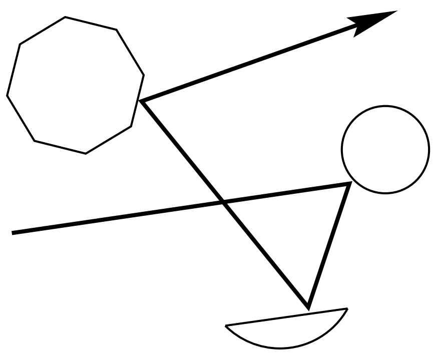
STOCHASTIC PIN-BALL

Geoffrey R. Grimmett

Abstract. A ball is propelled through a random environment of obstacles off which it rebounds with perfect reflection. What is the behaviour of the trajectory of the ball? We summarise known mathematical results concerning this model, which we call 'stochastic pin-ball', and which is known also as a 'Lorentz lattice gas' and a version of which is sometimes termed the 'Ehrenfest wind–tree model'. The rigorous theory is more extensively developed if the environment is allowed to include a positive density of space in which the ball behaves in the manner of a random walk. For a lattice model of this type, one may employ arguments of percolation theory in order to prove theorems concerning non-localisation, transience, and asymptotic normality, under certain assumptions on the environment.



1. The origins of stochastic pin-ball

There is a modern version of the game of bagatelle involving a ball which is propelled about an inclined plane and which suffers deflections as a result of collisions with protruding nails. Modern pin-ball is an electrified version of this game, with a variety of obstacles and with interaction with the player. In a simplified stochastic model for the motion of the ball, we position smooth obstacles about $\mathbb{R}^2$ at random, and we then project a ball through

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the ensuing environment, requiring that the ball be reflected at the obstacles on which it impacts. What can be said about the trajectory of the ball?

Such a mathematical model is often named after Hendrik Lorentz, following his 1905 papers [21] concerning the motion of an electron through a field of massive particles. The 'Lorentz lattice gas' has generated considerable interest amongst physicists (see [8, 9, 10, 24, 26, 27]), but very little is known about the rigorous mathematical theory. The apparent difficulty of the problem is due to the fact that the model postulates a dynamical system within a random environment; the asymptotic behaviour of the dynamical system can be rather sensitive to small variations in the environment.

Lorentz's exposition was developed by Ehrenfest [14], and a simple version of the lattice gas model has become known as the 'Ehrenfest wind–tree model', a title with a natural interpretation. Another modern interpretation of such a system is that of a ray of light shining through a medium of mirrors: reflecting bodies are placed randomly in $\mathbb{R}^d$, and the trajectory of a ray of light through the subsequent environment is studied. These interpretations have as common requirement the definition of a probability measure governing the dispositions and shapes of the obstacles (massive particles, mirrors, trees, etc.). Once this is prescribed, then one seeks to categorise the trajectory of the ball (electron, light ray, wind, etc.) using words of the type 'recurrent, transient, ergodic, (non-)localised, diffusive'.

Only fragmentary progress has been made with the required mathematics, and we summarise some of this in the following paragraphs.

1. *Periodic pin-ball in* $\mathbb{R}^2$. Circular obstacles are distributed about a finite box of $\mathbb{R}^2$ in the manner of a Poisson process, and the contents of the finite box are copied periodically in a tiling of the plane $\mathbb{R}^2$. Subject to certain assumptions, Bunimovitch and Sinai [3] have established a central limit theorem for the trajectory of the ball. See [25] for related material.

2. *Plane mirrors in* $\mathbb{R}^2$. Plane two-sided mirrors of unit length are distributed as follows about $\mathbb{R}^2$: they are centred at points of a Poisson process with intensity λ, and their orientations are chosen independently and randomly from a given countable set S having a certain property. We now project light from the origin. Harris [19] has proved that the light is a.s. localised (i.e., confined to a bounded region of $\mathbb{R}^2$) of $\lambda < \lambda_c$, and is non-localised with strictly positive probability when $\lambda > \lambda_c$. Here, λ_c is the critical density of the continuum percolation system of mirrors, viewed as unit rods. (Related results concerning continuum percolation may be found in [22].)

3. *Diagonal mirrors on* $\mathbb{Z}^2$. Each vertex of $\mathbb{Z}^2$ is designated a *mirror* with probability p and a *crossing* otherwise. Given that a vertex is a mirror, it is designated a north-west (nw) mirror with probability $\frac{1}{2}$ and a north-

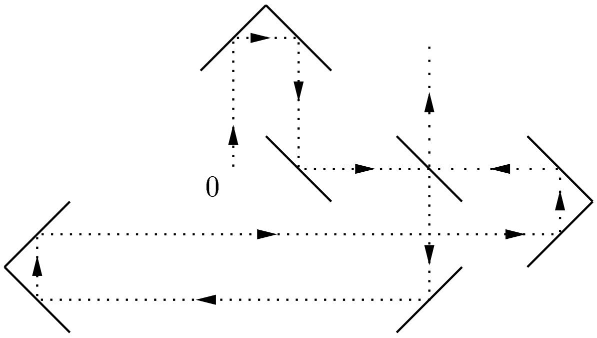

Fig. 1. A labyrinth of mirrors on the square lattice. The ray of light is reflected by the mirrors, and it is a problem is to determine, for a given density of mirrors, whether or not the light is a.s. restricted to a finite region.

east (ne) mirror otherwise. We now place small plane two-sided mirrors at those vertices which have been designated mirrors, in the directions of the designations (see Figure 1). Light is shone northwards (say) from the origin, and we ask for properties of the ensuing trajectory.

It is trivial that light is non-localised if $p = 0$, and it is known but non-trivial that light is a.s. localised if $p = 1$ (see [4, 15, 16]). The latter statement is proved by a simple but beautiful argument from percolation theory. Some partial progress for the case of general p has been made recently by Quas [23], but mathematicians have no proof of the physicists' conjecture ([10, 27]) that light is a.s. localised whenever $p > 0$.

4. *Pin-ball with scatterers.* Menshikov and Volkov [18] have proposed a model in which the obstacles are augmented by zones in which the pin-ball behaves in the manner of a random walk. In a lattice model, for example, a positive density of vertices are designated random walk (rw) points. When the ball arrives at a rw point, it chooses its exit direction uniformly at random from those available. Not surprisingly, this additional randomness provides a degree of flexibility in the environment which may be harnessed by mathematicians. The subsequent theory for lattice pin-ball has been developed in [2, 16], and is summarised in Sections 2–4 of this survey.

5. *Rotator pin-ball.* Ruijgrok and Cohen [24] have proposed a general study of mirror and 'rotator' models. In a rotator model in $\mathbb{R}^2$, the direction of the ball is rotated clockwise through an angle $\theta(x)$ when it arrives at a vertex x; here, the $\theta(x)$ are independent, identically distributed random variables. One may also allow a stochastic variation in the environment, in the sense that the $\theta(x)$ may be allowed to vary randomly as time passes. See [4, 5] for partial results.

2. Generalised pin-ball

Our model for generalised pin-ball involves a random environment of reflecting bodies distributed around the vertices of the d-dimensional cubic lattice $\mathbb{L}^d$. Each vertex is designated either a reflector (of a randomly chosen type) or a 'random walk point'. The interpretation of the term 'random walk point' is as follows: when a ball impacts on such a point, then it departs in a direction chosen randomly from the $2d$ available directions, this exit being chosen independently of everything else. Such models were introduced in [18] and have been studied further in [2, 16]. A similar model was proposed in [7] in the context of train sets, together with an application to the design of a computing machine.

There are many types of reflector, especially in three and more dimensions. The defining properties of a reflector ρ are that (i) to each incoming direction u there is assigned a unique outgoing direction $\rho(u)$, and (ii) the ball will retrace its path if the path's direction is reversed. Let $I = \{u_1, u_2, \dots, u_d\}$ be the set of positive unit vectors of $\mathbb{Z}^d$, and let $I^\pm = \{\alpha u_j : \alpha = \pm, 1 \le j \le d\}$. A *reflector* is defined to be a map $\rho : I^\pm \to I^\pm$ with the property that $\rho(-\rho(u)) = -u$ for all $u \in I^\pm$ (this condition is in response to the reversibility of light paths). We write $\mathcal{R}$ for the set of all reflectors. One particular reflector is special, namely the identity map satisfying $\rho(u) = u$ for all $u \in I^\pm$; we call this the *crossing*, and we denote it by $+$. Crossings do not deflect the ball.

A *random labyrinth* is defined as follows. Let p_{rw} and p_+ be non-negative reals such that $p_{\mathrm{rw}} + p_+ \le 1$, and let π be a probability mass function on the set $\mathcal{R}\backslash\{+\}$ of 'non-trivial' reflectors (that is, $\pi(\rho) \ge 0$ for $\rho \in \mathcal{R}\backslash\{+\}$ and $\sum_{\rho\in\mathcal{R}\backslash\{+\}} \pi(\rho) = 1$). Let $Z = (Z_x : x \in \mathbb{Z}^d)$ be a family of independent random variables, taking values in $\mathcal{R} \cup \{\varnothing\}$, with probabilities

$$\mathbb{P}(Z_x = \beta) = \begin{cases} p_{\mathrm{rw}} & \text{if } \beta = \varnothing, \\ p_+ & \text{if } \beta = +, \\ (1 - p_{\mathrm{rw}} - p_+)\pi(\rho) & \text{if } \beta = \rho \in \mathcal{R}\backslash\{+\}. \end{cases}$$

A vertex x is called a *crossing* if $Z_x = +$, and a *random walk (rw) point* if $Z_x = \varnothing$.

We now study admissible paths in the labyrinth Z. Consider a path in $\mathbb{L}^d$ which visits (in order) the vertices $x_0, x_1, \dots, x_n$; we allow the path to revisit a given vertex more than once, and to traverse a given edge more than once. This path is *admissible* if it conforms to the reflectors which it meets, which is to say that

$$x_{j+1} - x_j = Z_{x_j}(x_j - x_{j-1}) \quad \text{for all } j \text{ such that } Z_{x_j} \ne \varnothing.$$

If $p_{\mathrm{rw}} = 0$, then very little is known about such systems except that which has been summarised in Section 1. Henceforth, we assume that $p_{\mathrm{rw}} > 0$, and we define a 'random walk in the labyrinth Z'. Let x be a rw point. A walker starts at x, and flips a fair $2d$-sided coin in order to determine the direction of its first step. Henceforth, it is required to traverse admissible paths only, and it flips the coin to determine its exit direction from any rw point encountered. We write P_x^Z for the law of the random walk in the labyrinth Z, starting from a rw point x.

There is a natural equivalence relation on the set N of rw points of $\mathbb{Z}^d$, namely $x \leftrightarrow y$ if there exists an admissible path with endpoints x and y. Let C_x be the equivalence class containing the rw point x. We may follow the progress of a random walk starting at x by writing down (in order) the rw points which it visits, say X_0 $(= x), X_1, X_2, \ldots$. Now, given Z, $X = (X_n)$ is an irreducible Markov chain on the countable state space C_x; furthermore it is reversible with respect to the measure μ given by $\mu(y) = 1$ for $y \in C_x$. We say that x is *Z-localised* if $|C_x| < \infty$, and *Z-non-localised* otherwise. We call Z *localised* if all rw points are Z-localised, and we call Z *non-localised* otherwise. By a zero–one law, we have that $\mathbb{P}(Z$ is localised$)$ equals either 0 or 1.

We say that the rw point x is *Z-recurrent* if

$$P_x^Z(X_N = x \text{ for some } N \geq 1) = 1,$$

and *Z-transient* otherwise. The labyrinth Z is called *recurrent* if all rw points are Z-recurrent, and *transient* otherwise. By an appropriate zero–one law, we have that $\mathbb{P}(Z$ is recurrent$)$ equals either 0 or 1.

We now state four problems concerning the random labyrinth Z. Only partial information about these problems is known.

1. Decide when it is the case that

$$\mathbb{P}(Z \text{ is localised}) = 1.$$

2. If $\mathbb{P}(Z$ is non-localised$) = 1$, decide when it is the case that

$$\mathbb{P}\Big(0 \text{ is } Z\text{-recurrent} \,\Big|\, 0 \text{ is a rw point}\Big) = 1.$$

3. Decide when it is the case that

$$\mathbb{P}\Big(P_0^Z(|X_n|^2) \,\Big|\, 0 \text{ is a rw point}\Big) \sim cn \qquad \text{as } n \to \infty$$

for some $c > 0$. (Here, $m(Y)$ denotes the mean of Y under the measure m, and $|\cdot|$ denotes Euclidean distance.) There is also a 'pointwise', or 'quenched', version of this question.

4. If $\mathbb{P}(Z \text{ is non-localised}) = 1$ and $X_0 = 0$, $|C_0| = \infty$, decide when it is the case that (X_n) satisfies a central limit theorem, in the limit as $n \to \infty$.

Problems 3 and 4 are versions of the 'diffusivity' problem discussed in the physics literature (see, for example, [8, 9, 10, 24, 26, 27]). We note that the mean-square displacement of $|X_n|^2$ could (in principle) grow linearly with n even when the walk is localised. In contrast, one cannot have a full central limit theorem without non-localisation.

3. Non-localisation and recurrence

We concentrate in this section on the property of non-localisation for generalised pinball. Let $p_c = p_c(\mathbb{L}^d)$ denote the critical probability of site percolation on $\mathbb{L}^d$; see [15] for an account of percolation theory.

Theorem 3.1. *Let $d \geq 2$ and $p_{\mathrm{rw}} > 0$.*

(a) *The number M of infinite equivalence classes of $(N, \leftrightarrow)$ satisfies*

$$\textit{either } \mathbb{P}(M = 0) = 1 \textit{ or } \mathbb{P}(M = 1) = 1.$$

(b) *There exists a strictly positive constant $A = A(p_{\mathrm{rw}}, d)$ such that*

$$\mathbb{P}(Z \text{ is non-localised}) = 1 \tag{3.1}$$

if either $1 - p_{\mathrm{rw}} - p_+ < A$ or $p_{\mathrm{rw}} > p_c$.

Part (a) is proved by adapting the scheme of Burton and Keane [6] who proved the uniqueness of infinite clusters in percolation-type models. The details may be found in [2]. As for part (b), two related but distinct proofs have appeared in [16, 18]. The major difficulty is to prove non-localisation under the assumption that the density $1 - p_{\mathrm{rw}} - p_+$ is small. Of greatest value in the proofs of part (b) is the 'block method' of [18], which provides a powerful tool for controlling the geometry of the labyrinth, and which is useful for other problems too.

One may find cases of labyrinths which are localised, and also non-localised labyrinths which fall outside the conditions of part (b) of Theorem 3.1. See [16, 18] for the latter.

We turn now to the question of determining whether a labyrinth is transient or recurrent. For this problem, the most useful arguments appear to be those related to certain corresponding electrical networks; see [13, 17]. One may use block arguments, referred to above, in order to compare a random walk in a random labyrinth with a random walk on the infinite cluster of a certain related percolation model. Another feasible approach might be to employ the results of [1].

Theorem 3.2. *Let $p_{\mathrm{rw}} > 0$.*

(a) *If $d = 2$, the labyrinth Z is $\mathbb{P}$-a.s. recurrent.*

(b) *Let $d \geq 3$. There exists a strictly positive constant $A = A(p_{\mathrm{rw}}, d)$ such that: (3.1) holds, and in addition*

$$\mathbb{P}(Z \text{ is transient}) = 1,$$

whenever either $1 - p_{\mathrm{rw}} - p_+ < A$ or $p_{\mathrm{rw}} > p_{\mathrm{c}}$.

Part (a) is a minor extension of Theorem 3 of [18], proved similarly. The conclusion of part (b) has appeared in [16, 18] under the condition that $1 - p_{\mathrm{rw}} - p_+ < A$; when $p_{\mathrm{rw}} > p_{\mathrm{c}}$, the claim follows from the results and arguments of [17, 18].

4. Central limit theorem

When $p_{\mathrm{rw}} > 0$, the ball follows a type of random walk in a random environment, the environment being the rather rigid one provided by the pin-ball table. Whenever the walk is non-localised, it is natural to seek a central limit theorem (CLT) for its displacement after n units of time have elapsed. The basic methodology of the theorem which follows is the CLT of Kipnis and Varadhan [20], together with its application to percolation by DeMasi, Ferrari, Goldstein, and Wick [11, 12]. Numerous complications arise in applying such techniques in the present setting.

Suppose that the origin 0 is a rw point. As before, we consider the sequence $X_0\ (= 0), X_1, X_2, \dots$ of rw points visited in sequence by a random walk in Z beginning at the origin 0. For $\epsilon > 0$, we let

$$X^\epsilon(t) = \epsilon X_{\lfloor \epsilon^{-2} t \rfloor} \qquad \text{for } t \geq 0,$$

and we are interested in the behaviour of the process $X^\epsilon(\cdot)$ in the limit as $\epsilon \downarrow 0$. We shall study X^ϵ under the probability measure $\mathbb{P}_0$, defined as the measure $\mathbb{P}$ conditional on the event $\{0 \text{ is a rw point, and } |C_0| = \infty\}$.

Theorem 4.1. *Let $d \geq 2$ and $p_{\mathrm{rw}} > 0$. There exists a strictly positive constant $A = A(p_{\mathrm{rw}}, d)$ such that the following holds whenever either $1 - p_{\mathrm{rw}} - p_+ < A$ or $p_{\mathrm{rw}} > p_{\mathrm{c}}$:*

(a) $\mathbb{P}(0 \text{ is a rw point, and } |C_0| = \infty) > 0$, *and*

(b) *as $\epsilon \downarrow 0$, the re-scaled process $X^\epsilon(\cdot)$ converges $\mathbb{P}_0$-dp to $\sqrt{\delta} W$, where W is a standard Brownian motion in $\mathbb{R}^d$ and δ is a strictly positive constant.*

The convergence '$\mathbb{P}_0$-dp' means that

$$P_0^Z\big(f(X^\epsilon)\big) \to E\big(f(W)\big) \quad \text{in } \mathbb{P}_0\text{-probability}, \tag{4.1}$$

for all bounded continuous functions on the appropriate Skorohod path-space $D([0,\infty),\mathbb{R}^d)$. Here, E stands for the canonical expectation operator.

We finish with some remarks concerning Theorem 4.1 (full details of the proof of which may be found in [2, 16, 18]). In applying the CLT of [12, 20], one needs certain information about the geometry of the pin-ball table. Non-trivial reflectors (i.e., reflectors other than crossings) may have complicated geometries, and so one works as much as possible on volumes of space which contain only rw points and crossings. The geometry of such regions may be controlled as follows. First, one states an appropriate property of a large block, and then one utilises arguments from percolation theory (see [15]) to describe the set of 'good' blocks. More precisely, when the density of good blocks exceeds the critical percolation probability p_c, then there exists a.s. an infinite cluster of good blocks, all of whose rw points lie in a single inter-communicating class within the pin-ball table. The validity of a CLT then follows in a fairly straightforward manner from the results of [12, 20]. It is however a substantial problem to prove that the diffusion constant δ is *strictly* positive, and this may be achieved using arguments relating random walks to electrical networks.

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Statistical Laboratory, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, UK

E-mail address: g.r.grimmett@statslab.cam.ac.uk

URL: http://www.statslab.cam.ac.uk/~grg/