MULTISCALE ANALYSIS IN DISORDERED SYSTEMS: PERCOLATION AND CONTACT PROCESS IN A RANDOM ENVIRONMENT

ABEL KLEIN*
Department of Mathematics
University of California
Irvine, CA 92717
U.S.A.

Abstract. Multiscale analysis is a technique used in the study of disordered systems in the presence of phenomena similar to Griffiths singularities. In this article we illustrate the use of a multiscale analysis by applying it to a very simple model: percolation in a random environment. We also describe the application of this technique to continuous time percolation and contact processes in random environments.

Key words: Disordered systems, random enviroment, Griffiths singularities, percolation, contact process.

1. Introduction

Multiscale analysis is a technique used in the study of disordered systems in the presence of phenomena similar to Griffiths singularities. Typically, the corresponding homogeneous system exhibits two different types of behavior (phases) that can be obtained by varying one or more parameters: an ordered phase characterized by the existence of long range order in the system, and a localized phase characterized by the decay of some correlation function. In the presence of disorder, each phase may manifest itself in infinitely many arbitrarily large regions in which the system's parameters will be on the range characteristic of these phases. As a consequence, even if the system is not ordered as a whole, there may be finite but arbitrarily large regions inside which the system is strongly correlated, giving raise to phenomena similar to Griffiths singularities (Griffiths 1969).

Such phenomena appears in random Schrödinger operators, in models of classical and quantum statistical mechanics with random parameters, percolation and contact processes in random environments, etc. In all these models the standard tools to obtain the exponential decay of correlation functions are typically expansions, which fail to converge due to the existence of those arbitrarily large regions inside which the system is strongly correlated (so terms may be large, there may be small divisors, etc.).

One way of dealing with such singular regions is to modify the desired expansion, by means of a multiscale analysis which provides good estimates on the typical distances between singular regions. An expansion is then performed outside the

* Partially supported by the NSF under grant DMS 92-08029.

singular regions, the contribution of the singular regions must be estimated, and it must be shown that the decay obtained outside the singular regions dominates their contribution.

Such a scheme was first used by Fröhlich and Spencer (1983) and Fröhlich, Martinelli, Scoppola, and Spencer (1985) to prove localization for random Schrödinger operators. In this approach singular regions are defined for each realization of the disorder, probabilistic estimates are established for the geometrical layout of the singular regions and for the behavior of the system inside the singular regions, and a modified expansion is performed in each realization with typical geometry and behavior.

A simpler multiscale analysis was introduced by von Dreifus (1987) and Spencer (1988), and used by von Dreifus and Klein (1989, 1991), also to prove localization for random Schrodinger operators. In this multiscale analysis a simple geometrical layout for the singular regions is predetermined and the probabilistic estimates are simple and elementary. These ideas were used by Campanino and Klein (1991), Campanino, Klein, and Perez (1991), Klein and Perez (1992) and Klein (1992, 1993) to study quantum spin systems, continuous time percolation and contact processes in random environments.

In this article we will first illustrate the use of such a multiscale analysis by applying it to a very simple model: percolation in a random environment. After that we will describe the application of this technique to continuous time percolation and contact processes in random environments (Klein 1992).

2. Percolation in a Random Environment

2.1. The Model and Results

Let us consider a bond percolation model on $\mathbb{Z}^d$ (e.g., Grimmett 1989); the occupation probability of a bond (or edge) $\langle x, y\rangle$ being denoted by $p_{x,y}$. The collection of bonds with endpoints on a subset Λ of $\mathbb{Z}^d$ will be denoted by $\mathbb{B}(\Lambda)$, we will call $\mathbf{p} = \{p_{x,y} : \langle x, y\rangle \in \mathbb{B}(\mathbb{Z}^d)\}$ the *environment*. The corresponding percolation probability measure will be denoted by $\mathcal{P}_{\mathbf{p}}$.

If $x, y \in \Lambda \subset \mathbb{Z}^d$, we will write $x \overset{\Lambda}{\longleftrightarrow} y$ for the event that x is connected to y by a path of occupied bonds in Λ, here and elsewhere we omit Λ in case $\Lambda = \mathbb{Z}^d$. The *connectivity function* in the region Λ is defined by

$$G_{\mathbf{p},\Lambda}(x, y) = \mathcal{P}_{\mathbf{p}}(\{x \overset{\Lambda}{\longleftrightarrow} y\}).$$

By a *cluster* we will mean a maximal collection of sites in $\mathbb{Z}^d$ all connected to each other by occupied bonds.

The environment is *homogeneous* if all $p_{x,y} = p$, in which case we will write $\mathcal{P}_p$ and $G_p(x, y)$ for the the corresponding percolation probability measure and connectivity function. In this case, there exists a critical probability $p_c = p_c(d)$, with $p_c(1) = 1$ and $0 < p_c(d) < 1$ for $d \geq 2$, such that for $p > p_c$ we have percolation, i.e., existence of an infinite cluster with non-zero probability), while for $p < p_c$ there is no percolation and the connectivity function decays exponentially with the distance:

there exist $m_p > 0$ and $C_p < \infty$ such that

$$G_p(x, y) \leq C_p e^{-m_p|x-y|} \tag{1}$$

for all x, $y \in \mathbb{Z}^d$ (e.g., Grimmett 1989). An application of the FKG inequality shows that (1) implies the same estimate with $C_p = 1$.

But things are not so simple in an inhomogeneous environment $\mathbf{p}$. Let $p_+ = \sup_{\langle x,y\rangle} p_{x,y}$ and $p_- = \inf_{\langle x,y\rangle} p_{x,y}$. Clearly, if $p_- > p_c$ we have percolation, and if $p_+ < p_c$ we have exponential decay of the connectivity function and no percolation. But if $p_- < p_c < p_+$ the situation becomes more complicated; in this case we need more information about the environment.

We now turn to random environments, and take the $p_{x,y}$, $\langle x, y\rangle \in \mathbb{B}(\mathbb{Z}^d)$, to be independent identically distributed random variables. We will denote the underlying probability measure and expectation by $\mathbf{P}$ and $\mathbf{E}$. We now define p_+ and p_- to be the supremum and infimum of the essential range of the random variable $p_{x,y}$, respectively. If $p_- < p_c < p_+$, the system will develop phenomena similar to Griffiths singularities. Indeed, with probability one (with respect to $\mathbf{P}$), if for any length scale L we look for hypercubes in $\mathbb{Z}^d$ with sides of length L, we will always find infinitely many in which all $p_{x,y} < p_c$, and infinitely many in which all $p_{x,y} > p_c$. In the latter the system will try to form large clusters, while in former the clusters will like to be small relative to L for L sufficiently large.

But if $p_1 < p_c$ and $\mathbf{P}(p_{x,y} \leq p_1)$ is sufficiently close to one, the infinitely many regions in which the system would like to form large clusters would be typically located at very large distances from each other, so we should expect that we would not have percolation and that the connectivity function should decay exponentially with the distance. But the usual expansions will not converge, so we will perform a modified expansion using a multiscale analysis. We will prove:

THEOREM 2.1. *Let $p_1 < p_c$, $q = q(p_1) = \mathbf{P}(p_{x,y} \leq p_1)$. There exists $q_1 = q_1(p_1) < 1$, such that if $q > q_1$ we can pick $m(q) > 0$, with $\lim_{q \nearrow 1} m(q) = m_{p_1}$, for which we have for $\mathbf{P}$-almost every environment $\mathbf{p}$ and every $x \in \mathbb{Z}^d$ that*

$$G_{\mathbf{p}}(x, y) \leq C_{x,\mathbf{p}} e^{-m(q)|x-y|} \tag{2}$$

for all $y \in \mathbb{Z}^d$, with $C_{x,\mathbf{p}} < \infty$.

Remark. For bond percolation we do not need a multiscale analysis, since

$$\mathbf{E}\mathcal{P}_{\mathbf{p}} = \mathcal{P}_{\overline{p}}, \tag{3}$$

where $\overline{p} = \mathbf{E}p_{x,y}$. Thus, if $\overline{p} < p_c$, we obtain (1) with $p = \overline{p}$. It follows, by an application of Chebyshev's inequality and the Borel–Cantelli Lemma, that for any $m < m_{\overline{p}}$ and every $x \in \mathbb{Z}^d$ we have (2) for all $y \in \mathbb{Z}^d$, with $C_{x,\mathbf{p}} = C_{x,\mathbf{p}}(m) < \infty$ for $\mathbf{P}$-almost every $\mathbf{p}$. In general, (3) (or something like it) does not hold. But since bond percolation is probably the simplest model in which we can perform a modified expansion using a multiscale analysis, we will use it for pedagogical reasons.

2.2. The Expansion for Homogeneous Environments

Before we can perform a modified expansion, we need an expansion to modify. The one we will use is based on the so called *Hammersley–Simon–Lieb inequality*: Let $\Lambda \subset \Lambda' \subset \mathbb{Z}^d$, $x \in \Lambda$ and $y \in \Lambda' \setminus \Lambda$. In any environment $\mathbf{p}$ we have (from now on the subscript $\mathbf{p}$ will be omitted, unless needed for emphasis):

$$G_{\Lambda'}(x,y) \le \sum_{z \in \partial\Lambda} G_\Lambda(x,z)\, G_{\Lambda'}(z,y), \tag{4}$$

where

$$\partial\Lambda = \{z \in \Lambda : \langle z,w\rangle \in \mathbb{B}(\mathbb{Z}^d) \text{ for some } w \notin \Lambda\}.$$

We will actually use the following consequence of (4), which we will call the HSL inequality: If $x \in \Lambda \subset \Lambda' \subset \mathbb{Z}^d$, $y \in \Lambda' \setminus \Lambda$, we have

$$G_{\Lambda'}(x,y) \le G_\Lambda(x,\partial)\, G_{\Lambda'}(z_1,y) \tag{5}$$

for some $z_1 \in \partial\Lambda$, where $G_\Lambda(x,\partial) = \sum_{z\in\partial\Lambda} G_\Lambda(x,z)$.

Given $L > 0$, $x \in \mathbb{Z}^d$, we set $|x| = |x|_\infty$ and

$$\Lambda_L(x) = \{y \in \mathbb{Z}^d : |y-x| \le L\}.$$

LEMMA 2.2. *Let us consider percolation in a homogeneous environment. Suppose that at some length scale L we have*

$$G_{\Lambda_L(0)}(0,\partial) \le \rho < 1. \tag{6}$$

Then for all x, $y \in \mathbb{Z}^d$ we have

$$G(x,y) \le Ce^{-m|x-y|} \tag{7}$$

with $m = -L^{-1}\log\rho$ and $C = e^{mL}$.

Proof. By the homogeneity of the model, (6) holds with any x substituted for 0. So let x, $y \in \mathbb{Z}^d$, with $|x-y| \ge L$. For any positive integer $n < L^{-1}|x-y|$, we can apply (5) and (6) n times to obtain

$$G(x,y) \le \rho^n G(z,y) \tag{8}$$

for some $z \in \mathbb{Z}^d$. Since $G(z,y) \le 1$, we get

$$G(x,y) \le \rho^{(|x-y|/L)-1}, \tag{9}$$

which is just (7). □

PROPOSITION 2.3. *Let us consider percolation in a homogeneous environment. Then there exists $\widehat{p} > 0$ such that (1) holds for all $p < \widehat{p}$.*

Proof. For any scale L, we have

$$\lim_{p\to 0} G_{\Lambda_L(0)}(0,\partial) = 0. \tag{10}$$

If we let $\widehat{p}(L) = \sup\{p : G_{\Lambda_L(0)}(0,\partial) < 1\}$ and take $\widehat{p} = \sup_L \widehat{p}(L)$, the proposition follows from the previous lemma. (It is easy to see that $\widehat{p} \ge (2d-1)^{-1}$.) □

2.3. Geometrical Layout and Probabilistic Estimates

We start the multiscale analysis by establishing *a priori* a geometrical layout for the singular regions with appropriate probabilistic estimates.

Let L_k, $k = 0, 1, 2, \ldots$, be an increasing sequence of length scales and let p_1 be as in Theorem 2.1. In a given environment $\mathbf{p}$ we will say that a site $x \in \mathbb{Z}^d$ is 0-*regular* if $p_{y,z} \leq p_1$ for all $\langle y, z\rangle \in \mathbb{B}(\Lambda_{L_0}(x))$.

We now proceed inductively. For $k = 1, 2, \ldots$ a site $x \in \mathbb{Z}^d$ is k-*regular* if there are no two $(k-1)$-*singular* (i.e., not $(k-1)$-regular) sites $y, z \in \Lambda_{L_k}(x)$ with $|y-z| \geq L_{k-1}$. We call $\Lambda \subset \mathbb{Z}^d$ a k-*regular region* if it contains no k-singular sites, otherwise we call Λ a k-*singular region.*

From now on we consider a random environment and introduce probabilities P_k such that $\mathbf{P}(0 \text{ is } k\text{-singular}) \leq P_k$ for all $k = 0, 1, 2, \ldots$. In this case we will also have $\mathbf{P}(x \text{ is } k\text{-singular}) \leq P_k$ for all x by the translation invariance of $\mathbf{P}$. A simple way to do it is the following scheme due to von Dreifus (1987).

LEMMA 2.4. *Let $L_0 > 0$, $r > 2d$ and $1 < \alpha < 2r/(r + 2d)$. Set $L_{k+1} = L_k^{\alpha}$ and $P_k = L_k^{-r}$ for $k = 0, 1, 2, \ldots$. Suppose we have*

$$\mathbf{P}(0 \text{ is } 0\text{-singular}) \leq P_0. \tag{11}$$

Then we also have

$$\mathbf{P}(0 \text{ is } k\text{-singular}) \leq P_k \tag{12}$$

for all $k = 1, 2, \ldots$ if L_0 is large enough, how large depending only on d, α and r.

Proof. We proceed by induction on k. Suppose (12) holds. Then

$$\mathbf{P}(0 \text{ is } (k+1)\text{-singular}) \leq (2L_{k+1} + 1)^{2d} P_k^2 \tag{13}$$

since the events $\{x \text{ is } k\text{-singular}\}$ and $\{y \text{ is } k\text{-singular}\}$ are independent if $|x - y| \geq L_k$, and the number of such pairs is smaller than $(2L_{k+1} + 1)^{2d}$. But the right hand side is easily seen to be dominated by P_{k+1} if L_0 is large enough. $\square$

From now on we will assume the setup given in Lemma 2.4. The following consequence is of particular interest, since it gives the typical geometric layout of singular sets.

PROPOSITION 2.5. *Suppose (12) holds for all $k = 0, 1, 2, \ldots$. Then for any $b \geq 1$ and $x \in \mathbb{Z}^d$ let*

$$\overline{k}_b(x) = \sup\{k : \Lambda_{bL_{k+1}}(x) \text{ is a } k\text{-singular region}\}. \tag{14}$$

Then $\overline{k}_b(x) < \infty$ with probability one.

Proof. It follows from Lemma 2.4 that

$$\mathbf{P}\Big(\Lambda_{bL_{k+1}}(x) \text{ is a } k\text{-singular region}\Big) \leq (2bL_{k+1} + 1)^d P_k \leq \frac{(3b)^d}{L_k^{r-\alpha d}}. \tag{15}$$

Since $r > \alpha d$, the desired result follows from the Borel–Cantelli Lemma. $\square$

2.4. The Multiscale Expansion

The main result of the deterministic part of the multiscale analysis is given by

THEOREM 2.6. *Let us fix an environment* $\mathbf{p}$ *and let* $m_0 < m_{p_1}$. *Then for any* k*-regular site* $x \in \mathbb{Z}^d$, $k = 0, 1, 2, \ldots$, *we have*

$$G_{\Lambda_{L_k}(x)}(x, \partial) \leq e^{-m_k L_k}, \tag{16}$$

where m_k *is defined inductively for* $k = 1, 2, \ldots$ *by*

$$m_k = m_{k-1} - \frac{6m_0}{L_{k-1}^{\alpha-1}} \tag{17}$$

with

$$m_k \geq d\,\frac{\log L_k}{L_k}, \tag{18}$$

and L_0 *sufficiently large, how large depending only on* d, p_1, m_0 *and* α.

Proof. The proof is by induction on k. If $k = 0$, since $G_\Lambda(x, y)$ is increasing in Λ, (16) follows immediately from (1) with $p = p_1$ in case L_0 is large enough, in which case we also have (18) for $k = 0$.

Let us assume that (16), (17) and (18) hold for for k, we will prove they hold for $k+1$. So let x be $(k+1)$-regular, $y \in \partial\Lambda_{L_{k+1}}(x)$, we will estimate $G_{\Lambda_{L_{k+1}}(x)}(x, y)$. We will proceed as in the proof of Lemma 2.2, with some modifications (if $\Lambda_{L_{k+1}}(x)$ was a k-regular region, the same proof would work). By the definition of $(k+1)$-regularity, there exists $u \in \Lambda_{L_{k+1}}(x)$ such that $\widetilde{\Lambda} = \Lambda_{L_{k+1}}(x) \setminus \Lambda_{2L_k - 1}(u)$ is a k-regular region. If $z \in \widetilde{\Lambda} \cap \Lambda_{L_{k+1} - L_k - 1}(x)$, we use (5) and the induction hypothesis (16) to get

$$G_{\Lambda_{L_{k+1}}(x)}(z, y) \leq e^{-m_k L_k} G_{\Lambda_{L_{k+1}}(x)}(z_1, y) \tag{19}$$

for some $z_1 \in \partial\Lambda_{L_k}(z)$. If $z \in \Lambda_{2L_k - 1}(u)$ and $\Lambda_{2L_k}(u) \subset \Lambda_{L_{k+1} - L_k - 1}(x)$, we use (5) with $\Lambda = \Lambda_{2L_k}(u)$ and $\Lambda' = \Lambda_{L_{k+1}}(x)$ to get

$$G_{\Lambda_{L_{k+1}}(x)}(z, y) \leq |\partial\Lambda_{2L_k}(u)| G_{\Lambda_{L_{k+1}}(x)}(z_1, y) \leq L_k^d\, G_{\Lambda_{L_{k+1}}(x)}(z_1, y), \tag{20}$$

for some $z_1 \in \partial\Lambda_{2L_k}(u) \subset \widetilde{\Lambda} \cap \Lambda_{L_{k+1} - L_k - 1}(x)$, since we always have the obvious bound

$$G_\Lambda(z, z') \leq 1. \tag{21}$$

The second inequality is true for L_0 large. But now we can apply inequality (19) to $G_{\Lambda_{L_{k+1}}(x)}(z_1, y)$ obtaining

$$G_{\Lambda_{L_{k+1}}(x)}(z, y) \leq L_k^d e^{-m_k L_k} G_{\Lambda_{L_{k+1}}(x)}(z_2, y) \leq G_{\Lambda_{L_{k+1}}(x)}(z_2, y), \tag{22}$$

for some $z_2 \in \partial\Lambda_{L_k}(z_1)$, the second inequality following from the induction hypothesis (18).

We now estimate $G_{\Lambda_{L_{k+1}}(x)}(x,y)$ as in the proof of Lemma 2.2, using either (19) or (22) as appropriate when possible, plus (21) at the end. We get

$$G_{\Lambda_{L_{k+1}}(x)}(x,y) \leq \exp\left\{-m_k L_k \left(\frac{L_{k+1}-4L_k}{L_k} - 1\right)\right\}, \tag{23}$$

since we can use (19) at least N times, where N is the integer satisfying

$$\frac{L_{k+1}-4L_k}{L_k} - 1 \leq N < \frac{L_{k+1}-4L_k}{L_k}.$$

As (23) is valid for any $y \in \partial\Lambda_{L_{k+1}}(x)$, and $|\partial\Lambda_{L_{k+1}}(x)| \leq L_{k+1}^d$ for L_0 large, we get

$$G_{\Lambda_{L_{k+1}}(x)}(x,\partial) \leq L_{k+1}^d e^{-m_k(L_{k+1}-5L_k)} \leq e^{-m_{k+1}L_{k+1}}, \tag{24}$$

with m_{k+1} given by (17) satisfying (18), if L_0 is sufficiently large, how large depending only on d, p_1, m_0 and α. □

We can now prove Theorem 2.1. Given $m < m_{p_1}$ we set $m_0 = \frac{1}{2}(m + m_{p_1})$, and pick L_0 large enough so that we can apply Lemma 2.4 and Theorem 2.6, with

$$m_\infty \equiv \lim_{k\to\infty} m_k = m_0 \left(1 - 6\sum_{k=0}^{\infty} L_k^{-(\alpha-1)}\right) \geq \tfrac{1}{2}(m + m_0). \tag{25}$$

Since

$$\mathbf{P}(0 \text{ is } 0\text{-singular}) \leq (1-q)\,(2L_0+1)^d, \tag{26}$$

we have (11) if $q > 1 - \{L_0^r(2L_0+1)^d\}^{-1}$. We can thus apply Proposition 2.5 with $b = (m_\infty - m)^{-1}$. So let $x \in \mathbb{Z}^d$, for any $y \in \mathbb{Z}^d$ we can find a unique k such that

$$bL_k \leq |y-x| < bL_{k+1}. \tag{27}$$

If $k > \overline{k}_b(x)$, we have that $\Lambda_{bL_{k+1}}(x)$ is a k-regular region, so the proof of Lemma 2.2 applies inside $\Lambda_{bL_{k+1}}(x)$, with $L = L_k$ and $\rho = e^{-m_k L_k}$, yielding

$$G(x,y) \leq e^{-m_k|y-x| + m_k L_k} \leq e^{-(m_k - b^{-1})|y-x|} \leq e^{-m|y-x|}. \tag{28}$$

This finishes the proof of Theorem 2.1. □

3. Continuous Time Percolation and Contact Process in a Random Environment

Let us consider the inhomogeneous continuous time percolation process on $\mathbb{Z}^d \times \mathbb{R}$ defined as follows.

Let $\boldsymbol{\delta} = \{\delta(x) > 0; x \in \mathbb{Z}^d\}$ and $\boldsymbol{\lambda} = \{\lambda(x,y) > 0 : \langle x,y\rangle \in \mathbb{B}(\mathbb{Z}^d)\}$, where $\mathbb{B}(\mathbb{Z}^d)$ denotes the collection of bonds (or edges) $\langle x,y\rangle$ in $\mathbb{Z}^d$, i.e., unoriented pairs of sites $x,y \in \mathbb{Z}^d$ with $\|x-y\|_2 = 1$. Along each vertical line $\{x\} \times \mathbb{R}^d$ we put *cuts* at times given by a Poisson point process with intensity $\delta(x)$, and between each pair of adjacent vertical lines $\{x\}\times\mathbb{R}$ and $\{y\}\times\mathbb{R}$, $\langle x,y\rangle \in \mathbb{B}(\mathbb{Z}^d)$, we place *bridges* at times given by a Poisson point process with intensity $\lambda(x, y)$. All these Poisson processes are independent of each other.

Given a realization of all these Poisson processes (i.e., a locally finite collection of *cuts* and *bridges*), we consider the subset of $\mathbb{R}^{d+1}$ we obtain by taking $\mathbb{Z}^d \times \mathbb{R}$, removing all *cuts*, and adding all *bridges*, and decompose it into connected components. We call these connected components *clusters*. We say that $(x, t), (y, s) \in \mathbb{Z}^d \times \mathbb{R}$ are connected if they belong to the same cluster; in this case we write $(x, t) \leftrightarrow (y, s)$. Notice that this happens if and only if there is a path from (x, t) to (y, s) made out of *uncut* segments of vertical lines and *bridges*.

More generally, if $\mathbf{W} \subset \mathbb{R}^{d+1}$ we can substitute $\mathbf{W}$ for $\mathbb{R}^{d+1}$ in the above considerations. Thus (x, t) is connected to (y, s) in $\mathbf{W}$, and we write $(x, t) \leftrightarrow_{\mathbf{W}} (y, s)$, if the connection lies in $\mathbf{W}$. Given subsets A and B of $\mathbb{Z}^d \times \mathbb{R}$, we say that $A \leftrightarrow_{\mathbf{W}} B$ if there exist $(x, t) \in A$, $(y, s) \in B$, such that $(x, t) \leftrightarrow_{\mathbf{W}} (y, s)$. If $\mathbf{W} = \mathbb{R}^{d+1}$ we omit $\mathbf{W}$.

Given such a inhomogeneous environment $\boldsymbol{\delta}$, $\boldsymbol{\lambda}$, we will denote by $\mathbf{Q} = \mathbf{Q}_{\boldsymbol{\delta},\boldsymbol{\lambda}}$ the percolation probability measure, i.e., the probability measure on the space of configurations (locally finite collection of *cuts* and *bridges*). The connectivity function in the region $\mathbf{W} \subset \mathbb{R}^{d+1}$ is then defined by

$$G_{\mathbf{W}}^{\boldsymbol{\delta},\boldsymbol{\lambda}}((x, t), (y, s)) = \mathbf{Q}_{\boldsymbol{\delta},\boldsymbol{\lambda}}\big((x, t) \leftrightarrow_{\mathbf{W}} (y, s)\big). \tag{29}$$

We may omit $\boldsymbol{\delta}, \boldsymbol{\lambda}$ from the notation. If $\Lambda \subset \mathbb{Z}^d$, $I \subset \mathbb{Z}$, we will write $G_{\Lambda \times I}$ for $G_{\widetilde{\Lambda} \times I}$, where $\widetilde{\Lambda} = \Lambda \cup \{[x, y]; x, y \in \Lambda, \|x - y\|_2 = 1\}$. We will also write $|(x, t)| = \|(x, t)\|_\infty = \max\{\|x\|_\infty, |t|\}$.

Notice that the scaling $(\boldsymbol{\delta}, \boldsymbol{\lambda}) \to (a\boldsymbol{\delta}, a\boldsymbol{\lambda})$, where $a > 0$, does not change the behavior of such systems; it corresponds to a simple re-scaling of the time variable.

This inhomogeneous percolation process appears as the limit of the percolation processes on $\mathbb{Z}^d \times n^{-1}\mathbb{Z}$ studied by Campanino, Klein, and Perez (1991). The homogeneous version was independently studied by Bezuidenhout and Grimmett (1991).

In a homogeneous environment, i.e., $\delta(x) \equiv \delta > 0$, $\lambda(x, y) \equiv \lambda > 0$, we always have a non-trivial phase diagram. The relevant parameter is $\xi = \lambda/\delta$. In any dimension $d = 1, 2, \ldots$ there exists $0 < \xi_c(d) < \infty$, such that if $\xi < \xi_c(d)$ the connectivity function decays exponentially, i.e.,

$$G((x, t), (y, s)) \leq C e^{-m|(x-y,t-s)|}, \tag{30}$$

for some $m > 0$, $C < \infty$, and if $\xi > \xi_c(d)$ there is percolation (Bezuidenhout and Grimmett 1991, Campanino, Klein, and Perez 1991). If $d = 1$, we actually have $\xi_c(1) = 1$ (Bezuidenhout and Grimmett 1991).

Again life is not so simple in a inhomogeneous environment $(\boldsymbol{\delta}, \boldsymbol{\lambda})$. Let

$$\begin{aligned} \underline{\xi}(x) &= \frac{1}{\delta(x)} \min\big\{\lambda(x, y) : y \text{ satisfying } \|x - y\|_2 = 1\big\}, \\ \overline{\xi}(x) &= \frac{1}{\delta(x)} \max\big\{\lambda(x, y) : y \text{ satisfying } \|x - y\|_2 = 1\big\}. \end{aligned}$$

It follows from the monotonicity properties of $G((x, t), (y, s))$ with respect to each $\delta(x)$ and $\lambda(x, y)$, that if $\sup_x \overline{\xi}(x) < \xi_c$ we have exponential decay, whereas if

$\inf_x \underline{\xi}(x) > \xi_c$ we have percolation. The interesting non-trivial cases are thus when the above conditions are not satisfied, in particular when we can find sites where $\overline{\xi}(x) < \xi_c$ and sites where $\underline{\xi}(x) > \xi_c$, so the system exhibits the type of behavior associated with Griffiths singularities. This typically happens in *disordered environments.*

If we consider the oriented percolation process with the *cuts* as above, but replacing the *bridges* by *one-way bridges*, i.e., each Poisson process of bridges is replaced by two independent Poisson processes with intensities $\lambda_1(x,y) > 0$ and $\lambda_2(x,y) > 0$, the first giving one-way bridges from $\{x\} \times \mathbb{R}$ to $\{y\} \times \mathbb{R}$, and the second from $\{y\} \times \mathbb{R}$ to $\{x\} \times \mathbb{R}$, and uncut segments can only be traversed in the direction of increasing time, then we obtain the graphical representation of the inhomogeneous contact process (see Bezuidenhout and Grimmett 1991). The above considerations apply to the contact process (e.g., Liggett 1985).

Given the inhomogeneous contact process with $\boldsymbol{\delta}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$, we can consider the percolation process with intensities $\boldsymbol{\delta}$, $\boldsymbol{\lambda}$, where $\lambda(x,y) = \lambda_1(x,y) + \lambda_2(x,y)$, (i.e., we make all bridges *two-ways*). Then clearly no percolation (i.e., no infinite cluster) in the percolation process implies extinction (i.e., no infinite *oriented* cluster) in the contact process, and survival of the contact process implies percolation in the percolation process.

We will consider these percolation and contact processes in a random environment by taking the $\boldsymbol{\delta} = \{\delta(x) : x \in \mathbb{Z}^d\}$ and $\boldsymbol{\lambda} = \{\lambda(x,y) : \langle x,y \rangle \in \mathbb{B}(\mathbb{Z}^d\}$ ($\boldsymbol{\delta}$, $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ for the contact process) to be independent families of independent identically distributed strictly positive random variables. We will use $\mathbf{P}$ and $\mathbf{E}$ to denote the probability measure and expectation associated with these random variables. We will also use δ and λ for representative random variables. In such environments we will have to deal with phenomena similar to Griffiths singularities. But the situation is very different from the simple one we encountered in the previous section, since now the disorder is *frozen* in the continuous time direction. *If* $\mathbf{P}(\underline{\xi}(x) > \xi_c) > 0$*, the connectivity function cannot have exponential decay in the time direction, since given any exponential rate of decay we can find, with probability one, singular regions inside which the connectivity function exhibits a slower rate of decay in the time direction than the given rate.*

The one-dimensional ($d = 1$) contact process in a random environment was studied by Liggett (1991, 1992), who gave conditions on the probability distributions for extinction and survival. Another one-dimensional survival result is due to Bramson, Durrett, and Schonmann (1991). Andjel (1993) exhibited examples of survival in two or more dimensions.

Campanino, Klein, and Perez (1991) (see their Theorem 4.1) used a multiscale analysis to give the first proof of no percolation and decay of the connectivity function for the multidimensional continuous time percolation process in a random environment, thus also proving extinction for the multidimensional contact process in a random environment. Klein (1992) extended their proof to a larger class of probability distributions. Campanino, Klein, and Perez (1991) also proved percolation for the continuous time percolation process in a random environment for $d \geq 2$. Aizenman, Klein, and Newman (1993) gave a proof of percolation in the one-dimensional case.

Continuous time percolation and contact processes were studied in a quasi-periodic environment by Jitomirskaya and Klein (1993), who also used a multiscale analysis to show no percolation and decay of the connectivity function.

Aizenman, Klein, and Newman (1993) found probability distributions (in any dimension) under which there is always percolation. Their result is:

PROPOSITION 3.1. *Consider the continuous time percolation process on $\mathbb{Z}^d \times \mathbb{R}$ in a random environment $\boldsymbol{\delta}$, $\boldsymbol{\lambda}$. Suppose either $d = 1$, $\mathbf{E}(\delta) + \mathbf{E}(\lambda^{-1}) < \infty$, and*

$$\lim_{u\to\infty} \frac{u}{\log u} \mathbf{P}\left\{\log \frac{1}{\delta} > u\right\} = \infty; \tag{31}$$

or $d \geq 2$, $\lambda \geq \lambda_0 > 0$ and $\delta \leq \delta_0 < \infty$ for some λ_0, δ_0, and

$$\lim_{u\to\infty} u^d \mathbf{P}\left\{\log \frac{1}{\delta} > u\right\} = \infty. \tag{32}$$

Then there is percolation with probability one.

Proposition 3.1 tells us that conditions on the probability distributions of δ and λ are needed for absence of percolation. Unlike the situation in Theorem 2.1, it is not enough to require

$$\mathbf{P}\big(\overline{\xi}(x) \leq \xi_1\big) \text{ sufficiently close to 1 for some } \xi_1 < \xi_c. \tag{33}$$

In fact, Proposition 3.1 suggests the conjecture that a sufficient additional condition for no percolation should be

$$\mathbf{E}\big\{[\log(1+\lambda)]^\beta\big\} + \mathbf{E}\big\{[\log(1+\delta^{-1})]^\beta\big\} < \infty \tag{34}$$

for some $\beta > d$.

The following result is proven in Klein (1992). Notice the manifestation of the Griffiths-type singularities in the less than exponential rate of decay in the time direction in (38).

THEOREM 3.2. *Let $d = 1, 2, \ldots,$ and consider the continuous time percolation process on $\mathbb{Z}^d \times \mathbb{R}$ in a random environment $\boldsymbol{\delta}, \boldsymbol{\lambda}$. Let*

$$\beta > 2d^2\left(1 + \sqrt{1 + \frac{1}{d}} + \frac{1}{2d}\right) \tag{35}$$

and suppose

$$\Gamma = \max\Big\{\mathbf{E}\big\{[\log(1+\lambda)]^\beta\big\}, \mathbf{E}\big\{[\log(1+\delta^{-1})]^\beta\big\}\Big\} < \infty. \tag{36}$$

Then we can find $\epsilon = \epsilon(d, \beta, \Gamma) > 0$ such that, if

$$\mathbf{E}\big\{[\log(1 + (\lambda/\delta))]^\beta\big\} < \epsilon, \tag{37}$$

we have no percolation with probability one.

In fact, there exists $q(\beta, d) > 1$ *such that, given* $m > 0$ *and* q *with* $1 < q < q(\beta, d)$, *we can find* $\epsilon = \epsilon(d, \beta, \Gamma, m, q) > 0$ *for (37) for which we have, with probability one, that for every* $x \in \mathbb{Z}^d$,

$$G((x,t),(y,s)) \le C_{x,\boldsymbol{\delta},\boldsymbol{\lambda}} \exp\Big\{-m\big|\big(x-y, [\log(1+|t-s|)]^q\big)\big|\Big\} \tag{38}$$

for all $y \in \mathbb{Z}^d$, $t, s \in \mathbb{R}$, *with* $C_{x,\boldsymbol{\delta},\boldsymbol{\lambda}} < \infty$.

Condition (37) is only used to get (33) and may be replaced by it. Notice also that

$$\mathbf{E}\big\{[\log(1+(\lambda/\delta))]^\beta\big\} \le 2^{\beta+1}\widetilde{\Gamma}, \tag{39}$$

where

$$\widetilde{\Gamma} \equiv \min_{\gamma>0} \max\Big\{\mathbf{E}\big\{[\log(1+\gamma\lambda)]^\beta\big\}, \mathbf{E}\big\{[\log(1+(\gamma\delta)^{-1})]^\beta\big\}\Big\} \le \Gamma. \tag{40}$$

Theorem 3.2 immediately implies the following result about the contact process in a random environment:

COROLLARY 3.3. *Let* $d = 1, 2, \ldots,$ *and consider the d-dimensional contact process in a random environment* $\boldsymbol{\delta}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$. *Let* β *satisfy (35) and* $\boldsymbol{\delta}, \boldsymbol{\lambda}$ *satisfy (36), where* $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$. *Then there exists* $\epsilon = \epsilon(d, \beta, \Gamma) > 0$ *such that if (37) holds, the contact process becomes extinct with probability one.*

Theorem 3.2 is proven by a multiscale analysis, albeit not as simple as the one in Section 2. In this article we will only describe the results of the multiscale analysis and refer to Klein (1992) for the full proof.

For $X = (x,t) \in \mathbb{Z}^d \times \mathbb{R}$, $L > 0$, $T > 0$, let $B_{L,T}(X) = \Lambda_L(x) \times [t-T, t+T]$. We fix ν such that $0 < \nu < 1$, and set $B_L(X) = B_{L,e^{L^\nu}}(X)$. For $Y \in B_{L,T}(X)$ let

$$\begin{aligned} G_{B_{L,T}(X)}(Y, \partial) \;=\; & \sum_{Z \in \Lambda_L(x) \times \{t-T, t+T\}} G_{B_{L,T}(X)}(Y, Z) \\ & + \sum_{\langle z,z'\rangle \in \partial\Lambda_L(x)} \lambda(z,z') \int_{t-T}^{t+T} G_{B_{L,T}(X)}(Y,(z,s))ds, \end{aligned} \tag{41}$$

where now

$$\partial\Lambda = \big\{\langle z, z'\rangle \in \mathbb{B}(\mathbb{Z}^d) : z \in \Lambda,\ z' \notin \Lambda\big\}.$$

If $Y \in B_{L,T}(X) \subset \mathbf{W}$ and $Z \in \mathbf{W} \setminus B_{L,T}(X)$, we have the following consequence of the HSL inequality:

$$G_{\mathbf{W}}(Y,Z) \le G_{B_{L,T}(X)}(Y,\partial)\, G_{\mathbf{W}}(Z_1, Z) \tag{42}$$

for some Z_1 either in $\Lambda_L(x) \times \{t-T, t+T\}$ or of the form (z_1, s) with $z_1 \in \Lambda_L(x)$, $\langle z_1, z_1'\rangle \in \partial\Lambda_L(x)$ for some z_1', and $s \in [t-T, t+T]$.

In a homogeneous environment we may use (42) to obtain (30), as in Lemma 2.2. As before we need to develop a multiscale analysis to extend the argument to a random environment.

Given $m > 0$, $L > 1$, a site $x \in \mathbb{Z}^d$ will be called (m, L)-*regular* if

$$G_{B_L((x,0))}((x,0), \partial) \leq e^{-mL}. \tag{43}$$

The main result of the multiscale analysis is given by

THEOREM 3.4. *Let $d = 1, 2, \ldots,$ and consider the continuous time percolation process on $\mathbb{Z}^d \times \mathbb{R}$ in the random environment $\boldsymbol{\delta}$, $\boldsymbol{\lambda}$. Let*

$$\beta > 2d^2 \left(1 + \sqrt{1 + \frac{1}{d}} + \frac{1}{2d}\right), \tag{44}$$

and suppose (36) holds. Set

$$\alpha = d + \sqrt{d^2 + d},$$

and choose ν and p such that

$$\frac{\alpha d(\alpha + \beta + 1)}{\beta(\alpha - d + \alpha d)} \; < \; \nu < 1, \tag{45}$$

$$\alpha d \; < \; p < \frac{\beta(\nu(\alpha - d + \alpha d) - \alpha d) - \alpha d}{\alpha}.$$

Let m_0 and m_∞ be given, and satisfy $0 < m_\infty < m_0$. There exists a number $\overline{L} = \overline{L}(d, \beta, \Gamma, \nu, p, m_0, m_\infty) < \infty$ such that, if for some $L_0 > \overline{L}$ we have

$$\mathbf{P}\big(0 \text{ is } (m_0, L_0)\text{-regular}\big) \geq 1 - L_0^{-p}, \tag{46}$$

then, setting $L_{k+1} = L_k^\alpha$, $k = 0, 1, \ldots,$ we also have

$$\mathbf{P}\big(0 \text{ is } (m_\infty, L_k)\text{-regular}\big) \geq 1 - L_k^{-p} \tag{47}$$

for all $k = 0, 1, 2, \ldots$.

Inequality (45) can be satisfied because of (44). Indeed, let

$$f(\theta) = \frac{\theta d(\theta + 1)}{\theta - d} \quad \text{for } \theta > d.$$

It is easy to see that $f(\theta)$ attains its minimum at $\theta = \alpha \equiv d + \sqrt{d^2 + d}$, and $f(\alpha) = 2d^2(1 + \sqrt{1 + d^{-1}} + (2d)^{-1})$. Thus (44) just says that $\beta > f(\alpha)$, and (45) says that we picked $0 < \nu < 1$ and $p > \alpha d$ such that

$$\beta > \frac{\alpha(p + d)}{\alpha\nu - d(\alpha(1 - \nu) + \nu)} = \frac{\alpha(p + d)}{\nu(\alpha - d + \alpha d) - \alpha d}.$$

The proof of Theorem 3.4 proceeds by induction on k. In addition to the HSL inequality, the proof uses the Harris–FKG and the van den Berg–Kesten inequalities.

The main difficulty is in estimating the decay of the connectivity function in the time direction, *inside the singular regions.* Such an estimate is needed since the singular regions are cylinders infinitely extended in the time direction, as the disorder is frozen in that direction. Using (36) it is shown that a suitable estimate holds in each scale with good probability.

Since we can obtain (46) by taking ϵ in (37) sufficiently small, Theorem 3.2 follows from Theorem 3.4 by

THEOREM 3.5. *Let $d = 1, 2, \ldots$, and consider the continuous time percolation process on $\mathbb{Z}^d \times \mathbb{R}$ in the random environment $\boldsymbol{\delta}$, $\boldsymbol{\lambda}$. Let ν, α, p, m_∞, L_0 be such that $0 < \nu < 1$, $\alpha > 1$, $p > \alpha d$, $m_\infty > 0$, $L_0 > 1$. Set $L_{k+1} = L_k^\alpha$, $k = 0, 1, 2, \ldots$. Suppose*

$$\mathbf{P}\big(0 \text{ is } (m_\infty, L_k)\text{-regular}\big) \geq 1 - L_k^{-p} \tag{48}$$

for all $k = 0, 1, 2, \ldots$. Then, for any m such that $0 < m < m_\infty$, we have, with probability one, that for every $x \in \mathbb{Z}^d$,

$$G((x,t),(y,s)) \leq C_x \exp\Big\{-m\big|(x-y, [\log(1+|t-s|)]^{1/\nu})\big|\Big\}$$

for all $y \in \mathbb{Z}^d$, $t, s \in \mathbb{R}$, with $C_x = C_x(\boldsymbol{\delta}, \boldsymbol{\lambda}, m) < \infty$.

References

Aizenman, M., Klein, A., and Newman, C. M. (1993). Percolation methods for disordered quantum Ising models. To appear in *Phase Transitions: Mathematics, Physics, Biology, …* (R. Kotecky ed.), World Scientific.

Andjel, E. (1993). Survival of multidimensional contact process in random environments. *Boletim da Sociedade Brasileira de Matemática* **23**, 109–119.

Bezuidenhout, C. and Grimmett, G. (1991). Exponential decay for subcritical contact and percolation processes. *Annals of Probability* **19**, 984–1009.

Bramson, M., Durrett, R., and Schonmann R. H. (1991). The contact process in a random environment. *Annals of Probability* **19**, 960–983.

Campanino, M. and Klein, A. (1991). Decay of two-point functions for $(d+1)$-dimensional percolation, Ising and Potts model with d-dimensional disorder. *Communications in Mathematical Physics* **135**, 483–497.

Campanino, M., Klein, A., and Perez, J. F. (1991). Localization in the ground state of the Ising model with a random transverse field. *Communications in Mathematical Physics* **135**, 499–515.

Dreifus, H. von (1987). On the effects of randomness in ferromagnetic models and Schrödinger operators. Ph.D. Thesis, New York University.

Dreifus, H. von and Klein, A. (1989). A new proof of localization in the Anderson tight binding model. *Communications in Mathematical Physics* **124**, 285–299.

Dreifus, H. von and Klein, A. (1991). Localization for random Schrödinger operators with correlated potentials. *Communications in Mathematical Physics* **140**, 133–147.

Fröhlich, J. and Spencer, T. (1983). Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Communications in Mathematical Physics* **88**, 151–184.

Fröhlich, J., Martinelli, F., Scoppola, E., and Spencer, T. (1985). Constructive proof of localization in the Anderson tight binding model. *Communications in Mathematical Physics* **101**, 21–46.

Griffiths, R. (1969). Non-analytic behavior above the critical point in a random Ising ferromagnet. *Physical Review Letters* **23**, 17–19.

Grimmett, G. (1989). *Percolation.* Springer-Verlag, New York.

Jitomirskaya, S. and Klein, A. (1993). Ising model in a quasi-periodic transverse field, percolation and contact processes in quasi-periodic environments. *Journal of Statistical Physics*, to appear.

Klein, A. (1992). Extinction of contact and percolation processes in a random environment. *Annals of Probability*, to appear.

Klein, A. (1993). Disordered quantum spin processes, percolation and contact processes. To appear in *Phase Transitions: Mathematics, Physics, Biology, ...* (R. Kotecky, ed.), World Scientific.

Klein, A. and Perez, J. P. (1992). Localization in the ground state of a disordered array of quantum rotators. *Communications in Mathematical Physics* **147**, 241–252.

Liggett, T. M. (1985). *Interacting Particle Systems.* Springer-Verlag, New York.

Liggett, T. M. (1991). Spatially inhomogeneous contact processes. In *Spatial Stochastic Processes. A Festschrift in honor of the Seventieth Birthday of Ted Harris*, Birkhäuser, Boston, 105–140.

Liggett, T. M. (1992). The survival of one dimensional contact processes in random environments. *Annals of Probability* **20**, 696–723.

Spencer, T. (1988). Localization for random and quasi-periodic potentials. *Journal of Statistical Physics* **51**, 1009–1019.