CONSTRUCTIVE METHODS IN MARKOV CHAIN THEORY

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Abstract. General criteria are given for the ergodicity, recurrence, and transience of countable Markov chains. Conditions are given for the continuity, in the parameter, of the stationary probabilities of families of such chains. All criteria are closely connected with the well known criterion of Foster for the ergodicity of Markov chains and are given in terms of semimartingales. A complete classification is obtained for the random walks in $\mathbb{Z}_+^2$. The zero drift case inside $\mathbb{Z}_+^2$ and almost zero drift one-dimensional processes constitute new directions of the development.

Key words: Ergodicity, recurrence, transience, random walks, semimartingales, stability.

1. Introduction

The goal of this paper is the illustration of some constructive martingale criteria for classification of Markov chains and random walks in $\mathbb{Z}_+^2$. The basic method of obtaining these criteria consists of the construction of Lyapounov functions. These methods enable us to say when a Markov chain is ergodic, null recurrent or transient, i.e., to solve the problem of the complete classification. Some of ideas given in this paper for random walks in $\mathbb{Z}_+^2$ can be used for solving some analogous problems in $\mathbb{Z}_+^n$ ($n \geq 3$).

We will show martingale criteria, their equivalents for Markov chains and then some applications for random walks.

The notion of the *Lyapounov function* or *test function* is close to the well known Lyapounov function for ordinary differential equations and goes back to Foster 1953, as far as we know. Although his examples are now trivial, his ideas and criteria for ergodicity and for transience became basic for later extensions. There exist now many technical generalisations of these criteria, some of which we will give in this paper. Generalized Foster's criteria for ergodicity were given in Malyshev (1991). In Filonov (1989) a new martingale proof was proposed with an important extension to random times.

In Fayolle et al. (1993) these results were summarized and simplified. The first results for random $\mathbb{Z}_+^2$ walks appeared already 20 years ago (Malyshev 1972). In Malyshev et al. (1979), Menshikov (1974), Ignatyuk (1991) these ideas were applied to $\mathbb{Z}_+^n$ ($n \geq 3$). But the classification was obtained only for $\mathbb{Z}_+^3$ and $\mathbb{Z}_+^4$ and only for random walks with non zero drifts.

The consideration of the zero drift case required new general martingale criteria for Markov chains and a search for new Lyapounov functions. The zero drift case in $\mathbb{Z}_+^2$ and almost zero drift one-dimensional processes constitute new directions of the development, initiated by Lamperti (1960) 30 years ago. They are directly related to several works of R. Williams, S. R. S. Varadhan and others.

2. Criteria involving Semi-Martingales

Let $(\Omega, \mathcal{F}, P)$ be a given probability space and $\{\mathcal{F}_n, n \geq 0\}$ an increasing family of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \ldots \subset \mathcal{F}_n \ldots \subset \mathcal{F}$.

Let $\{S_i, i \geq 0\}$ be a sequence of real non-negative random variables, such that S_i is $\mathcal{F}_i$-measurable, $\forall i \geq 0$. Moreover, S_0 will be taken constant, which does not restrict the generality.

Denote by τ the $\mathcal{F}_n$-stopping time representing the epoch of the first entry into $[0, C]$, i.e., $\tau = \inf\{n \geq 1 : S_n \leq C\}$. Introduce the stopped sequence $\tilde{S}_n = S_{n \wedge \tau}$, where

$$n \wedge \tau = \begin{cases} n & \text{if } n \leq \tau \\ \tau & \text{if } n > \tau. \end{cases}$$

Theorem 2.1. *Assume that $S_0 > C$ and, for some $\epsilon > 0$ and all $n \geq 0$,*

$$\mathbf{E}(\tilde{S}_{n+1} \,|\, \mathcal{F}_{n-1}) \leq \tilde{S}_n - \epsilon \mathbf{1}(\tau > n) \quad a.s. \tag{1}$$

Then

$$\mathbf{E}(\tau) < \frac{S_0}{\epsilon} < \infty. \tag{2}$$

Here we used the classical notation for the indicator function

$$\mathbf{1}_A = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem is the generalization of Theorem 2.1, which is useful in the investigation of the ergodicity of random walks on $\mathbb{Z}_+^n$ $(n \geq 3)$.

Let $\{N_i, i \geq 1\}$ be a random sequence of positive integers which is *predictable*, i.e., N_{i+1} is $\mathcal{F}_i$-measurable, and such that $N_0 = 0$, $N_i - N_{i-1} \geq 1$ a.s., $\forall i \geq 1$. Introduce $Y_0 = S_0$, $Y_i = S_{N_i}$, $i \geq 1$, the stopping time

$$\sigma = \inf\{i \geq 1 : Y_i \leq C\},$$

and the stopped sequences $\tilde{Y}_i = Y_{i \wedge \sigma}$, $\tilde{N}_i = N_{i \wedge \sigma}$, $i \geq 1$.

Theorem 2.2. *Assume that for some $\epsilon > 0$ and all $n \geq 0$,*

$$\mathbf{E}(\tilde{Y}_{n+1} \,|\, \mathcal{F}_{N_n}) \leq \tilde{Y}_n - \epsilon(\tilde{N}_{n+1} - \tilde{N}_n) \quad a.s. \tag{3}$$

Then

$$\mathbf{E}(\tau) \leq \frac{S_0}{\epsilon}. \tag{4}$$

Theorem 2.3. *Suppose that, for $n \geq 1$ and some positive real M,*

$$\mathbf{E}(\tilde{S}_n \mid \mathcal{F}_{n-1}) \geq \tilde{S}_{n-1} \quad a.s., \tag{5}$$

$$\mathbf{E}[|\tilde{S}_n - \tilde{S}_{n-1}| \mid \mathcal{F}_{n-1}] \leq M \quad a.s. \tag{6}$$

Then $\mathbf{E}(\tau) = \infty$.

The proofs of these theorems can be found in Malyshev et al. (1979) and Fayolle et al. (1993).

3. Criteria for Countable Markov Chains

Let us consider a time homogeneous Markov chain $\mathcal{L}$ with a countable state space $\mathcal{A} = \{\alpha_i, i \geq 0\}$. $\mathcal{L}$ is supposed to be irreducible and aperiodic. The position of the chain at time n is ξ_n.

Theorem 3.1. *The Markov chain $\mathcal{L}$ is recurrent if and only if there exist a positive function $f(\alpha)$, $\alpha \in \mathcal{A}$, and a finite set A, such that*

$$\mathbf{E}[f(\xi_{m+1}) - f(\xi_m) \mid \xi_m = \alpha_i] \leq 0, \quad \forall \alpha_i \notin A,$$

and $f(\alpha_j) \to \infty$ as $j \to \infty$.

Theorem 3.2. *The Markov chain $\mathcal{L}$ is transient if and only if there exist a positive function $f(\alpha), \alpha \in \mathcal{A}$ and a set $A \in \mathcal{A}$ such that the following inequalities are fulfilled*

$$\mathbf{E}[f(\xi_{m+1}) - f(\xi_m) \mid \xi_m = \alpha_i] \leq 0, \quad \forall \alpha_i \notin A,$$

$$f(\alpha_k) < \inf_{\alpha_j \in A} f(\alpha_j), \quad \text{for at least one } \alpha_k \notin A.$$

Theorem 3.3. (Foster) *The Markov chain $\mathcal{L}$ is ergodic if and only if there exist a positive function $f(\alpha)$, $\alpha \in \mathcal{A}$, a number $\epsilon > 0$ and a finite set $A \in \mathcal{A}$ such that*

$$\mathbf{E}[f(\xi_{m+1}) - f(\xi_m) \mid \xi_m = \alpha_j] \leq -\epsilon, \quad \alpha_j \notin A;$$

$$\mathbf{E}[f(\xi_{m+1}) \mid \xi_m = \alpha_i] < \infty, \quad \alpha_i \in A.$$

The following theorem is a generalization of Foster's theorem, in the same way that Theorem 2.2 was a generalization of Theorem 2.1.

Theorem 3.4. *The Markov chain $\mathcal{L}$ is ergodic if and only if there exist a positive function $f(\alpha)$, $\alpha \in \mathcal{A}$, a number $\epsilon > 0$, a positive integer-valued function $k(\alpha)$, $\alpha \in \mathcal{A}$, and a finite set A, such that the following inequalities hold:*

$$\mathbf{E}f[(\xi_{m+k(\xi_m)}) - f(\xi_m) \mid \xi_m = \alpha_i] \leq -\epsilon k(\alpha_i), \quad \alpha_i \notin A; \tag{7}$$

$$\mathbf{E}[f(\xi_{m+k(\xi_m)} \mid \xi_m = \alpha_i] < \infty, \quad \alpha_i \in A. \tag{8}$$

Theorem 3.5. *For an irreducible Markov chain $\mathcal{L}$ to be non-ergodic, it is sufficient that there exist a function $f(\alpha), \alpha \in \mathcal{A}$, and constants C and d such that*

1. $\mathbf{E}[f(\xi_{m+1}) - f(\xi_m) \,|\, \xi_m = \alpha] \geq 0$, *for every m, all $\alpha \in \{f(\alpha) > C\}$, where the sets $\{\alpha : f(\alpha) > C\}$ and $\{\alpha : f(\alpha) \leq C\}$, are non empty;*
2. $\mathbf{E}[|f(\xi_{m+1}) - f(\xi_m)| \,|\, \xi_m = \alpha] \leq d$, *for every m, $\forall \alpha \in \mathcal{A}$.*

It is easy to understand that Theorem 2.1 is the martingale analogy of Foster's Theorem 3.3, Theorem 2.2 is the analogy of Theorem 3.4, and Theorem 2.3 is the analogy of Theorem 3.5.

The proofs of Theorems 3.1 and 3.2 are based on the martingale technique (see for example Foster (1953) and Fayolle et al. (1993)).

Theorems 3.1–3.5 are very useful in the classification of random walks on $\mathbb{Z}_+^2$ with non-zero drifts. In this case we make use of ϵ-linear Lyapounov functions and so the additional conditions on moments in these theorems are fulfilled. In the case with zero mean jumps inside $\mathbb{Z}_+^2$ we have to use some quadratic forms and functionals of quadratic forms. So condition 2 of Theorem 3.5 is not fulfilled. The following Theorem 3.6 helps us to solve this problem (see Fayolle et al. (1992) and Fayolle et al. (1993)).

Theorem 3.6. *For an irreducible Markov chain $\mathcal{L}$ to be null recurrent, it is sufficient that there exist two functions $f(x)$ and $\phi(x)$, $x \in \mathcal{A}$, and a finite subset $A \in \mathcal{A}$, such that the following conditions hold:*

1. $f(x) \geq 0,\ \phi(x) \geq 0,\ \forall x \in \mathcal{A}$;
2. *For some positive α, γ, with $1 < \alpha \leq 2$, $f(x) \leq \gamma(\phi(x))^\alpha$, $x \in \mathcal{A}$;*
3. $\phi(x_i) \to \infty$, *for $i \to \infty$,* $\sup_{x \notin A} f(x) > \sup_{x \in A} f(x)$;
4. (a) $\mathbf{E}[f(\xi_{n+1}) - f(\xi_n) \,|\, \xi_n = x] \geq 0$, $\forall x \notin A$;
   (b) $\mathbf{E}[\phi(\xi_{n+1}) - \phi(\xi_n) \,|\, \xi_n = x] \leq 0$, $\forall x \notin A$;
   (c) $\sup_{x \in \mathcal{A}} \mathbf{E}[|\phi(\xi_{n+1}) - \phi(\xi_n)|^\alpha \,|\, \xi_n = x] = C < \infty$.

4. Classification of Random Walks on $\mathbb{Z}_+^2$

Consider a discrete time homogeneous irreducible and aperiodic Markov chain $\mathcal{L} = \{\xi_n, n \geq 0\}$. Its state space is the lattice in the positive quarter plane $\mathbb{Z}_+^2 = \{(i,j) : i, j \geq 0\}$ and it satisfies the recursive equation

$$\xi_{n+1} = [\xi_n + \theta_{n+1}]^+,$$

where the distribution of θ_{n+1} depends only on the position of ξ_n in the following way (*maximal space homogeneity*)

$$p[\theta_{n+1} = (i,j) \,|\, \xi_n = (k,l)] = \begin{cases} p_{ij}, & \text{for } k, l \geq 1, \\ p'_{ij}, & \text{for } k \geq 1, l = 0, \\ p''_{ij}, & \text{for } k = 0, l \geq 1, \\ p^0_{ij}, & \text{for } k = l = 0. \end{cases}$$

Moreover we shall make, for the one-step transition probabilities, the following assumptions:

CONDITION A (LOWER BOUNDEDNESS)

$$\begin{cases} p_{ij} = 0, \text{ if } i < -1 \text{ or } j < -1; \\ p'_{ij} = 0, \text{ if } i < -1 \text{ or } j < 0; \\ p''_{ij} = 0, \text{ if } i < 0 \text{ or } j < -1. \end{cases}$$

CONDITION B (FIRST MOMENT CONDITION)

$$E[\| \theta_{n+1} \| \mid \xi_n = (k,l)] \le C < \infty, \quad \forall (k,l) \in \mathbb{Z}_+^2,$$

where $\|z\|$, $z \in \mathbb{Z}_+^2$, denotes the euclidean norm and C is an arbitrary but strictly positive number.

Notation: We shall use lower case greek letters $\alpha, \beta, \ldots$ to denote arbitrary points of $\mathbb{Z}_+^2$, and then $p_{\alpha\beta}$ will mean the one-step transition probabilities of the Markov chain $\mathcal{L}$ and $\alpha > 0$ means

$$\alpha_x > 0, \alpha_y > 0, \text{ for } \alpha = (\alpha_x, \alpha_y).$$

Also, from the homogeneity conditions, one can write

$$\theta_{n+1} = (\theta_x, \theta_y), \text{ given that } \xi_n = (x, y).$$

Define the vector

$$M(\alpha) = (M_x(\alpha), M_y(\alpha))$$

of the one-step mean jumps (drifts) from the point α. Setting

$$\alpha = (\alpha_x, \alpha_y), \quad \beta = (\beta_x, \beta_y),$$

we have

$$M_x(\alpha) = \sum_\beta p_{\alpha\beta}(\beta_x - \alpha_x), \quad M_y(\alpha) = \sum_\beta p_{\alpha\beta}(\beta_y - \alpha_y).$$

Condition B ensures the existence of $M(\alpha)$, for all $\alpha \in \mathbb{Z}_+^2$. By the homogeneity condition A, only four drift vectors are different from zero

$$M(\alpha) = \begin{cases} M, & \text{for } \alpha_x, \alpha_y > 0; \\ M', & \text{for } \alpha = (\alpha_x, 0), \ \alpha_x > 0; \\ M'', & \text{for } \alpha = (0, \alpha_y), \ \alpha_y > 0; \\ M^0, & \text{for } \alpha = (0,0). \end{cases}$$

The following theorem was proved by Malyshev 1972 under the additional condition: the jumps are bounded with probability 1. In Fayolle et al. (1993) it was proved for unbounded jumps.

Theorem 4.1. *Assume conditions A and B are satisfied.*
(a) *If $M_x < 0$, $M_y < 0$, then the Markov chain $\mathcal{L}$ is*
  (i) *ergodic if $M_x M'_y - M_y M'_x < 0$, and $M_y M''_x - M_x M''_y < 0$;*
  (ii) *non-ergodic if either $M_x M'_y - M_y M'_x \ge 0$ or $M_y M''_x - M_x M''_y \ge 0$.*

(b) *If* $M_x \geq 0$, $M_y < 0$, *then the Markov chain* $\mathcal{L}$ *is*
  (i) *ergodic if* $M_x M_y^{'} - M_y M_x^{'} < 0$*;*
  (ii) *transient if* $M_x M_y^{'} - M_y M_x^{'} > 0$.
(c) (*case symmetric to case* (b)) *If* $M_y \geq 0$, $M_x < 0$, *then the Markov chain* $\mathcal{L}$ *is*
  (i) *ergodic if* $M_y M_x^{''} - M_x M_y^{''} < 0$*;*
  (ii) *transient if* $M_y M_x^{''} - M_x M_y^{''} > 0$.
(d) *If* $M_x \geq 0$, $M_y \geq 0$, $M_x + M_y > 0$, *then the Markov chain* $\mathcal{L}$ *is transient.*

The proof of this theorem is based on the construction of ϵ-linear Lyapounov functions for which the conditions of Theorems 3.1–3.3, 3.5 are fulfilled correspondingly.

5. Zero Drift

We consider the Markov chain $\mathcal{L}$ which was introduced in the previous section, but satisfying the stronger

Condition C (Second moment condition)

$$E[\| \theta_{n+1} \|^2 \mid \xi_n = (k,l)] \leq B < \infty, \quad \forall (k,l) \in \mathbb{Z}_+^2.$$

Until recently nothing was precisely known for the case $M = 0$. In fact, this problem in many respects, is of a very different nature. In particular, the intuition does not provide us with any evidence that the random walk could be ergodic, when $M = 0$.

There is a crucial difference between the cases $M \neq 0$ and $M = 0$: indeed, the case $M \neq 0$ is in a sense locally *linear* and $M = 0$ is locally *quadratic*. The local second order effects are well caught by functional of quadratic Lyapounov functions.

For $M = 0$, we will obtain the ergodicity conditions in terms of the second moments and the covariance of the one-step jumps inside $\mathbb{Z}_+^2$,

$$\lambda_x = \sum_{ij} i^2 p_{ij}, \ \lambda_y = \sum_{ij} j^2 p_{ij}, \ R = \sum_{ij} ij p_{ij},$$

and of the angles ϕ_x, ϕ_y. Here ϕ_x is the angle between $M^{'}$ and the negative x-axis, ϕ_y is the angle between M'' and the negative y-axis. Thus, if $\phi_x \neq \frac{1}{2}\pi$ and $\phi_y \neq \frac{1}{2}\pi$, then

$$\tan\phi_x = -\frac{M_y'}{M_y'}, \quad \tan\phi_y = -\frac{M_x''}{M_y''}.$$

Theorem 5.1. (i) *If* $\phi_x \geq \frac{1}{2}\pi$ *or* $\phi_y \geq \frac{1}{2}\pi$, *then the random walk* $\mathcal{L}$ *is non-ergodic.* (ii) *If* $\phi_x < \frac{1}{2}\pi$ *and* $\phi_y < \frac{1}{2}\pi$, *then the random walk* $\mathcal{L}$ *is ergodic if*

$$\lambda_x \tan\phi_x + \lambda_y \tan\phi_y + 2R \equiv -\lambda_x \frac{M_y^{'}}{M_x^{'}} - \lambda_y \frac{M_x^{''}}{M_y^{''}} + 2R < 0, \tag{9}$$

and non-ergodic if

$$\lambda_x \tan\phi_x + \lambda_y \tan\phi_y + 2R > 0. \tag{10}$$

(iii) *If (10) holds together with* $\phi_x + \phi_y \leq \frac{1}{2}\pi$, *then the random walk is null recurrent.*

Remark 1. It follows easily from the statement of the theorem that the mean first entrance time of $\mathcal{L}$ to the boundary, when starting from some arbitrary point $\alpha > 0$ at finite distance, is finite (resp. infinite) if $R < 0$ (resp. $R > 0$), since in this case the vectors $M^{'}$ and $M^{''}$ can be properly chosen to satisfy (9) (resp. (10)).

Remark 2. It is clear from the formulation of the theorem that we do not consider the limiting situation

$$\lambda_x \tan\phi_x + \lambda_y \tan\phi_y + 2R = 0,$$

which would impose assumptions of third order.

The necessary and sufficient conditions for transience were proved recently (Asymont et al. 1993). The structure of Lyapounov functions was more difficult.

Theorem 5.2. *Let the following conditions hold for these random walks:*

$$M_x'' > 0 \text{ and } M_y' > 0.$$

Then the random walk is recurrent if

$$\lambda_x \cot\phi_y + \lambda_y \cot\phi_x + 2R \geq 0$$

(perhaps, with the exception of the case: $\lambda_x \cot\phi_y + R = \lambda_y \cot\phi_x + R = 0$*).*

The walk is transient if

$$\lambda_x \cot\phi_y + \lambda_y \cot\phi_x + 2R < 0.$$

Remark 1. If the assumption of Theorem 5.2 holds we have

$$\cot\phi_x = -\frac{M_x^{'}}{M_y^{'}}, \quad \cot\phi_y = -\frac{M_y^{''}}{M_x^{''}}.$$

Remark 2. We do not consider the cases $M_y^{'} = 0$ and $M_x^{''} = 0$ which are more trivial.

Remark 3. The case $\lambda_x \cot\phi_y + R = \lambda_y \cot\phi_x + R = 0$ cannot be classified by the method to be used. We think that the knowledge of the first moments on the axes is not sufficient to solve the classification problem and it is necessary to consider also the second moments on the axes.

6. Random Walks with 'Almost Zero' Mean Jumps

Let us consider a discrete-time Markov chain $\{X_n\}$, with state space $\mathbb{R}_+$. Let us define for $x \in \mathbb{R}_+$

$$\mu_i(x) = E[(X_{n+1} - X_n)^i \mid X_n = x].$$

Let $\tau_{x_0} \geq 0$ be the time at which the process first enters the interval $[0, A)$, when $X_0 = x_0$.

Theorem 6.1. *Suppose that for some* $\epsilon, p > 0$ *and all sufficiently large* x

$$2x\mu_1(x) + (2p-1)\mu_2(x) \leq -\epsilon,$$
$$\mu_2(x) = O(1), \quad \mu_{2p}(x) = o(x^{2p-2}).$$

Then for any sufficiently large A *and all* x_0 *we have* $E(\tau_{x_0}^p) < \infty$*.*

Theorem 6.2. *Suppose for some $\epsilon, p > 0$ that*

$$2x\mu_1(x) + (2p-1)\mu_2(x) \geq \epsilon > 0$$

for all large x; suppose also that $\mu_{2p}(x)$ exists and

$$\mu_1(x) = O(x^{-1}),\ \mu_2(x) = O(1),\ \mu_{2p}(x) = o(x^{2p-2}).$$

Then for all sufficiently large A, for every $x_0 > A$, we have $E(\tau_{x_0}^p) = \infty$.

These two theorems generalize the results of Lamperti (1960, 1963), which he proved only for integer positive p.

After some generalizations we can apply these two theorems to random walks on $\mathbb{Z}_+^2$ with zero mean jumps inside, and can obtain almost necessary and sufficient conditions when $E(\tau^p) < \infty$. Here $p > 0$, and τ is the time at which the random walk first enters some finite set. In the previous section we considered only the cases when $p = 1$ (ergodicity). But the construction of Lyapounov functions in the case of zero drift is more difficult. The paper in this region (Aspandiiarov et al. 1993) is in preparation.

7. Stability

The present section is devoted to the continuity of stationary distribution for families of homogeneous irreducible and aperiodic Markov chains. First we give a necessary and sufficient condition for this continuity, and also constructive sufficient conditions for the continuity of the stationary probabilities in terms of *test* functions. These results were obtained by Malyshev and Menshikov (1979).

Let us consider a family of homogeneous irreducible aperiodic Markov chain $\{\mathcal{L}^\nu\}$ with discrete time and countable set of states $\mathcal{A} = \{0, 1, \ldots\}$, for $\nu \in D$, where D is an open subset of the real line. By $p_{ij}(t, \nu)$ we denote the t-step transition probability from the point i to point j in $\mathcal{L}^\nu$. Everywhere in this section we assume that the $p_{ij}(1, \nu)$ are continuous in ν for all $\nu \in D$ and $i, j \in \mathcal{A}$. For the sake of brevity, we will write

$$p_{ij}(\nu) \stackrel{\text{def}}{\equiv} p_{ij}(1, \nu).$$

It is easy to prove that the $p_{ij}(t, \nu)$ are continuous functions of ν ($\nu \in D$) for every natural number t and all $i, j \in \mathcal{A}$.

On the set $\mathcal{A}$ let $\{\pi_j(\nu)\}$, $j \in \mathcal{A}$, $\nu \in D$, be a given family of distributions, where D is some open subset of the real line. We have

$$\sum_{j \in \mathcal{A}} \pi_j(\nu) = 1, \quad (\nu \in D).$$

Definition. The family of distributions $\{\pi_j(\nu)\}$, ($j \in \mathcal{A}$, $\nu \in D$), satisfies Condition (λ) at the point $\nu_0 \in D$ if, for any $\epsilon > 0$, there exist $\delta > 0$ and a finite set $B^\epsilon \subset \mathcal{A}$ such that

$$\sum_{j \in \mathcal{A} \setminus B^\epsilon} \pi_j(\nu) < \epsilon,$$

for all ν with $|\nu - \nu_0| < \delta$.

Let the chain $\mathcal{L}^\nu$ be ergodic for every ν belonging to some neighborhood $U_0 \subset D$ of zero.

Theorem 7.1. *The stationary probabilities $\pi_j(\nu)$ depend on ν continuously at $\nu = 0$ for all $j \in \mathcal{A}$ if and only if the family of distributions $\{\pi_j(\nu)\}$ satisfies Condition (λ) at the point $\nu = 0$.*

Before proving this, we make the following remark. Following Prohorov 1956 we form the metric space $D(\mathcal{A})$. To that end, we define the distance $L(\mu_1, \mu_2)$ between any two measures μ_1 and μ_2 on $\mathcal{A} = \{0, 1, \ldots, n\}$, so that convergence in the sense of this distance is equivalent to weak convergence of measures. The collection of all measures on $\mathcal{A}$ together with the function $L(\mu_1, \mu_2)$ forms the metric space $D(\mathcal{A})$. Still in accordance with Prohorov (1956) we introduce the following definition.

Definition. A set T of measures on $\mathcal{A}$ satisfies Condition (χ) if:

$(\chi 1)$ the values $\mu(\mathcal{A})$, $\mu \in T$, are bounded;

$(\chi 2)$ for any given $\epsilon > 0$, there exists a finite set k_ϵ of points such that $\mu(\mathcal{A} \setminus K_\epsilon) < \epsilon$, for all $\mu \in T$.

In Prohorov (1956) it is proved that for $T \subset D(\mathcal{A})$ to be compact, it is necessary and sufficient that Condition (χ) be satisfied. For $\{\pi_j(\nu)\}$, Condition (χ) obviously implies Condition (λ). Therefore, as a result of Theorem 7.1, we have the following theorem.

Theorem 7.2. *In order that the stationary probabilities $\pi_j(\nu)$ depend continuously on ν for all $j \in \mathcal{A}$ it is sufficient that the family $\{\pi_j(\nu)\}$ of distributions be compact in $D(\mathcal{A})$.*

The following theorem will be formulated in terms of *test* functions. The continuity of stationary probabilities of random walks in $\mathbb{Z}_+^n$ can be studied by means of the results of this section.

Assume that on the set $\mathcal{A} = \{0, 1, \ldots\}$ there is a given family $f^\nu = \{f_i^\nu\}$ for $i \in \mathcal{A}$, $\nu \in D$ of real functions, where

$$\inf_{i \in \mathcal{A}, \nu \in D} f_i^\nu \geq 0.$$

Theorem 7.3. *Assume that the following conditions are satisfied for some $\delta > 0$, some $\gamma > 1$, and finite non-empty set $B \subset \mathcal{A}$:*

1. $\sum_{j=0}^{\infty} p_{ij}(\nu) f_i^\nu - f_i^\nu < -\delta$, $i \notin B$, $\nu \in D$,
2. $\sup_{i \in B, \nu \in D} \sum_{j=0}^{\infty} p_{ij}(\nu) (f_j^\nu)^\gamma = \lambda_\gamma < \infty$,
3. $\sup_{i \in B, \nu \in D} \sum_{j=0}^{\infty} p_{ij}(\nu) |f_j^\nu - f_i^\nu|^\gamma = C_\gamma < \infty$,
4. $f_i^\nu \to \infty$ *uniformly in* $\nu \in D$ *as* $i \to \infty$.

Then the chains $\mathcal{L}^\nu$ are ergodic for every $\nu \in D$, and the stationary probabilities $\pi_j(\nu)$ are continuous in ν for $\nu \in D$ and $j \in \mathcal{A}$.

In Malyshev and Menshikov (1979) this theorem was applied to analysis of continuity of the family random walks in $\mathbb{Z}_+^n$. Analyticity conditions for stationary probabilities for general Markov chains and for random walks in $\mathbb{Z}_+^n$ are also stated in this paper in terms of Lyapounov functions.

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