqq\log\alpha(2r)-\log\alpha(r)-\log(2Mr)+\log\left(1-\frac{\pi(Mr)}{\alpha(2r)}\right).$$

Use (4.12) and (4.13) to show that

$$0\leqq\frac{\pi(Mr)}{\alpha(2r)}\leqq(4c_1/c_3)\exp((4c_4-Mc_2)r)$$

and pick M large enough that $4c_4-Mc_2<0$; now divide (4.14) by r and let $r\to\infty$ to obtain $\liminf_{r\to\infty} r^{-1}\log\beta(r)\geqq-\alpha$, which, in conjunction with inequality (4.1), yields (4.7).

5. Proofs of main results

Proof of Theorem 1. For each integer i let $\chi_i(k)$ be the indicator function of the event that the site $(i, 0)$ is joined by an open path to the line $y = m(k)$, where $m(k)$ is a prescribed function of k. Clearly $E(\chi_i(k)) = \alpha(m)$. Let $S_k = \sum_{i=0}^{k} \chi_i(k)$; S_k is a random variable taking non-negative integer values, and it follows by standard inequalities that

$$1 - (\operatorname{Var}(S_k)/(ES_k)^2) \leqq P(S_k > 0) \leqq ES_k.$$

The first part of Theorem 1 follows immediately since $\{\uparrow T(m, k)\} \subseteq \{S_k > 0\}$ and $E(S_k) = (k+1)\alpha(m)$.

In the proof of the second assertion of the theorem, we claim first that

$$(5.1) \qquad (\operatorname{Var}(S_k)/(ES_k)^2) \to 0 \quad \text{whenever } k\alpha(m)/m \to \infty.$$

Expand $\operatorname{Var}(S_k)$ and use the stationarity of $\chi_i(k)$ $(-\infty < i < \infty)$ to show that

$$(5.2) \quad \operatorname{Var}(S_k) = (k+1)\alpha(m)(1-\alpha(m)) + 2\sum_{i=1}^{k} (k+1-i)\operatorname{Cov}(\chi_0(k), \chi_i(k)).$$

Note that each covariance is non-negative by the FKG inequality (see the discussion in Seymour and Welsh (1978)). We require an upper bound on $\operatorname{Cov}(\chi_0(k), \chi_i(k))$. Let $A_i = \{0 \sim L_{[\frac{1}{2}i]}\}$ and $B_i = \{(i, 0) \sim L_{[\frac{1}{2}i]}\}$. Then, abbreviating $\chi_i(k)$ to χ_i,

$$\begin{aligned}
\operatorname{Cov}(\chi_0, \chi_i) &= E(\chi_0\chi_i) - \alpha(m)^2 \\
&= P(\chi_0\chi_i = 1, A_i^c \cap B_i^c) - \alpha(m)^2 + P(\chi_0\chi_i = 1, A_i \cap B_i^c) \\
&\quad + P(\chi_0\chi_i = 1, A_i^c \cap B_i) + P(\chi_0\chi_i = 1, A_i \cap B_i) \\
&\leqq P(\chi_0 = 1, A_i^c)P(\chi_i = 1, B_i^c) - \alpha(m)^2 + 3\min(P(A_i), E(\chi_0)) \\
&\leqq 3\min(\alpha([\tfrac{1}{2}i]), \alpha(m))
\end{aligned}$$

by the independence of $\{\chi_0 = 1\} \cap A_i^c$ and $\{\chi_i = 1\} \cap B_i^c$. Split the summation in (5.2) into values of i less than $2m$ and values of i which are at least $2m$ to obtain

$$(5.3) \quad \begin{aligned}
\sum_{i=1}^{k} (k+1-i)\operatorname{Cov}(\chi_0, \chi_i) &\leqq 3(k+1)\Big(\sum_{i<2m} \alpha(m) + \sum_{i \geqq 2m} \alpha([\tfrac{1}{2}i])\Big) \\
&\leqq 3(k+1)\Big(2m\alpha(m) + 2\sum_{i \geqq 0} \alpha(m+i)\Big).
\end{aligned}$$

We use (4.4) with $r = m$, $s = i$ and $k = Ms + (M-1)r$ to bound the summation in (5.3) by

$$(5.4) \quad \begin{aligned}
\sum_{i \geqq 0} \alpha(m+i) &\leqq \alpha(m) \sum_{i \geqq 0} \left(2M(m+i)\alpha(i) + \frac{\pi(M(m+i))}{\alpha(m)}\right) \\
&\leqq \alpha(m)(A_1 m + A_2)
\end{aligned}$$

for constants A_1 and A_2 depending on M, where we have used (4.12) and (4.13) and chosen M to be large enough as before. Combine (5.2), (5.3) and (5.4) to obtain

$$\frac{\operatorname{Var}(S_k)}{(ES_k)^2} \leqq \frac{(k+1)}{k}\left(\frac{(1-\alpha(m))}{k\alpha(m)} + \frac{12(A_2+A_3m)}{k\alpha(m)}\right)$$
$$\to 0 \quad \text{if} \quad k\alpha(m)/m \to \infty,$$

where A_3 depends only on M. This completes the proof of (5.1), and we have shown that the probability that some site $(i, 0)$ with $0 \leqq i \leqq k$ is joined by an open path to the line $y = m(k)$ approaches 1 as $k \to \infty$. It remains to show that, with probability approaching 1, such a path exists within $T(m, k)$. Let $F_i(k)$ be the event that $(i, 0)$ is joined to the line $y = m(k)$ by an open path, no section of which crosses $T(m, k)$ from bottom to top. Then

$$\sum_{i=0}^{k} P(F_i(k)) = 2\sum_{i=0}^{m} P(F_i(k)) + 2\sum_{i=m+1}^{[\frac{1}{2}k]} P(F_i(k))$$
$$\leqq 2(m+1)\alpha(m) + 2\sum_{i>m} \alpha(i),$$

since, if $m+1 \leqq i \leqq [\frac{1}{2}k]$ then $F_i(k)$ occurs only if $(i, 0)$ is joined by an open path to at least one of the lines $x = 0$ and $x = k$. Therefore, by (5.4),

$$\sum_{i=0}^{k} P(F_i(k)) \leqq 2(m+1)\alpha(m) + 2\alpha(m)(A_1m + A_2)$$
$$\to 0 \quad \text{as} \quad k \to \infty$$

by the second inequality in (4.13).

Proof of Theorem 2. Suppose $a\alpha_p > 1$. Then

$$\log(k\alpha(m)) = (1 + am^{-1}\log\alpha(m))\log k$$
$$\sim (1 - a\alpha_p)\log k \to -\infty,$$

showing that $k\alpha(m) \to 0$. Then use Theorem 1.

Now, suppose $a\alpha_p < 1$. Then $\log(k\alpha(m)/m) \sim (1 - a\alpha_p)\log k - \log m \to \infty$, showing that $k\alpha(m)/m \to \infty$. Use Theorem 1 again.

Proof of Theorem 3. Suppose first that $a\alpha_q > 1$. Let A_k be the event that $(k, 0)$ is joined to the curve $y = f(x)$ by a closed path lying entirely in $Z_2(f)$. We claim that $P(A_k$ infinitely often$) = 0$. Pick a' such that $a > a' > \alpha_q^{-1}$ and let W_k^c denote the set of all sites y such that $(k, 0) \overset{c}{\to} y$. Write

$$B_k = \{(k, 0) \text{ is joined to line } y = a'\log k \text{ by a closed path}\}$$
$$C_k = \{r(W_k^c) \geqq \tfrac{1}{2}k\}.$$

For all k bigger than some K, we have that

$$A_k \subseteq B_k \cup C_k, \tag{5.5}$$

since, if A_k occurs but B_k does not occur, then W_k^c contains a site which is further distant from $(k, 0)$ than the intersection of the lines $y = a' \log k$ and $y = a \log(x+1)$. These meet at a point whose x coordinate has the value $x = k^{a'/a} - 1$, and which is thus at least distance $k - k^{a'/a}$ away from $(k, 0)$; but $a' < a$ and so $k - k^{a'/a} \geqq \frac{1}{2}k$ for all large k. Sum (5.5) to obtain

$$\begin{aligned}\sum_{k \geqq K} P(A_k) &\leqq \sum_{k \geqq K} (\alpha_q(a' \log k) + \pi_q(\tfrac{1}{2}k)) \\ &= \sum_{k \geqq K} (\pi_q(\tfrac{1}{2}k) + k^{-a'\alpha_q} \exp(o(\log k))) < \infty\end{aligned}$$

by (4.5) and (4.6) and the fact that $a'\alpha_q > 1$. Use the Borel–Cantelli lemma to deduce that $P(A_k$ infinitely often$) = 0$, showing that almost surely only finitely many closed paths join the x axis to the line $y = f(x)$. Thus, in the dual of $Z_2(f)$ there almost surely exists an infinite open cluster and the second assertion of Theorem 3 follows.

Suppose now that $a\alpha_q < 1$ and pick δ such that $a < a(1+\delta) < \alpha_q^{-1}$. Let c_1 and c_2 be chosen to satisfy (4.12) and let the increasing sequence $\{x_k : k = 1, 2, \cdots\}$ be given by $x_k = k^{1+\delta}$.

Let T_k be the sponge comprising all sites and bonds of Z_2 lying within the rectangular region with corners $(x_k, 0)$, $(x_{k+1}, 0)$, $(x_k, a \log x_{k+1})$, $(x_{k+1}, a \log x_{k+1})$ but lying on neither the left- nor right-hand sides. Let y_k be the integer closest to $\frac{1}{2}(x_k + x_{k+1})$ and write

$A_k = \{T_k$ is traversible vertically by a closed path$\}$

$B_k = \{(y_k, 0)$ is joined to line $y = a \log x_{k+1}$ by a closed path$\}$

$C_k = \{(y_k, 0)$ is joined to line $y = a \log x_{k+1}$ by a closed path, some segment of which traverses T_k vertically$\}$

$D_k = \{r(W_k^c) \geqq \frac{1}{2}(x_{k+1} - x_k) - 2\}$

where W_k^c denotes now the set of sites y such that $(y_k, 0) \overset{c}{\to} y$. Now $A_k \supseteq C_k$ and $B_k \subseteq C_k \cup D_k$, since if B_k occurs but C_k does not occur then $(y_k, 0)$ is joined by a closed path to some site on one of the lines $x = x_k + 1$ and $x = x_{k+1} - 1$. Thus $\sum P(A_k) \geqq \sum (P(B_k) - P(D_k))$.

Now, for any large K we have that

$$\begin{aligned}\sum_{k \geqq K} P(B_k) &= \sum_{k \geqq K} \alpha_q(a \log x_{k+1}) \\ &= \sum_{k \geqq K} (k+1)^{-a(1+\delta)\alpha_q} \exp(o(\log k)) \\ &= \infty\end{aligned}$$

since $a(1+\delta)\alpha_q < 1$. Also, by (4.12),

$$\begin{aligned}\sum_{k\geqq K} P(D_k) &\leqq \sum_{k\geqq K} \pi_q(\tfrac{1}{2}(x_{k+1}-x_k)-2)\\ &\leqq \sum_{k\geqq K} \pi_q((1+\delta)k^\delta-2)\\ &\leqq \sum_{k\geqq K} c_1 \exp(-c_2((1+\delta)k^\delta-2))\\ &<\infty.\end{aligned}$$

Hence $\sum P(A_k)=\infty$; but the A_k are independent and the second Borel–Cantelli lemma asserts that $P(A_k \text{ infinitely often})=1$. Thus $Z_2(f)$ is almost surely traversed by infinitely many disjoint closed paths, and use of the dual technique shows that the dual of $Z_2(f)$ almost surely contains no infinite open cluster. This completes the proof of Theorem 3.

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