# DIFFUSION IN RANDOM AND NON-LINEAR PDE'S

A. KUPIAINEN
*Mathematics Department*
*Helsinki University*
*P.O. Box 4*
*Helsinki 00014*
*Finland*

**Abstract.** We review Renormalization Group methods developed for the study of large time asymptotics of non-linear parabolic PDE's, random walks in random environments and certain non-Markovian random walks.

**Key words:** Renormalization, diffusion, nonlinear pde, random walk in random environment.

## 1. Introduction

Many of the most difficult problems in non-linear analysis and probability theory are ones with infinitely many degrees of freedom that are strongly interacting or dependent. Such problems are sometimes also approximately scale invariant and the proper understanding and use of this property can be the key to the solution.

Diffusive behaviour is a prime example of such phenomena. A quantity $u(x,t)$ (e.g., solution of a PDE or probability distribution of a random walk) is diffusive, if

$$u(x,t) \sim t^{-\alpha/2} f^* \left( \frac{x}{\sqrt{t}} \right). \tag{1}$$

Often such a limit law is *universal*: the $\alpha$ and $f^*$ will not change under suitable perturbations of the problem.

The connection between scaling and universality was first understood in the physics of quantum fields and critical phenomena using Renormalization Group theory. Here I would like to discuss some simple and less simple applications of these ideas to the mathematics of non-linear parabolic PDE's and disordered or non-Markovian random walks.

## 2. Non-Linear Parabolic PDE's

As a first and simplest example of the RG strategy (Barenblatt 1979, Goldenfeld et al. 1990, Bricmont et al. 1992), we will consider nonlinear heat equations of the type

$$\partial_t u = \Delta u + F(u, \nabla u, \nabla\nabla u). \tag{2}$$

where, $u(x,t) \in \mathbb{R}$ (or $\mathbb{R}^m$ in general), $x \in \mathbb{R}^d$, $t \in \mathbb{R}^+$. We want to prove global existence of solutions to (2) and study the possibility of diffusive asymptotics for the solution, i.e., whether, as $t \to \infty$ (1) holds and how it depends on the initial data and concrete form of (2).

The idea of the RG is to turn the question about asymptotics to a question about stability of fixed points of a dynamical system. Briefly, the idea is as follows.

The RG map $\mathcal{R}_L$, for $L > 1$ is defined in a suitable Banach space $B$ of initial data $f(x) = u(x, 1)$ and a suitable space of non-linearities $F$. These depend on the particular problem and we will give concrete examples below.

$\mathcal{R}_L$ consists of two operations. First, one solves (2) up to the finite time $L^2$, i.e., proves a local existence and uniqueness theorem . The second step consists of a scale transformation, modelled after the expected asymptotics (1): we scale the solution and correspondingly the equation; let

$$\mathcal{R}_L(f, F) = (f_L, F_L)$$

where

$$f_L(x) = L^{\alpha} u(Lx, L^2)$$

and

$$F_L(u, v, w) = L^{2+\alpha} F(L^{-\alpha} u, L^{-\alpha-1} v, L^{-\alpha-2} w).$$

This scaling assures the semigroup property

$$\mathcal{R}_{L^n} = \mathcal{R}_L^n$$

on a common domain.

Suppose now one is able to prove that $\mathcal{R}_L$ maps our space of data and equations (or some ball in it) to itself, and that further, that we get convergence in the appropriate norms

$$\mathcal{R}_L^n(f, F) \to (f^*, F^*)$$

as $n \to \infty$, where $(f^*, F^*)$ is a fixed point for $\mathcal{R}_L$. Then we have the asymptotics (1): $L^{n\alpha} u(L^n x, L^{2n}) \sim f^*(x)$, i.e., setting $t = L^{2n}$ (1) follows. Universality, i.e., independence of the asymptotics on $f$ and $F$ is then explained in terms of a dynamical systems picture: if $(f, F)$ lies on the stable manifold of the fixed point, all the corresponding equations and data have the same asymptotics.

As a trivial example, consider the linear equation $F = 0$, i.e., the heat equation. Anticipating the result, put $\alpha = d$, and get for the Fourier transform $\hat{f}$ of $f$

$$(\widehat{\mathcal{R}_L f})(k) = e^{(1-L^{-2})k^2} \hat{f}(k/L) \tag{3}$$

which converges for an integrable $f$ as $L \to \infty$ to $\hat{f}(0)\hat{f}^*$, a multiple of the Gaussian fixed point $f^*(x) = Ce^{-x^2/4}$. Note, how we in this case have a one parameter family of fixed points, and the location where we end on this curve depends on the initial data.

In this case, we also see explicitly the stability of the fixed point: since $\mathcal{R}$ is linear, it equals its derivative and this linear operator is, from (3)

$$\mathcal{R}_L = L^{\mathcal{L}_0}$$

where

$$\mathcal{L}_0 = \Delta + \frac{\xi}{2} \cdot \nabla + \frac{d}{2}; \tag{4}$$

$\mathcal{L}_0$ is conjugate to the Schrödinger operator of the harmonic oscillator

$$e^{\xi^2/8} \mathcal{L}_0 e^{-\xi^2/8} = \Delta - \frac{\xi^2}{16} + \frac{d}{4}. \tag{5}$$

Thus $\mathcal{L}_0$ is self-adjoint on its domain in $L^2(\mathbb{R}^d, d\mu)$, where $d\mu(\xi) = e^{\xi^2/4} d\xi$, and it has a pure point spectrum $\{-\frac{1}{2}m \mid m = 0, 1, \ldots\}$. We have one neutral direction, corresponding to the multiple of the fixed point, and the rest of the spectrum is contractive. This discreteness of the spectrum of the derivative of the RG at a fixed point will be true more generally: the scaling achieves this.

Linear diffusion equations can of course be understood in many ways, so let us now consider the non-linear equations, and start by studying the stability of the Gaussian fixed point $(Af^*, 0)$ in the space of the data and equations. For simplicity, consider just the simple non-linearity and $d = 1$ (for the general treatment, see Bricmont et al. 1992)

$$u_t = u_{xx} + \lambda u^p \tag{6}$$

(where $u^p \equiv |u|^{p-1}u$). With $\alpha = 1$ and $F = u^p$ we have $F_L = L^{3-p}F$, i.e., our Gaussian fixed point seems stable, if $p > 3$. This indeed is easy to prove, for a much larger class of $F$'s, for small data that decays suitably at infinity (Bricmont et al. to appear): the equation has asymptotics given by $Af^*$, where the only dependence on the data and $F$ is in the constant $A$.

For $p \leq 3$ the situation is more interesting. We need to take $\lambda \leq 0$, otherwise the solution blows up in finite time (see below). Then, for $p = 3$,

$$n^{\frac{1}{2}} \mathcal{R}_L^n(f, F) \to (Af^*, F)$$

i.e., we get a logarithmic correction:

$$u(x,t) \sim A(t \log t)^{-\frac{1}{2}} f^*(xt^{-\frac{1}{2}}).$$

Finally, for $p < 3$, the Gaussian fixed point is unstable in the $F$ direction, and we need to change $\alpha$. Equation (6) is invariant under the scaling $u \to u_L$ with

$$u_L(x,t) = L^{2/(p-1)} u(Lx, L^2 t)$$

which suggests setting

$$f_L(x) = L^{2/(p-1)} u(Lx, L^2)$$

and a corresponding definition of $F_L$. Then $\mathcal{R}_L(f_\gamma, F^*) = (f_\gamma, F^*)$ for $F^*(u) = -u^p$, where $f_\gamma$ is a one parameter family of *non-Gaussian* fixed points of the RG. These come as scale invariant solutions of the equation (6) (this is just the fixed point condition in the $f$ variable):

$$u(x,t) = t^{-1/(p-1)} f(xt^{-\frac{1}{2}}). \tag{7}$$

(6) implies that $f$ solves the ordinary differential equation

$$f^{''} + \frac{1}{2}xf^{'} + \frac{f}{p-1} - f^p = 0. \tag{8}$$

The theory of positive solutions of (8) has been developed in Brezis et al. (1986), Galaktionov et al. (1986), Kamin and Peletier (1985). The main result is that, for any $p > 1$, there exist smooth, everywhere positive solutions, $f_\gamma$, of (8) with $f_\gamma^{'}(0) = 0$ and $f_\gamma(0) = \gamma$ for $\gamma$ larger than a certain critical value $\gamma_p$ (but not too large). Actually, for $p < 3$, $\gamma_p > 0$, while $\gamma_p = 0$ for $p \geq 3$.

The existence of a critical $\gamma_p$ can be understood intuitively by viewing (8) as Newton's equation for a particle of mass one, whose 'position' as a function of 'time' is $f(x)$. The potential is then

$$U(f) = \frac{f^2}{2(p-1)} - \frac{f^{p+1}}{p+1}$$

and the 'friction term' $\frac{1}{2}xf^{'}$ depends on the 'time' $x$. Hence, if $f^{'}(0) = 0$ and $f(0) = \gamma$ is large enough, the time it takes to approach zero is long and, by then, the friction term has become sufficiently strong to prevent 'overshooting'. However, as $p$ increases, the potential becomes flatter and one therefore expects $\gamma_p$ to decrease with $p$.

These solutions have the asymptotics

$$f_\gamma(x) \sim |x|^{-2/(p-1)}$$

as $|x| \to \infty$ if $\gamma > \gamma_p$, while, for $\gamma = \gamma_p$, it decays at infinity as

$$f_{\gamma_p}(x) \sim |x|^{\{2/(p-1)\}-1} e^{-x^2/4}.$$

To study whether these solutions govern the long time asymptotics of (6), we need to study the derivative of the RG at these fixed points. This is given by the linear operator

$$d\mathcal{R}_L(f_\gamma) = L^{\mathcal{L}}$$

where

$$\mathcal{L} = \mathcal{L}_0 + V,$$

with

$$V(\xi) = -pf_\gamma^{p-1}(\xi) + \frac{1}{p-1} - \frac{1}{2}.$$

$\mathcal{L}$ is now the harmonic oscillator with a potential added that is bounded. It is now a non-trivial fact, that $e^{\tau(\mathcal{L}_0+V)}$ is contractive in a suitable Banach space (Bricmont and Kupiainen 1992). It turns out that all the fixed points $f_\gamma$ are stable under perturbations that fall off at infinity faster than $|x|^{-2/(p-1)}$, i.e., we have the asymptotics (1), with $f^* = f_\gamma$, and all the details of the initial data and the $F$ are erased in the limit.

The RG approach is not restricted to the study of the diffusive approach to zero as above: it can be used to study approach to more general attractors: universal stationary patterns (Bricmont and Kupiainen 1993a), moving fronts (Bricmont and Kupiainen 1993b) and the blowup of solutions of (6) with large data (Bricmont and Kupiainen 1993c). We will however now turn to applications to linear, but random equations.

## 3. Random Walk in Random Environment

Let us consider a stochastic version of the equations of the previous section, namely diffusion in a random environment. One model of this is given by a Markov process on the state space $\mathbb{Z}^d$, with generator

$$\Delta - \vec{b} \cdot \vec{\nabla}$$

where $\Delta$ is the Laplacian on $\mathbb{Z}^d$ and $\vec{b}$ is a random vector 'field'. For example, we could take $\vec{b}(x)$ for $x \in \mathbb{Z}^d$ to be i.i.d. with covariance

$$\mathrm{E}(b_\alpha(x) b_\beta(y)) = \epsilon^2 \delta_{\alpha\beta} \delta_{xy}.$$

Thus, the transition probablity $P(x,t)$ in time $t$ from origin to $x$ satisfies the Fokker–Planck equation

$$\partial_t P = \Delta P + \vec{\nabla} \cdot (\vec{b} P)$$

with the initial condition

$$P(x,0) = \delta_{x0}.$$

We would like to inquire again, whether a diffusive limit is attained for large times, e.g.,

$$\lim_{t\to\infty} t^{d/2} P(t^{\frac{1}{2}} x, t) = A e^{-dx^2/(Dt)} \quad \text{a.s.} \tag{9}$$

for some $D > 0$, or, whether the diffusion constant exists and is constant a.s.

$$\lim_{t\to\infty} t^{-1} \sum_x P(x,t) x^2 = D. \tag{10}$$

Since the equation is linear, we may write

$$P(x,t) = e^{t(\Delta + \nabla \cdot b)}(0,x) = p^t(0,x) \tag{11}$$

where $p(x,y) = e^{\Delta + \nabla \cdot b}(x,y)$ can be interpreted as a transition matrix for a random walk on $\mathbb{Z}^d$ and we may thus write (11) as

$$P(x,t,p) = \sum_{\omega : 0 \to x} \prod_{i=0}^{t-1} p(\omega(i), \omega(i+1)) \tag{12}$$

where we emphasized the dependence on the (random) transition matrix $p$. (12) is then a model of *random walk in random environment* (RWRE). Versions of this have been studied quite extensively both heuristically and rigorously (for references see Bricmont and Kupiainen (1991). In the asymmetric case, discussed below, the validity of (9) and (10) turns out to be subtle: Sinai showed (Sinai 1982) that for

$d = 1$ the walk is subdiffusive, i.e., $D = 0$ under very general assumptions on the $p$. For $d > 2$ diffusion is expected on heuristic grounds (Luck 1983, Derrida and Luck 1983, Fisher 1984) and we sketch below an argument leading to the proof of this (Bricmont and Kupiainen 1991).

The RG of Section 2 is now the following recursion for the transition probabilities: we solve for time $L^2$ and scale

$$(\mathcal{R}_L p)(x, y) = L^d p^{L^2}(Lx, Ly) = L^d \sum_{\omega} \prod_{i=0}^{L^2-1} p(\omega(i), \omega(i+1)) \tag{13}$$

with $\omega(0) = Lx$, $\omega(L^2) = Ly$, and then the following relation holds

$$\begin{aligned} P(x, t, p) &= L^{-d} P(L^{-1}x, L^{-2}t, \mathcal{R}_L p) \\ &\equiv L^{-d} \sum_{\omega:\omega(i) \in (L^{-1}\mathbb{Z})^d} L^{-d(T_1-1)} \prod_{i=0}^{t_1-1} (\mathcal{R}_L p)(\omega(i-1), \omega(i)) \end{aligned} \tag{14}$$

where $\omega(0) = 0$, $\omega(t_1) = L^{-1}x$, $t_1 = L^{-2}t$. The powers of $L$ in (13) are of course chosen because we expect the long time limit to be diffusive. The ones in (14) become very natural, provided we note that, since $\omega$ now are walks in $(L^{-1}\mathbb{Z})^d$, due to the scaling involved in $\mathcal{R}_L p$, it is natural to replace $\sum_\omega$ by an 'integral'.

The claims (9) and (10) can now be restated in terms of the map $\mathcal{R}$: given a random matrix $p$ as above, show that almost surely

$$\mathcal{R}_L^n p \to p^* \tag{15}$$

where $p^*$ is a Gaussian fixed point of $\mathcal{R}_L$ (see below), which are given by

$$p^*(x, y) = A e^{-d(x-y)^2/(2D)} \tag{16}$$

where $x, y \in \mathbb{R}^d$ and $A$ normalizes $p$ to a probability density $\int dy\, p^*(x, y) = 1$. The connection of (9) and (15) is the iteration of (14):

$$P(x, t, p) = L^{-nd} P(L^{-n}x, L^{-2n}t, \mathcal{R}_L^n p) \tag{17}$$

where the right-hand side refers to walks on $L^{-n}\mathbb{Z}^d$ with transition probability density $\mathcal{R}_L^n p \equiv p_n$.

The RG maps an 'environment' $p_n$ to another $p_{n+1}$. Thus $p_n$, $n \geq 1$, are random variables, being functions of $p$. The meaning of (10) is that *the rescaled long-time transition probability densities for our RWRE are given as the transition probability densities in the rescaled time of a RWRE with renormalized p's* (note that trivially $\mathcal{R}$ maps transition *probability densities* to transition *probability densities*: for all $n$ $\int dy\; p_n(x, y) = 1$ holds).

Universality would now be the claim, that apart from $D$, the renormalized diffusion constant, the limit (15) is independent on the $p$ we start with. In terms of RG, such $p$'s are on the stable manifold of the one parameter family of Gaussian fixed points (16). Note the important difference to Section 2: there the diffusion constant was not renormalized by the non-linearity, whereas the prefactor $A$ was.

Here $A$ is always fixed by the probability normalization, but the diffusion constant is renormalized.

Let us now study the stability of the non-random Gaussian fixed point (16), under random perturbations. We could study our original model (11), with $\epsilon$ small, but let us rather look at the more familiar nearest neighbour walk. Thus consider a $p$ of the form

$$p(x,y) = \begin{cases} \frac{1}{2d} + b(x,y) & |x-y| = 1 \\ 0 & |x-y| \neq 1. \end{cases} \tag{18}$$

For $b = 0$, (18) defines the simple random walk. For $p$ to be probabilities, we need $\sum_y b(x,y) = 0$ . The $\{b(x,y)\}_{x,y\in\mathbb{Z}^d}$ is a family of random variables of whose distribution we assume the following.

(i) We take $b(x,\cdot)$ and $b(x',\cdot)$ to be i.i.d. if $x \neq x'$ with mean zero $Eb(x,y) = 0$. Note, in particular, that $b(x,y)$ and $b(y,x)$ are independent: the environment is *asymmetric*.

(ii) We require the distribution of the $b$ to be invariant under rotations of the lattice.

(iii) We next require that $b$ in (18) is a 'small' perturbation in the following sense: the generating function of $b$ satisfies for $\epsilon$ small

$$Ee^{tb(x,y)} \le e^{t^2\epsilon^2}.$$

Thus, the variance is small.

(iv) Finally, we impose a condition on the probability that the $p(x,y)$'s are near zero:

$$\text{Prob}\left(p(x,y) \le \frac{1}{2d}e^{-N}\right) \le e^{-\Gamma N}, \quad N \in \mathbb{N}. \tag{19}$$

This is designed to avoid the walk getting 'trapped' in some region of $\mathbb{Z}^d$; see the discussion below: $\Gamma$ in (19) will be taken large.

With these assumptions and $d > 2$ one can prove (Bricmont and Kupiainen 1991) (9) and (10). In fact one can moreover prove weak convergence to Brownian motion almost surely. Here we just want to see how the $d > 2$ emerges from the linear RG analysis and how the trap condition (19) emerges.

For this, we compute (13) perturbatively in $b$:

$$\begin{aligned} p_1(x,y) = {} & L^d T^{L^2}(Lx - Ly) \\ & + L^d \textstyle\sum_{u,v} \sum_{t=0}^{L^2} T^t(Lx-u) T^{L^2-t}(Ly-v) b(u,v) + O(b^2). \end{aligned}$$

Here we denoted the $b$ independent part in (18) by $T$. The first term, call it $T_L$, in (20) is straightforward. It can be calculated by the Fourier transform as

$$\hat{T}_L(k) = \hat{T}(k/L)^{L^2}. \tag{20}$$

Since from (18) $\hat{T}(k) = d^{-1}\sum_{\alpha=1}^d \cos k_\alpha$ we have, as $L \to \infty$,

$$\hat{T}_L(k) \to \exp\left[-\frac{k^2}{2d}\right] = \hat{p}^*(k) \tag{21}$$

where $p^*$ is as in (16) with $D = 1$. This is the familiar approach to Gaussian fixed point we saw already when discussing the diffusion equation. If $b = 0$, this is the full RG and the argument is just a variation of the central limit theorem.

When $b$ is not zero, we will define an effective $b$ at each step of the iteration by dividing $p_n$ into a 'deterministic' and a 'random' part:

$$p_n(x,y) = T_n(x-y) + b_n(x,y)$$

where

$$\begin{aligned} T_n(x-y) &= Ep_n(x,y) \\ b_n(x,y) &= p_n(x,y) - Ep_n(x,y) \end{aligned}$$

and we have used the translation invariance of the distribution of $b$. Evidently, $\int dy T_n(y) = 1$ and $\int dy b_n(x,y) = 0 = Eb_n(x,y)$.

The aim is to show that $b_n \to 0$, so that eventually only the $T$ iteration survives. Obviously, at each scale, $b_n$ will modify the diffusion constant. If $b_n$ goes to zero sufficiently fast, we shall obtain a sequence of approximations $D_n$ converging to the true diffusion constant $D$.

Thus consider the second term in (20). This involves a sum of walks from $Lx$ to $u$ and from its nearest neighbour $v$ to $Ly$. Thus, since the total time is $L^2$, the main contribution comes when $|x-y|$ is $O(1)$ and $u$ within distance $L$ from $x$. Since the probability of hitting $u$ is $O(L^{-d})$ as is the one of hitting $y$, and there are $L^2$ times, the term linear in $b$ in (20) seems roughly to be $L^{2-d}\sum_u c(u)$ where the sum is over $L^d$ independent random variables of mean zero and covariance $\epsilon^2$. Thus this should have covariance $L^{4-d}\epsilon^2$, i.e., bigger than that of $b$ in $d = 3$. However, we have not yet used the crucial fact that $p$'s are transition probabilities, i.e., that $\sum_y b(x,y) = 0$. This implies that in (20) the second $T$ has effectively a derivative $\nabla_u$, that brings an extra $L^{-1}$. Hence altogether from the linear analysis we expect that

$$Eb_1(x,y)^2 \sim e^{-|x-y|} L^{2-d}\epsilon^2.$$

A more quantitative analysis confirms this expectation. Thus we expect the effective disorder $b_n$ to stay local (with exponential tails) and its variance to go to zero exponentially in $n$ if $d > 2$.

There are various hard problems, however, when one tries to extend this linear analysis to the full control of the RG. We shall discuss briefly only the main one, namely the problem of the traps.

Because of the asymmetry of the $p$'s, the environment may produce traps, i.e., regions that are easy to enter, but hard to exit. Indeed, the simplest trap consists of nearest neighbours $x$ and $y$ for which $p(x,y), p(y,x) \sim 1 - e^{-N}$ for $N$ large and thus $p(u,v) \sim e^{-N}$ for $u = x, y$, $v \neq x, y$. Thus if the walk at some time enters $x$ or $y$, it wants to stay there, since exiting is strongly suppressed. But, by the asymmetry, it is possible (and, indeed, likely for $\epsilon$ small) that $p(u,v) \sim (2d)^{-1}$ for $v = x, y$, $u \neq x, y$. Thus the trap $\{x, y\}$ is easy to enter and hard to exit. The time the walk wants to stay in the trap is $\sim e^N$ and thus diffusive behaviour is unlikely, if the density of the traps is higher than $e^{-\gamma N}$ , $\gamma$ sufficiently small. The condition (20) assures that the density is small and diffusion likely.

However, things are more complicated, since such traps can exist in all scales. One can expect this already from the linear analysis above. Recall, that $b_1 \sim L^{1-d} \sum b(x)$ and thus even if the covariance of $b_1$ contracts, there are large fluctuations: supposing even that $|b| \leq \epsilon$ a.s., $b_1$ can be as large as $L\epsilon$ and upon iteration $b_n$ can become trapping. Thus, in the control of the RG we need to show that such effective traps get more and more unlikely as $n$ increases (i.e., the effective $\Gamma_n$ analogous to (19) on the $n$th scale increases). This indeed is so (Bricmont and Kupiainen 1991).

## 4. Non-Markovian Walks

Let us finally discuss the application of the above RG ideas to non-Markovian random walks. We give three examples of this, namely the walk obtained by averaging over $p$ in (12), the true self-avoiding walk, and certain deterministic models for diffusion.

### 4.1. Average of RWRE

Let us consider the expression (12) for $P$ and take the expectation of it over the randomness

$$EP(x,t,p) = E\sum_{\omega}\prod(T+b) = \sum_{\omega} p(\omega) \tag{22}$$

where the weight of the walk is given by

$$p(\omega) = \sum_{I\subset[0,t-1]} T_{I^c} E b_I \tag{23}$$

and we used the notation $T_{I^c} = \prod_{s\in I^c} T(\omega(s),\omega(s+1))$ and $b_I$ similarly.

By independence,

$$Eb_I = \prod_{\alpha} U_{I_\alpha}$$

where we wrote $\bigcup_{s\in I}\omega(s) = \bigcup_\alpha x_\alpha$ and $I_\alpha = \omega^{-1}(x_\alpha)\cap I$ i.e., we divide the times $I$ into sets $I_\alpha$ where $\omega(s) = x_\alpha$. Therefore we get the following representation:

$$p(\omega) = \sum_{I_\alpha} T_{I^c}\prod_{\alpha} V_{I_\alpha} \tag{24}$$

where $\{I_\alpha\}$ is a family of subsets of $[1,t]$, $|I_\alpha| \geq 2$ and $V_{I_\alpha} = 0$ unless all the $\omega(s)$ for $s\in I_\alpha$ are near each other (actually the same point). One should think of (24) as a general representation for the weight of a non-Markovian random walk. It consists of the Markovian part $T$ and interactions among the times when the walk crosses itself.

(24) turns out to be a form invariant under the RG. Before we discuss this, we want to consider two other examples that lead to the same representation.

### 4.2. The True Self-avoiding Walk

Here

$$p(\omega) = \prod_{s=1}^{t} p_s(\omega) \tag{25}$$

where $\omega$ is a nearest neighbour walk and

$$p_s(\omega) = \frac{e^{-\lambda n_s(\omega(s+1);\omega)}}{\sum_{|y-\omega(s)|=1} e^{-\lambda n_s(y;\omega)}} \tag{26}$$

and $n_s(x,\omega) = |\{s' \le s \,|\, \omega(s') = x\}|$. Thus the walk does not want to enter a region which it has visited before. This true self-avoiding walk is a random walk, unlike the standard self-avoiding or self-suppressing walk: we have $\sum_{\omega(t+1)} p_t(\omega) = 1$.

This model was perturbatively studied in Amit et al. (1983) and Peliti (1984), who concluded that it should, for small disorder, be diffusive for $d > 2$. For $d = 1$ it was recently proved (Toth 1993) that the mean square distance scales as $t^{\frac{2}{3}}$, i.e., the walk is super-diffusive. Here we are interested in developing methods to prove diffusive behaviour in $d > 2$ .

Writing now $p_s(\omega) = (2d)^{-1} + q_s(\omega)$ (we will consider $\lambda$ small in (26)), it is not hard to expand

$$q_s(\omega) = \sum_{I \subset [0,s]} b(I) \tag{27}$$

where $b(I)$ depends only on $\omega(s')$, $s' \in I$, and is non-zero only if $|\omega(s) - \omega(s')| = 1$ for all $s' \in I$. Therefore, expanding now $p(\omega) = \prod(T + q)$ in powers of $q$ inserting (27) we end up having again a representation (24), where $V_I$ is again localized near $\omega(I)$.

### 4.3. Deterministic Diffusion

We consider a lattice model for the Lorentz gas, where a particle is moving on $\mathbb{Z}^d$ in the presence of random configuration of scatterers. At discrete times $t \in \mathbb{Z}^+$ the particle has a position $\omega(t)$ and a velocity $v(t)$, with $|v(t)| = 1$. At the following time, $\omega(t+1) = \omega(t) + v(t)$ and $v(t+1)$ is determined by the presence and type of scatterer at $\omega(t)$.

We describe the scatterer by giving a function $p : V \times V \to \{0, 1\}$ with $\sum_{v'} p(v, v')$ $= 1$ (i.e., given $v$, there is a unique $v'$). Here $V$ is the set of unit vectors in $\mathbb{Z}^d$. We require $p(v, -v) = 0$. This means there is no back scattering: if there is a non-zero probability for the back scattering, it is easy to see, that every orbit is periodic (Figotin 1992): eventually the particle will encounter such a scatterer and then retrace its steps until it encounters another one and for subsequent times the motion will repeat itself. We require furthermore that $p(v, v') = p(-v', -v)$, which means the motion is reversible. Call $\mathcal{S}$ the the set of such $p$'s. Now, let $\{p_x\}_{x \in \mathbb{Z}^d}$ be i.i.d. random variables, taking values in $\mathcal{S}$. We require that the distribution of $p_x$'s is isotropic (i.e., that there is no preferred direction) and moreover

$$\mathrm{Prob}(p_x = \mathrm{id}) = 1 - \epsilon$$

for small $\epsilon$ where $\mathrm{id}(v, v') = \delta_{vv'}$ is the absence of scattering. This means that we have a small density of scatterers.

Given now such a set of $p_x$'s together with the initial velocity $v(0)$ and starting point, say origin, the motion is completely determined. The 'probability' for a path (walk) $\omega : [0, t] \to \mathbb{Z}^d$ with $\omega(0) = 0$ is

$$P(\omega) = \prod_{s=1}^{t} p_{\omega(s)}(v(s-1), v(s)).$$

Of course this takes value 1 only for one $\omega$, namely the actual trajectory.

One would now like to prove that, with high probability the motion is diffusive. Note, that there is always a non-zero density of closed orbits in environments that we have described. In fact, for $d = 2$ and $\epsilon = 1$ every orbit is periodic (Bunimovich and Troubetskoy 1992). However, if $d > 2$, we expect diffusion to occur in the complement of these orbits (note also, that one cannot enter a closed orbit from outside). An easier problem is to study the diffusion if we average over the scatterers: try to show, that

$$\lim t^{-1} E \sum_{\omega} P(\omega)\omega(t)^2 \neq 0. \tag{28}$$

Thus, let us consider the non-Markovian random walk with probabilities

$$p(\omega) = EP(\omega). \tag{29}$$

Writing

$$p_x(v, v') = T(v, v') + b_x(v, v') \tag{30}$$

with $T(v - v') = Ep_x(v, v')$ , inserting (30) to (29) and expanding in powers of $b$ (note that $Eb^2 = O(\epsilon)$) we again end up with an expansion of the form (24).

### 4.4. The Renormalization Group

The RG that we apply to (24) consists now of 'blocking' in time as before and in space. Thus, given a walk $\omega'$ on $L^{-1}\mathbb{Z}^d$, we set

$$p'(\omega') = \sum_{\omega \to \omega'} p(\omega) \tag{31}$$

with the constraint on the sum that $\omega(L^2 s) = L\omega'(s)$. To get (31) back to the form (24), we write, given $I_\alpha$ in (24), $[L^{-2} \bigcup_\alpha I_\alpha] = \bigcup_\alpha J_\alpha$, where $[\cdot]$ means the integer part and $J_\alpha$ are unions of $[L^{-2} I_\alpha]$'s, where $[L^{-2} I_\alpha]$ and $[L^{-2} I_\beta]$ are in the same $J_\gamma$ if they intersect. Thus we end up with

$$p'(\omega') = \sum_{J_\alpha, Z_\alpha} T_{L, I^c} \prod_\alpha V'_{J_\alpha} \tag{32}$$

where $V'$ are given by the formula (24) with above constraints and $T_L$ by (20)–(21). The only difference with (24) is now, that $|J_\alpha|$ can be 1. Such $V'_J$ are no more interactions coupling several times, since they only depend on $\omega'_s$, $\omega'_{s+1}$ where $J = \{s\}$. These $W$'s thus renormalize the $T_L$. We set

$$T' = T_L + V'_{\{s\}}. \tag{33}$$

The picture we would now like to establish is the following. The $V_I$ are irrelevant under the RG iteration, if $|I| > 1$. We expect $|V'_I| \sim L^{2-d} |V_I|$. The reason for this is similar to the one behind our computation of the variance of $b_1$ above (actually, $Eb_1^2$ is just $V_I$ for $|I| = 2$). The reason for the exponent is the fact that we are dealing with probabilities that sum to one. For the regular self-avoiding walk that fits into the present scheme we would find the exponent $4 - d$.

Thus $T$ above gets renormalized less and less as $n \to \infty$ and we end up with the representation (24) with no $V$ and $T = p^*$, i.e., the fixed point Markovian walk.

There are, however, again hard problems to deal with. There are expanding directions in our RG flow: those correspond to interactions $V_I$ for $I$ a union of many small components. These correspond to the walk returning to the same region during many disjoint time intervals. Upon iteration, these collect walks returning repeatedly to a $L^n$ cube. These $V$ do not contract under the RG before a scale is reached where the blocked time intervals form a connected set. Thus we need again to keep track of large 'fields', like the traps in the RWRE. It also turns out, that one has to localize the $V$'s not only in time but also in space.

The picture presented here is work in progress.

## References

Amit, D. J., Parisi, G., and Peliti, L. (1983). *The Physical Review* **B27**, 1635.

Barenblatt, G. I. (1979). *Similarity, Self-similarity and Intermediate Asymptotics*, Consultants Bureau, New York.

Brezis, H., Peletier L. A., and Terman D. (1986). A very singular solution of the heat equation with absorption. *Archive for Rational Mechanics and Analysis* **95**, 185–209.

Bricmont, J. and Kupiainen, A. (1991). Random walk in asymmetric random environment, *Communications in Mathematical Physics* **142**, 345–420.

Bricmont, J. and Kupiainen, A. (1992). Renormalization group and the Ginzburg–Landau equation. *Communications in Mathematical Physics* **150**, 193–208.

Bricmont, J. and Kupiainen, A. (1993a). Stability of moving fronts in the Ginzburg–Landau equation. *Communications in Mathematical Physics*, to appear.

Bricmont, J. and Kupiainen, A. (1993b). Universality in blow-up for nonlinear heat equations. Preprint, IHES.

Bricmont, J. and Kupiainen, A. (1993c). Stable non-Gaussian diffusive profiles. Preprint, IHES.

Bricmont, J., Kupiainen, A., and Lin, G. (1992). Renormalization group and asymptotics of solutions of nonlinear parabolic equations. *Communications in Pure and Applied Mathematics*, to appear.

Bunimovich, L. T. and Troubetzkoy, S. E. (1992). Recurrence properties of Lorentz lattice gas cellular automata. Preprint.

Derrida, D. and Luck, J. M. (1983). Diffusion on a random lattice: weak-disorder expansion in arbitrary dimension. *The Physical Review* **B 28**, 7183.

Figotin, A. (1992). The localization properties of a random stationary flow on a lattice. *Journal of Statistical Physics* **66**, 1599.

Fisher, D. (1984). Random walks in random environments. *The Physical Review* **A 30**, 960.

Galaktionov, V. A., Kurdyumov, S. P., and Samarskii, A. A. (1986). On asymptotic 'eigenfunctions' of the Cauchy problem for a nonlinear parabolic equation. *Mathematics of the USSR–Sbornik* **54**, 421–455.

Goldenfeld, N., Martin, O., Oono, Y., and Lin, F. (1990). Anomalous dimensions and the renormalisation group in a nonlinear diffusion process, *Physical Review Letters* **64**, 1361–1364.

Kamin, S. and Peletier, L. A. (1985). Large time behaviour of solutions of the heat equation with absorption. *Annali della Scuola Normale Superiore di Pisa* **12**, 393–408.

Kong, X. P. and Cohen, E. G. D. (1991). *Physica* **D47**, 9–18.

Luck, J. M. (1983). Diffusion in a random medium: A renormalization group approach. *Nuclear Physics* **B 225**, 169.

Peliti, L. (1984). Self-avoiding walks. *Physics Reports* **103**, 225–231.

Sinai, Y. G. (1982). Limiting behavior of a one-dimensional random walk in a random medium. *Theory of Probability and its Applications* **27**, 256.

Toth, B. (1993). Limit theorem for the local time of the bond-true self-avoiding walk on $Z$. Preprint.