In memory of Roland L. Dobrushin

DECAY OF CORRELATIONS IN SUBCRITICAL POTTS AND RANDOM-CLUSTER MODELS

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Abstract. We prove exponential decay for the tail of the radius R of the cluster at the origin, for subcritical random-cluster models, under an assumption slightly weaker than that $\mathrm{E}(R^{d-1}) < \infty$ (here, d is the number of dimensions). Specifically, if $\mathrm{E}(R^{d-1}) < \infty$ throughout the subcritical phase, then $\mathrm{P}(R \geq n) \leq \exp(-\alpha n)$ for some $\alpha > 0$. This implies the exponential decay of the two-point correlation function of subcritical Potts models, subject to a hypothesis of (at least) polynomial decay of this function. Similar results are known already for percolation and Ising models, and for Potts models when the number q of available states is sufficiently large; indeed the hypothesis of polynomial decay has been proved rigorously for these cases. In two dimensions, the hypothesis that $\mathrm{E}(R) < \infty$ is weaker than requiring that the susceptibility be finite, i.e., that the two-point function be summable. The principal new technique is a form of Russo's formula for random-cluster models reported by Bezuidenhout, Grimmett, and Kesten. For the current application, this leads to an analysis of a first-passage problem for random-cluster models, and a proof that the associated time constant is strictly positive if and only if the tail of R decays exponentially.

1. Introduction

The probability theory of phase transition in physical systems is fairly developed (see the papers published in [14]). For a variety of models of interest, it turns out that there is a unique point of phase transition, which separates a 'subcritical' phase from a 'supercritical' phase. Throughout the subcritical phase, one often finds that the correlation functions decay exponentially over large distances. In contrast, they are bounded away from zero in the supercritical phase. This general picture of statistical mechanics has been verified in many probabilistic systems, including the percolation and Ising models. Such percolation/Ising systems may be incorporated together with Potts models within the broader class of 'random-cluster models', and the latter class of models provides a beautiful general setting for studying such systems. In particular, one may ask whether or not the exponential decay of the connectivity function characterises the subcritical phases of all random-cluster models. The current paper is directed at this question.

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Decay rates are fundamental to understanding the structure of models of statistical physics. One of the major thrusts of the modern theory of Gibbs states is directed towards a control of correlation functions over large spatial scales. This programme was initiated in part in a famous paper of Dobrushin and Pecherski [11], who established in a certain context that polynomial decay of correlation functions implies exponential decay. Such results have provided stepping stones towards proofs of full exponential decay; see [18, 24, 27] for further examples of such theorems. In the present paper, we prove a similar result in the general context of the random-cluster model (otherwise known as the Fortuin–Kasteleyn representation).

In advance of presenting the technical details, we state briefly the main result of this paper. (For formal definitions, the reader is referred to Section 2.) Let p and q be the parameters of a random-cluster model on $\mathbb{Z}^d$ where $d \geq 2$; here, p is the edge parameter, and q is the cluster-weighting parameter. Suppose $q \geq 1$, and let $p_c(q)$ be the critical value of p, i.e.,

$$p_c(q) = \sup\{p : \phi_{p,q}(0 \leftrightarrow \infty) = 0\},$$

where $\phi_{p,q}$ is the appropriate probability measure, and $\{0 \leftrightarrow \infty\}$ is the event that the origin is in an infinite open cluster. [For cognoscenti, we remark that $\phi_{p,q}$ is the random-cluster measure obtained using 'free boundary conditions'.] Writing $\{0 \leftrightarrow \partial\Lambda_n\}$ for the event that the open cluster at the origin intersects the sphere of radius n, it is presumably the case that

$$\phi_{p,q}(0 \leftrightarrow \partial\Lambda_n) \leq e^{-\alpha n} \tag{1.1}$$

for some $\alpha = \alpha(p, q)$ satisfying

$$\alpha(p, q) > 0 \quad \text{if} \quad p < p_c(q). \tag{1.2}$$

Inequalities of the form (1.1) have been proved in the special cases when $q = 1$, $q = 2$, and q is sufficiently large. These cases correspond respectively to the percolation model ([13]), the Ising model ([2, 4, 6]), and Potts models with large q ([21, 22, 23]). Although the arguments used in these three special situations have certain features in common, there is no unified proof, and in particular no proof which extends to general values of q.

For percolation and Ising models, the exponential decay of the two-point function was first proved in two stages. Initially, it was shown that exponential decay is valid whenever the susceptibility is finite, i.e., whenever the two-point connectivity function (or *correlation function* in the case of the Ising model) is summable; and later it was proved that the susceptibility is indeed finite throughout the subcritical phase. (This was achieved by Hammersley [18] and Aizenman–Barsky [3] for percolation, and by Simon–Lieb [24, 27] and Aizenman–Barsky–Fernández [4] for the Ising model. In the case of percolation, a direct argument, avoiding the first stage, was discovered by Menshikov [25, 26].) In proofs of exponential decay for the percolation model, the BK inequality plays a central role (see [7, 13]). When $q = 2$, this role is played by the Simon–Lieb inequality (see [24, 27]). No such method is known for general q, although various attempts have been made to fill the gap (see [9, 15]).

In this paper, we establish the first stage of the above programme in the general setting of random-cluster models. We prove that

$$\text{if} \quad \limsup_{n\to\infty}\Big\{n^{d-1}\phi_{p,q}(0 \leftrightarrow \partial\Lambda_n)\Big\} < \infty \quad \text{when} \quad p < p_c(q) \tag{1.3}$$

then there exists $\alpha = \alpha(p, q)$, satisfying $\alpha(p, q) > 0$ when $p < p_c(q)$, such that

$$\phi_{p,q}(0 \leftrightarrow \partial\Lambda_n) \le e^{-\alpha n} \quad \text{for all large } n. \tag{1.4}$$

Next we discuss briefly the assumption (1.3). Hypothesis (1.3) requires that $\phi_{p,q}(0 \leftrightarrow \partial\Lambda_n)$ decay at least as fast as $1/n^{d-1}$, and is implied by the stronger statement that

$$\phi_{p,q}(R^{d-1}) < \infty \quad \text{when} \quad p < p_c(q), \tag{1.5}$$

where $R = \max\{n : 0 \leftrightarrow \partial\Lambda_n\}$ is the *radius* of the open cluster at the origin (and we use $\phi_{p,q}$ to denote *expectation* as well as *probability*); we shall return to this discussion just before the statement of Theorem 1 in Section 3. By elementary geometrical considerations, there exists a positive constant $\beta = \beta(d)$ such that

$$\beta|C|^{1/d} \le R + 1 \le |C| \tag{1.6}$$

where $C = \{x : 0 \leftrightarrow x\}$ is the open cluster at the origin. Therefore (1.6) is implied by the statement

$$\phi_{p,q}(|C|^{d-1}) < \infty \quad \text{when} \quad p < p_c(q), \tag{1.7}$$

which is equivalent, when $d = 2$, to the finiteness of the *susceptibility*

$$\chi(p, q) = \phi_{p,q}(|C|).$$

The relationship between random-cluster models and percolation/Ising/Potts models is well explored and documented elsewhere (see the references in [16]). The result described above has the following implication for ferromagnetic Potts models. If the two-point correlation function decays at least as fast as a certain negative polynomial, then it decays at least as fast as $e^{-\alpha n}$. Hypothesis (1.3) is not easily translated into an exactly equivalent statement for Potts models. Either of the following two conditions suffices for Potts models:
(a) the two-point correlation function decays at least as fast as $1/n^{2(d-1)}$,
(b) the finite-volume quantity $\pi^1_{\Lambda_n}(\sigma_0 = 1) - q^{-1}$ decays at least as fast as $1/n^{d-1}$.
(Here, π^1_Λ is a ferromagnetic Potts measure on $\{1, 2, \ldots, q\}^\Lambda$ having '1' boundary conditions, and σ_0 is the spin at the origin.)

For this study, it is natural to investigate a certain related first-passage problem arising as follows from the random-cluster model. Let F_n denote the minimum number of closed edges amongst paths of the lattice joining the origin to $\partial\Lambda_n$, i.e., F_n is the minimal number of extra edges required to be open in order that $\{0 \leftrightarrow \partial\Lambda_n\}$ occurs. It may be shown, using the ergodicity of $\phi_{p,q}$ (see [12, 16, 20]), that the limit

$$\mu(p, q) = \lim_{n\to\infty} \left\{n^{-1} F_n\right\} \tag{1.8}$$

exists and is constant ($\phi_{p,q}$-a.s.). It is presumably the case that

$$\mu(p, q) > 0 \quad \text{for } p < p_c(q). \tag{1.9}$$

We show in Theorem 4 that (1.9) holds if and only if $\phi_{p,q}(0 \leftrightarrow \partial\Lambda_n)$ decays exponentially as $n \to \infty$ when $p < p_c(q)$ (i.e., (1.1) and (1.2) hold). As noted earlier in a related context, exponential decay is proved only for $q = 1$, $q = 2$, and for sufficiently large q. The above first-passage problem has been studied in the case of percolation ($q = 1$) by Kesten [19], and related results are known for the two-dimensional Ising model (see [12] and its references).

A similar first-passage problem has been studied by Fontes and Newman [12]. By utilising one of their arguments, we shall establish sufficient conditions for the conclusion $\mu(p, q) > 0$. This in turn implies the required exponential decay.

Incidentally, the comparison inequalities (see [16], Thm 2.2) imply exponential decay for sufficiently small p. The problem is prove it all the way up to the critical point.

2. Random-cluster models

In this section, we introduce appropriate notation, and we define random-cluster measures. For general results and historical background, we refer the reader to [16] and the references therein.

We define a random-cluster measure on a finite graph $G = (V, E)$ as follows. Let $0 \le p \le 1$ and $q > 0$. The relevant sample space is the finite set $\Omega_E = \{0, 1\}^E$, containing configurations that allocate 0's and 1's to the edges of G. For $\omega \in \Omega_E$, we call an edge e *open* if $\omega(e) = 1$, and *closed* otherwise. The random-cluster measure on G, having parameters p and q, is the probability measure $\phi_{G,p,q}$ on Ω_E given by

$$\phi_{G,p,q}(\omega) = \frac{1}{Z_{G,p,q}} \left\{ \prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \quad \omega \in \Omega_E, \tag{2.1}$$

where $k(\omega)$ is the number of open components of ω (i.e., the number of components of the graph $(V, \eta(\omega))$, where $\eta(\omega)$ is the set of open edges under ω), and

$$Z_{G,p,q} = \sum_{\omega \in \Omega_E} \left\{ \prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega)} \tag{2.2}$$

is the normalising factor (or 'partition function').

We shall define a random-cluster measure on an infinite lattice by taking weak limits of such measures on finite boxes of the lattice. In advance of doing this, we present some notation which will be useful later. Let $\mathbb{L}$ be the d-dimensional hypercubic lattice having vertex set $\mathbb{Z}^d$ and edge set $\mathbb{E}$ containing all pairs of vertices which are euclidean distance 1 apart; we assume throughout that $d \ge 2$. We shall write $x = (x_1, x_2, \dots, x_d)$ for $x \in \mathbb{Z}^d$, and denote by $\langle x, y \rangle$ an edge joining vertices x and y. A path of $\mathbb{L}$ is an alternating sequence $x_0, e_0, x_1, e_1, \dots$ of distinct vertices x_i and edges e_j such that $e_j = \langle x_j, x_{j+1} \rangle$ for each j. If this path terminates at some x_n then it is said to join x_0 to x_n and to have length n; if a path has infinitely many vertices then it is said to connect x_0 to ∞. We write

$$\|x\| = \max_i \{|x_i|\} \quad \text{where } x = (x_1, x_2, \dots, x_d).$$

The basic configuration space is $\Omega = \{0, 1\}^{\mathbb{E}}$ endowed with the σ-field $\mathcal{F}$ generated by the finite-dimensional cylinders of Ω. A configuration ω ($\in \Omega$) is an assignment of 0 or 1 to each edge e ($\in \mathbb{E}$), and may be put into one–one correspondence with the set $\eta(\omega) =$

$\{e \in \mathbb{E} : \omega(e) = 1\}$ of 'open' edges in ω. The 'open paths' of a configuration ω are those paths of $\mathbb{L}$ all of whose edges are open. If A and B are sets of vertices, we write $\{A \leftrightarrow B\}$ for the event that there exists an open path joining some vertex of A to some vertex of B. Similarly we write $\{A \leftrightarrow \infty\}$ for the event that some vertex of A is the endpoint of an infinite open path. The complements of such events are denoted using the symbol $\not\leftrightarrow$.

For any subset E of $\mathbb{E}$, we write $\mathcal{F}_E$ for the σ-field of subsets of Ω generated by the finite-dimensional cylinders of E, so that $\mathcal{F} = \mathcal{F}_{\mathbb{E}}$. A *box* Λ is a subset of $\mathbb{Z}^d$ of the form

$$\Lambda = \prod_{i=1}^{d} [x_i, y_i]$$

for some $x, y \in \mathbb{Z}^d$, and where $[x_i, y_i]$ is interpreted as $[x_i, y_i] \cap \mathbb{Z}$. The box Λ generates a subgraph of $\mathbb{L}$ with vertex set Λ and edge set $\mathbb{E}_\Lambda$ containing all edges $\langle u, v\rangle$ with $u, v \in \Lambda$. Of particular interest are the boxes $\Lambda_n = [-n, n]^d$, for $n \geq 1$. The *boundary* ∂V of a set V of vertices is the set of all vertices x ($\in V$) which are adjacent to some vertex of $\mathbb{L}$ not in V.

For a box Λ, we write Ω_Λ^0 for the subset of Ω containing all configurations ω satisfying $\omega(e) = 0$ for $e \notin \mathbb{E}_\Lambda$.

Let $0 \leq p \leq 1$ and $q \geq 1$. We define $\phi^0_{\Lambda,p,q}$ to be the random-cluster measure on the finite graph $(\Lambda, \mathbb{E}_\Lambda)$ 'with boundary condition 0' (this is the equivalent of free boundary conditions for ferromagnetic systems). This is done basically as in (2.1), but on a slightly different probability space. More precisely, let $\phi^0_{\Lambda,p,q}$ be the probability measure on $(\Omega, \mathcal{F})$ satisfying

$$\phi^0_{\Lambda,p,q}(\omega) = \frac{1}{Z^0_{\Lambda,p,q}} \left\{ \prod_{e\in\mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)} \quad \text{for } \omega \in \Omega^0_\Lambda, \tag{2.3}$$

where $k(\omega, \Lambda)$ is the number of components of the graph $(\mathbb{Z}^d, \eta(\omega))$ which intersect Λ, and where $Z^0_{\Lambda,p,q}$ is the appropriate normalising constant

$$Z^0_{\Lambda,p,q} = \sum_{\omega\in\Omega^0_\Lambda} \left\{ \prod_{e\in\mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)} \right\} q^{k(\omega,\Lambda)}. \tag{2.4}$$

Note that $\phi^0_{\Lambda,p,q}(\Omega^0_\Lambda) = 1$.

The following facts are known and relevant (see [16]).

(a) The limit $\phi_{p,q} = \lim_{\Lambda\to\mathbb{Z}^d} \phi^0_{\Lambda,p,q}$ exists, in the sense of weak convergence of measures.
(b) The measure $\phi_{p,q}$ is ergodic.
(c) If $\phi^\xi_{\Lambda,p,q}$ is a random-cluster measure on Λ with some boundary condition ξ other than '0' (see [16]), then all weak limits as $\Lambda \to \mathbb{Z}^d$ of $\phi^\xi_{\Lambda,p,q}$ are equal to $\phi_{p,q}$, so long as $p < p_c(q)$, where $p_c(q)$ is the following *critical value*

$$p_c(q) = \sup\{p : \phi_{p,q}(0 \leftrightarrow \infty) = 0\}. \tag{2.5}$$

(d) Random-cluster measures (with $q \geq 1$) satisfy the FKG inequality.

The relationship between random-cluster models and Potts models is well documented elsewhere (see the references in [16]). We note here only that the q-state Potts model with pair-interaction J (> 0) corresponds to the random-cluster model with parameters

$p = 1 - e^{-J}$ and q. In particular, the two-point correlation function of the Potts model with spins σ satisfies

$$\langle \delta_{\sigma_0,\sigma_x} \rangle - q^{-1} = (1 - q^{-1})\phi_{p,q}(0 \leftrightarrow x),$$

where $\langle\cdot\rangle$ denotes averages with respect to the Potts measure on $\mathbb{L}$ arising from free boundary conditions, and $\delta_{i,j}$ is the Kronecker delta. Now,

$$\phi_{p,q}(0 \leftrightarrow x) \le \phi_{p,q}(0 \leftrightarrow \partial\Lambda_n) \quad \text{if } \|x\| = n,$$

so that upper bounds for $\phi_{p,q}(0 \leftrightarrow \partial\Lambda_n)$ imply upper bounds for the Potts correlation function.

3. Exponential decay

We are interested here in the rate of decay of connectivity functions in the subcritical phase, i.e., when $p < p_c(q)$. We prove exponential decay under a certain assumption which we introduce next. Let $q \ge 1$. For $0 \le p \le 1$, define

$$Z(p,q) = \limsup_{n\to\infty} \Big\{ n^{d-1} \phi_{p,q}(0 \leftrightarrow \partial\Lambda_n) \Big\}. \tag{3.1}$$

Now $Z(p,q)$ is non-decreasing in p, and we may therefore define

$$p_g(q) = \sup\{p : Z(p,q) < \infty\}. \tag{3.2}$$

Clearly $p_g(q) \le p_c(q)$, and it is generally believed that equality holds here. The critical point $p_g(q)$ plays the role of the quantity p_T in the percolation literature (see [13], p. 45), although $p_g(q)$ and p_T have different (but similar) definitions. As observed in Section 1, it is known that $p_g(q) = p_c(q)$ if $q = 1$, $q = 2$, or q is sufficiently large.

The condition $Z(p,q) < \infty$ amounts to assuming that the radius $R = \max\{\|x\| : 0 \leftrightarrow x\}$ has a tail decaying at least as fast as $n^{-(d-1)}$, and is a weaker assumption than the moment condition $\phi_{p,q}(R^{d-1}) < \infty$. [The expression $\mu(X)$ denotes the mean of the random variable X under the measure μ.] Actually $Z(p,q) = 0$ if $\phi_{p,q}(R^{d-1}) < \infty$, since

$$n^{d-1}\phi_{p,q}(0 \leftrightarrow \partial\Lambda_n) = n^{d-1}\phi_{p,q}(R \ge n) \le \sum_{k=n}^{\infty} k^{d-1}\phi_{p,q}(R = k).$$

There is a converse also. If $p < p_g(q)$ then $Z(p,q) < \infty$, implying that

$$n^c \phi_{p,q}(0 \leftrightarrow \partial\Lambda_n) \to 0 \quad \text{for all } c \text{ satisfying } c < d-1.$$

This in turn implies that $\phi_{p,q}(R^c) < \infty$ for all c satisfying $c < d - 1$ (see [17], Problem 5.6.18).

Theorem 1. *Let $0 < p < 1$ and $q \ge 1$, and suppose that $p < p_g(q)$. There exists $\alpha = \alpha(p,q)$ satisfying $\alpha(p,q) > 0$ such that*

$$\phi_{p,q}(0 \leftrightarrow \partial\Lambda_n) \le e^{-\alpha n} \quad \textit{for all large } n. \tag{3.3}$$

This theorem is proved in Section 5.

When $d=2$, it is believed that the critical point $p_c(q)$ coincides with the self-dual point $\kappa_q = \sqrt{q}/(1+\sqrt{q})$; see [16, 28]. It is known that $p_c(q) \ge \kappa_q$, but no rigorous proof of the converse inequality is available for general q (≥ 1). It would be sufficient to prove a 'reasonable' decay rate for $\phi_{p,q}(0 \leftrightarrow \partial\Lambda_n)$ as $n \to \infty$, when $p < p_c(q)$. Using Theorem 1, we find that $p_c(q) = \kappa_q$ if

$$\phi_{p,q}(0 \leftrightarrow \partial\Lambda_n) \le \frac{c(p)}{n} \quad \text{for all } n,$$

where $c(p) < \infty$ for $p < p_c(q)$.

4. Two lemmas, and a first-passage problem

Next we state and prove two fundamental inequalities. After this, we apply them in studying a first-passage problem.

First we review a fundamental formula of [8]. Fix $q \in (0,\infty)$, $p \in (0,1)$, and let ψ_p be the random-cluster measure with parameters p and q on the finite graph $G = (V, E)$; later we shall set $G = \Lambda$ and $\psi_p = \phi^0_{\Lambda,p,q}$. It is proved in [8] that, for any event A,

$$\frac{d}{dp}\,\psi_p(A) = \frac{1}{p(1-p)}\Big\{\psi_p(N1_A) - \psi_p(N)\psi_p(A)\Big\} \tag{4.1}$$

where 1_A is the indicator function of A, and N is the number of open edges (i.e., for $\omega \in \Omega_E = \{0,1\}^E$, we have $N(\omega) = \sum_e \omega(e)$). A version of this formula is often attributed to Russo in the case $q=1$ (percolation) although it was known earlier to those working in reliability theory (see the discussion in [13]).

There is a partial order on Ω_E given by: $\omega \le \omega'$ if and only if $\omega(e) \le \omega'(e)$ for all $e \in E$. A function $f : \Omega_E \to \mathbb{R}$ is called *increasing* if $f(\omega) \le f(\omega')$ whenever $\omega \le \omega'$, and is called *decreasing* if $-f$ is increasing. An event A $(\subseteq \Omega_E)$ is called *increasing* (resp. *decreasing*) if its indicator function 1_A is increasing (resp. decreasing).

Henceforth we assume that $q \ge 1$, so that ψ_p satisfies the FKG inequality. Suppose that A is an increasing event (but not the empty set $\varnothing$). For $\omega \in \Omega_E$, let $F_A(\omega)$ be the minimum number of additional edges necessary for A to occur; that is to say,

$$F_A(\omega) = \inf\left\{\sum_e \big\{\omega'(e) - \omega(e)\big\} : \omega' \ge \omega,\ \omega' \in A\right\}. \tag{4.2}$$

It may be checked that $N + F_A$ is an increasing random variable, and also that $F_A(\omega)1_A(\omega) = 0$ for all ω. Therefore, by the FKG inequality,

$$\psi_p(N1_A) = \psi_p\big((N+F_A)1_A\big) \ge \psi_p(N+F_A)\psi_p(A),$$

whence

$$\psi_p(N1_A) - \psi_p(N)\psi_p(A) \ge \psi_p(F_A)\psi_p(A).$$

Substituting this into (4.1), we obtain the following lemma.

Lemma 2. *Let $q \geq 1$ and $0 < p < 1$. For any increasing event A ($\neq \varnothing$),*

$$\frac{d}{dp}\{\log \psi_p(A)\} \geq \frac{\psi_p(F_A)}{p(1-p)}. \tag{4.3}$$

In the proof of Theorem 1, this inequality plays the role of inequalities (3.10) and (3.36) of [13], used by Menshikov [25, 26] to prove exponential decay for subcritical percolation models. Integrating (4.3) over the interval $[r, s]$, and using the facts that $p(1-p) \leq \frac{1}{4}$ and that F_A is a decreasing random variable, we find that

$$\begin{aligned} \psi_r(A) &\leq \psi_s(A) \exp\left\{-4\int_r^s \psi_p(F_A)dp\right\} \\ &\leq \psi_s(A) \exp\{-4(s-r)\psi_s(F_A)\}, \quad \text{if } r \leq s. \end{aligned} \tag{4.4}$$

There is a further relation between the probability of A and the mean of F_A.

Lemma 3. *Let $q \geq 1$ and $0 < r < s < 1$. Then, for any increasing event A,*

$$\psi_r(F_A \leq k) \leq \left(\frac{q}{s-r} \cdot \frac{(1-r)q}{r+(1-r)q}\right)^k \psi_s(A) \quad \textit{for all } k \geq 0. \tag{4.5}$$

This lemma is very closely related to the 'sprinkling' lemma of [5], a version of which is valid for random-cluster models; see also [13]. We shall make use of it in the following way. By (4.5) with $C = q^2(1-r)/\{(s-r)(r+(1-r)q)\}$,

$$\psi_r(F_A) = \sum_{k=0}^{\infty} \psi_r(F_A > k) \geq \sum_{k=0}^{K} \big(1 - C^k \psi_s(A)\big)$$

where $K = \max\{k : C^k \psi_s(A) \leq 1\}$. We sum this as usual, noting that $C > 1$, to find that

$$\psi_r(F_A) \geq \frac{-\log \psi_s(A)}{\log C} - \frac{C - \psi_s(A)}{C-1} \quad \text{if } r < s. \tag{4.6}$$

In advance of proving the latter lemma, we present an application of the two lemmas together. *Henceforth let $q \geq 1$.* Returning to the lattice $\mathbb{L}$, we set $A_n = \{0 \leftrightarrow \partial\Lambda_n\}$, and write F_n for F_{A_n}. As remarked in Section 1, Derrienic's theorem (see [12, 20]) implies the existence of the constant limit

$$\mu(p,q) = \lim_{n\to\infty}\left\{n^{-1}F_n\right\} \quad \phi_{p,q}\text{-a.s.} \tag{4.7}$$

Using a comparison inequality (see [16], Thm 2.2) we have that $\mu(p,q)$ is non-increasing in p, and we define

$$p_{\text{flow}}(q) = \sup\{p : \mu(p,q) > 0\}.$$

Next we define the *correlation length* $\xi(p,q)$ by

$$\xi(p,q)^{-1} = \lim_{n\to\infty}\left\{-\frac{1}{n}\log \phi_{p,q}\big(0 \leftrightarrow ne_1\big)\right\},$$

where e_1 is a unit vector in the direction of increasing first coordinate, and where the limit exists by the FKG inequality and subadditivity. (We adopt the convention that $\infty^{-1} = 0$.) Note that $\xi(p, q)$ is non-decreasing in p. Using the argument of [16], Thm 5.14, we have that

$$\lim_{n\to\infty} \left\{ -\frac{1}{n} \log \phi_{p,q}(A_n) \right\} = \xi(p, q)^{-1},$$

whence $\phi_{p,q}(A_n)$ decays exponentially if and only if $\xi(p, q) < \infty$. We define the further critical point

$$p_{\mathrm{corr}}(q) = \sup\{p : \xi(p, q) < \infty\}.$$

Theorem 4. *Let $q \geq 1$. It is the case that $p_{\mathrm{flow}}(q) = p_{\mathrm{corr}}(q)$.*

It is clear from the above observations that $p_{\mathrm{flow}}(q) = p_{\mathrm{corr}}(q) \leq p_{\mathrm{g}}(q) \leq p_{\mathrm{c}}(q)$, and it is a consequence of Theorem 1 that $p_{\mathrm{corr}}(q) = p_{\mathrm{g}}(q)$. It is believed also that $p_{\mathrm{g}}(q) = p_{\mathrm{c}}(q)$. As observed earlier, this is known only for $q = 1$, $q = 2$, and for sufficiently large q. The first-passage problem and the time constant $\mu(p, q)$ have been studied in detail when $q = 1$; see [19, 20]. Several authors have paid serious attention to a closely related question when $q = 2$ and $d = 2$, namely, the corresponding question for the two-dimensional Ising model, where the 'passage time' F_n is replaced by the minimum number of changes of spin along paths from the origin to $\partial\Lambda_n$; see [1, 12]. The time constant in the Ising case cannot exceed the corresponding random-cluster time constant $\mu(p, 2)$, since each edge of the Ising model having endpoints with unlike spins gives rise to a closed edge in the associated (coupled) random-cluster process.

In some of the following proofs we shall make use of Lemmas 2 and 3 applied to the infinite-volume random-cluster measures. Let A be an increasing (non-empty) cylinder event in the measurable space $(\Omega, \mathcal{F})$, and set $\psi_p = \phi^0_{\Lambda_M,p,q}$, where M is a positive integer. We apply (4.4) and (4.6) accordingly, noting that

$$\frac{q(1-r)}{s-r} < C < \frac{q}{s-r}.$$

Now, take the limit as $M \to \infty$, to obtain that, for $0 < r < s < 1$,

$$\phi_{r,q}(A) \leq \phi_{s,q}(A) \exp\Big\{-4(s-r)\phi_{s,q}(F_A)\Big\}, \tag{4.8}$$

$$\phi_{r,q}(F_A) \geq \frac{-\log \phi_{s,q}(A)}{\log\{q/(s-r)\}} - \frac{C}{C-1}. \tag{4.9}$$

Before turning to the proof of Theorem 4, we make one further observation. Inequalities (4.8) and (4.9), with $A = A_n$, imply that the correlation length $\xi(p, q)$ is *strictly* increasing in p whenever it is finite (cf. [13], Thm 5.14).

Proof of Lemma 3. Let $r < s$. We shall employ a suitable coupling of the measures ψ_r and ψ_s. Let $E = \{e_1, e_2, \ldots, e_m\}$ be the edges of the graph G, and let $U_1, U_2, \ldots, U_m$ be independent random variables having the uniform distribution on $[0, 1]$. We shall examine the edges in turn, to determine whether they are open or closed for the respective parameters r and s. The outcome will be a pair (π, ω) of configurations each lying in $\Omega = \{0, 1\}^E$ such that $\pi \leq \omega$. The configurations π, ω are random in the sense that they are functions of the U_j.

First, we declare

$$\begin{aligned}\pi(e_1) = 1 \quad &\text{if and only if} \quad U_1 < \psi_r(J_1),\\ \omega(e_1) = 1 \quad &\text{if and only if} \quad U_1 < \psi_s(J_1),\end{aligned}$$

where J_i is the set of configurations γ ($\in \Omega$) with $\gamma(e_i) = 1$. Note that $\psi_r(J_1) \le \psi_s(J_1)$ since $r < s$, and therefore $\pi(e_1) \le \omega(e_1)$.

Let M be an integer satisfying $1 \le M < m$. Having defined $\pi(e_i)$, $\omega(e_i)$ such that $\pi(e_i) \le \omega(e_i)$ (for $i \le M$), we define $\pi(e_{M+1})$ and $\omega(e_{M+1})$ as follows. We declare

$$\begin{aligned}\pi(e_{M+1}) = 1 \quad &\text{if and only if} \quad U_{M+1} < \psi_r\big(J_{M+1} \,\big|\, F_M(\pi)\big),\\ \omega(e_{M+1}) = 1 \quad &\text{if and only if} \quad U_{M+1} < \psi_s\big(J_{M+1} \,\big|\, F_M(\omega)\big),\end{aligned}$$

where $F_M(\gamma)$ is the set of configurations ν satisfying $\nu(e_i) = \gamma(e_i)$ for $1 \le i \le M$. We have that $\psi_r(J_{M+1} \mid F_M(\pi)) \le \psi_s(J_{M+1} \mid F_M(\omega))$ since $r < s$ and $\pi(e_i) \le \omega(e_i)$ for $1 \le i \le M$; this implies that $\pi(e_{M+1}) \le \omega(e_{M+1})$.

Continuing likewise, we obtain a pair (π, ω) of configurations satisfying:
(a) $\pi \le \omega$,
(b) π is distributed according to the measure ψ_r,
(c) ω is distributed according to the measure ψ_s.
We write μ for the probability measure associated with the U_j.

By a straightforward computation (cf. equation (3.10) of [16]),

$$\begin{aligned}\psi_p(J_i \mid D_i) &= \frac{p}{p + (1-p)q},\\ \psi_p(J_i \mid D_i^{\mathrm{c}}) &= p,\end{aligned}$$

where D_i is the event that there is no open path of $E \setminus \{e_i\}$ joining the endpoints of e_i, and D_i^{c} is the complement of D_i. Using conditional expectations, we deduce that, since $q \ge 1$, then

$$\frac{p}{p + (1-p)q} \le \psi_p(J_i \mid D) \le p \tag{4.10}$$

for any event D defined on the states of $E \setminus \{e_i\}$. It follows from the definition of the $\pi(e_i)$ and $\omega(e_i)$ that

$$\mu\big(\pi(e_{M+1}) = 0 \,\big|\, U_1, U_2, \dots, U_M\big) = 1 - \psi_r\big(J_{M+1} \,\big|\, F_m(\pi)\big) \le \frac{(1-r)q}{r + (1-r)q}.$$

By a similar argument,

$$\begin{aligned}&\mu\Big(\omega(e_{M+1}) = 1, \pi(e_{M+1}) = 0 \,\Big|\, U_1, U_2, \dots, U_M\Big)\\ &\qquad = \psi_s\big(J_{M+1} \,\big|\, F_M(\omega)\big) - \psi_r\big(J_{M+1} \,\big|\, F_M(\pi)\big) \ge \frac{s-r}{q}.\end{aligned}$$

A full derivation of the last inequality is obtainable as follows. Using Lemma 2 with $A = J_i$ (so that $F_{J_i} = 1_{J_i^c}$) together with (4.10),

$$\psi_p'(J_i) \geq \frac{\psi_p(J_i)\big(1-\psi_p(J_i)\big)}{p(1-p)} \geq \frac{1}{p+(1-p)q} \geq \frac{1}{q}.$$

Now integrate over the interval $[r,s]$ to obtain that

$$\psi_s(J_i) - \psi_r(J_i) \geq \frac{s-r}{q}. \tag{4.11}$$

Finally,

$$\psi_s\big(J_{M+1} \,\big|\, F_M(\omega)\big) - \psi_r\big(J_{M+1} \,\big|\, F_M(\pi)\big) \geq \psi_s\big(J_{M+1} \,\big|\, F_M(\omega)\big) - \psi_r\big(J_{M+1} \,\big|\, F_M(\omega)\big),$$

and the claim follows by applying (4.11) with $i = M+1$ to the graph obtained from G by contracting (resp. deleting) any edge e_i (for $1 \leq i \leq M$) with $\omega(e_i) = 1$ (resp. $\omega(e_i) = 0$). Cf. Theorem 2.3 of [16].

It follows from the above that

$$\mu\Big(\omega(e_{M+1}) = 1 \,\Big|\, \pi(e_{M+1}) = 0,\, U_1, U_2, \dots, U_M\Big) \geq \frac{s-r}{q} \cdot \frac{r+(1-r)q}{(1-r)q}. \tag{4.12}$$

Now fix a configuration ξ ($\in \Omega$) and a set B of edges such that $\xi(e) = 0$ for $e \in B$. We claim that

$$\mu\big(\pi = \xi,\, \omega(e) = 1 \text{ for } e \in B\big) \geq \left(\frac{s-r}{q} \cdot \frac{r+(1-r)q}{(1-r)q}\right)^{|B|} \mu(\pi = \xi). \tag{4.13}$$

This follows from the recursive construction of π and ω in terms of the family $U_1, U_2, \dots, U_m$, in the light of the bound (4.12).

Inequality (4.13) implies the claim of the lemma, as follows. Let ξ be a configuration satisfying $F_A(\xi) \leq k$. There exists a set $B = B_\xi$ of edges such that

(a) $|B| \leq k$,
(b) $\xi(e) = 0$ for $e \in B$,
(c) the configuration obtained from ξ by allocating state 1 to all edges in B lies in the event A.

If more than one such set B exists, we pick the earliest in some deterministic ordering of all subsets of E. Then, by (4.13),

$$\begin{aligned}
\psi_s(A) &\geq \mu\big(F_A(\pi) \leq k,\, \omega(e) = 1 \text{ for } e \in B_\pi\big) \\
&= \sum_{\xi : F_A(\xi) \leq k} \mu\big(\pi = \xi,\, \omega(e) = 1 \text{ for } e \in B_\xi\big) \\
&\geq \left(\frac{s-r}{q} \cdot \frac{r+(1-r)q}{(1-r)q}\right)^k \psi_r(F_A \leq k).
\end{aligned}$$

□

Proof of Theorem 4. Let $r < s < p_{\text{flow}}(q)$. There exists a constant $\gamma = \gamma(s,q)$ (> 0) such that

$$\phi_{s,q}(F_n) \geq n\gamma(s,q) \quad \text{for all } n \geq 1. \tag{4.14}$$

Now let $A = A_n = \{0 \leftrightarrow \partial\Lambda_n\}$. In conjunction with (4.14), (4.8) implies the exponential decay of $\phi_{r,q}(A_n)$, whence $r < p_{\text{corr}}(q)$. Therefore $p_{\text{flow}}(q) \leq p_{\text{corr}}(q)$.

Conversely, suppose that $r < s < p_{\text{corr}}(q)$. There exists $\alpha = \alpha(s,q)$ (> 0) such that $\phi_{s,q}(A_n) \leq e^{-\alpha n}$ for all n. By (4.9) with $A = A_n$ and some positive $\beta = \beta(r,s,q)$,

$$\phi_{r,q}(F_n) \geq \frac{-\log(e^{-\alpha n})}{\log\{q/(s-r)\}} - \beta = \frac{\alpha n}{\log\{q/(s-r)\}} - \beta,$$

whence $r < p_{\text{flow}}(q)$. Therefore $p_{\text{corr}}(q) \leq p_{\text{flow}}(q)$. □

5. Proof of Theorem 1

There are two stages in the proof. In the first stage, we use inequalities (4.4) and (4.6) in an iterative scheme in order to prove that $\phi_{p,q}(A_n)$ decays 'near-exponentially' when $p < p_{\text{g}}(q)$. In the second stage, we use Theorem 4 together with an argument developed by Fontes and Newman [12] to deduce full exponential decay. The conclusions of these two stages are summarised in the following two lemmas.

Lemma 5. *Let $0 < p < 1$ and $q \geq 1$, and suppose that $p < p_{\text{g}}(q)$. There exist constants $c(p)$, $\Delta(p)$, satisfying $c(p) > 0$, $0 < \Delta(p) < 1$, such that*

$$\phi_{p,q}(A_n) \leq \exp\bigl(-cn^{\Delta}\bigr) \qquad \textit{for all } n \geq 1.$$

We recall the flow constant $\mu(p,q)$ defined in (1.8) and (4.7). As before, C is the vertex set of the open cluster at the origin.

Lemma 6. *Let $0 < p < 1$ and $q \geq 1$. If $\phi_{p,q}\bigl(|C|^{2d+\epsilon}\bigr) < \infty$ for some $\epsilon > 0$, then $\mu(p,q) > 0$.*

Before embarking on the proofs of these lemmas, we make some remarks. First, Lemma 5 will be proved by an iterative scheme which may be continued further. If this is done, one obtains thereby a proof that $\phi_{p,q}(A_n)$ decays at least as fast as $\exp\bigl\{-\alpha_k(p)n/\log_k n\bigr)$ for any $k \geq 1$, where $\alpha_k(p) > 0$ and $\log_k n$ is the kth iterate of logarithm.

Secondly, the hypothesis of Lemma 6 is implied by the conclusion of Lemma 5, using (1.6). Therefore Lemmas 5 and 6 imply that $\mu(p,q) > 0$ when $p < p_{\text{g}}(q)$, whence Theorem 1 follows by Theorem 4.

Thirdly, essentially the only feature of the measure $\phi_{p,q}$ which enables Lemma 6 is the FKG property. More precisely, a version of Lemma 6 holds with $\phi_{p,q}$ replaced by any ergodic probability measure satisfying the FKG inequality. In addition, the moment condition may be relaxed just a little; see [10, 12].

Proof of Lemma 5.. We shall make central use of inequalities (4.4) and (4.6), in an iterative scheme. Rather than using these inequalities in the forms presented for finite graphs, we shall make use of their infinite-volume versions (4.8) and (4.9). In the following, we shall sometimes use *real* quantities when *integers* are required. It will be clear that this notational simplification has no ultimate effect on the validity of the proof. All o(1) and O(1) terms are to be interpreted in the limit $n \to \infty$.

Fix $q \geq 1$. For $p < p_{\mathrm{g}}(q)$, there exists $c_1(p)$ satisfying $c_1(p) > 0$ such that

$$\phi_{p,q}(A_n) \leq \frac{c_1(p)}{n^{d-1}} \quad \text{for all } n. \tag{5.1}$$

Let $r < s < t < p_{\mathrm{g}}(q)$. By (4.9),

$$\phi_{s,q}(F_n) \geq \frac{-\log \phi_{t,q}(A_n)}{\log C} + \mathrm{O}(1) \geq \frac{(d-1)\log n}{\log C} + \mathrm{O}(1)$$

where $1 < C = q/(t-s) < \infty$. Insert this into (4.8) to obtain that

$$\phi_{r,q}(A_n) \leq \frac{c_2(r)}{n^{d-1+\Delta_2(r)}} \quad \text{for all } n \tag{5.2}$$

for some strictly positive and finite $c_2(r)$ and $\Delta_2(r)$. This holds for all $r < p_{\mathrm{g}}(q)$, and is an improvement over (5.1).

Next we shall obtain an improvement of (5.2). Let m be a positive integer, and let $R_i = im$ for $0 \leq i \leq K$, where $K = \lfloor n/m \rfloor$. Let L_i be the event $\{\partial\Lambda_{R_i} \leftrightarrow \partial\Lambda_{R_{i+1}}\}$, and let $H_i = F_{L_i}$, the minimal number of extra edges needed for L_i to occur. Clearly,

$$F_n \geq \sum_{i=0}^{K-1} H_i, \tag{5.3}$$

since every path from 0 to $\partial\Lambda_n$ traverses each annulus $\Lambda_{R_{i+1}} \setminus \Lambda_{R_i}$. There exists a constant η (≥ 1) such that $|\partial\Lambda_R| \leq \eta R^{d-1}$ for all R. Therefore, by the translation invariance of $\phi_{p,q}$,

$$\phi_{p,q}(L_i) \leq |\partial\Lambda_{R_i}|\phi_{p,q}(A_m) \leq \eta n^{d-1}\phi_{p,q}(A_m). \tag{5.4}$$

Let $r < s < p_{\mathrm{g}}(q)$, and let $c_2 = c_2(s)$, $\Delta_2 = \Delta_2(s)$ where the functions $c_2(p)$ and $\Delta_2(p)$ are given as in (5.2). It follows from (5.2) and (5.4) that

$$\phi_{s,q}(L_i) \leq \eta n^{d-1} \frac{c_2}{m^{d-1+\Delta_2}} \leq \frac{1}{2} \tag{5.5}$$

if

$$m = \left\{(2\eta c_2)n^{d-1}\right\}^{1/(d-1+\Delta_2)}, \tag{5.6}$$

and we choose m accordingly (here and later, we assume that n is large). Now $H_i \geq 1$ if L_i does not occur, whence

$$\phi_{s,q}(F_n) \geq \sum_{i=0}^{K-1} \left\{1 - \phi_{s,q}(L_i)\right\} \geq \tfrac{1}{2}K \tag{5.7}$$

by (5.3) and (5.5). Also,

$$K = \lfloor n/m \rfloor \geq D n^{\Delta_3} \tag{5.8}$$

by (5.6), for appropriate positive constants D, Δ_3 satisfying $D > 0$, $0 < \Delta_3 < 1$. In conjunction with (4.8) and (5.7), this lower bound for K implies that

$$\phi_{r,q}(A_n) \leq \exp\left\{-c_3 n^{\Delta_3}\right\} \quad \text{for all } n. \tag{5.9}$$

where $c_3 = c_3(r) > 0$, $0 < \Delta_3 = \Delta_3(r) < 1$. This holds for all $r < p_{\mathrm{g}}(q)$. □

Proof of Lemma 6. We prove that $\mu(p, q) > 0$ by an argument to be found in [12]. Let Π_n be the set of all paths of $\mathbb{L}$ joining the origin to $\partial\Lambda_n$. With $T(\pi)$ denoting the number of closed edges in a path π, we have that

$$T(\pi) + 1 \geq \sum_{x \in \pi} \frac{1}{|C_x \cap \pi|}$$

where the sum is over all vertices x of π, and C_x is the open cluster at x. It follows by Jensen's inequality that

$$\frac{T(\pi)+1}{|\pi|} \geq \frac{1}{|\pi|} \sum_{x \in \pi} \frac{1}{|C_x|} \geq \left\{ \frac{1}{|\pi|} \sum_{x \in \pi} |C_x| \right\}^{-1}.$$

Therefore,

$$\frac{F_n + 1}{n} \geq \inf_{\pi \in \Pi_n} \left\{ \frac{T(\pi)+1}{|\pi|} \right\} \geq K_n^{-1}$$

where

$$K_n = \sup_{\pi \in \Pi_n} \left\{ \frac{1}{|\pi|} \sum_{x \in \pi} |C_x| \right\}.$$

Using (4.7), we find that $\mu(p, q) \geq K^{-1}$ a.s., where

$$K = \limsup_{m \to \infty} \left[\sup \left\{ \frac{1}{|\pi|} \sum_{x \in \pi} |C_x| : |\pi| = m \right\} \right], \tag{5.10}$$

where the (inner) supremum is over all paths from the origin containing m vertices. We propose to show that $K < \infty$ a.s., whence $\mu(p, q) > 0$ as required.

Let $\{\widetilde{C}_x : x \in \mathbb{Z}^d\}$ be a collection of independent subsets of $\mathbb{Z}^d$ with the property that $\widetilde{C}_x$ has the same distribution as C_x. We claim, as in [12], that $\{|C_x| : x \in \mathbb{Z}^d\}$ is dominated stochastically by $\{M_x : x \in \mathbb{Z}^d\}$, where

$$M_x = \sup\{|\widetilde{C}_y| : y \in \mathbb{Z}^d,\ x \in \widetilde{C}_y\}.$$

We prove this inductively. Let $v_1, v_2, \ldots$ be a deterministic ordering of $\mathbb{Z}^d$. Given the random variables $\{\widetilde{C}_x : x \in \mathbb{Z}^d\}$, we shall construct a family $\{D_x : x \in \mathbb{Z}^d\}$ having the same joint distributions as $\{C_x : x \in \mathbb{Z}^d\}$ and satisfying (for each x) $D_x \subseteq \widetilde{C}_y$ for some y depending on

x. First, we set $D_{v_1} = \widetilde{C}_{v_1}$. Given $D_{v_1}, D_{v_2}, \dots, D_{v_n}$, we define $E = \bigcup_{i=1}^{n} D_{v_i}$. If $v_{n+1} \in E$, we set $D_{v_{n+1}} = D_{v_j}$ for some j such that $v_{n+1} \in D_{v_j}$. If $v_{n+1} \notin E$, we argue as follows. Let ΔE be the set of edges of $\mathbb{Z}^d$ having exactly one endpoint in E. We may find a (random) subset F of $\widetilde{C}_{v_{n+1}}$ such that F has the conditional distribution of $C_{v_{n+1}}$ *given that* all edges in ΔE are closed; we now set $D_{v_{n+1}} = F$. [It is here that we use the FKG inequality.] We obtain the stochastic domination accordingly.

It follows by (5.10) that

$$K \le \limsup_{m\to\infty} \left[\sup \left\{ \frac{1}{|\pi|} \sum_{x\in\pi} M_x : |\pi| = m \right\} \right] \quad \text{a.s.}$$

By Lemma 2 of [12, p. 760],

$$K \le 2 \limsup_{m\to\infty} \left[\sup \left\{ \frac{1}{|\Gamma|} \sum_{x\in\Gamma} |\widetilde{C}_x|^2 : |\Gamma| = m \right\} \right] \quad \text{a.s.}$$

where the (inner) supremum is over all animals Γ of $\mathbb{L}$ having m vertices and containing the origin. Using the result of [10], the right side is a.s. finite so long as $|\widetilde{C}_x|^2$ has finite $(d+\epsilon)$th moment for some $\epsilon > 0$. The conclusion of Lemma 6 follows. □

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