LONG TIME TAILS IN PHYSICS AND MATHEMATICS

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Abstract. In physics the name 'long time tail' is used to denote slow decay of an equilibrium autocorrelation function of a fluctuating local quantity (like density, momentum, energy) in a system of interacting particles. This paper discusses long time tails for the velocity autocorrelation function of a tagged particle. After an expository introduction we describe various theories from physics and mathematics explaining this phenomenon.

Key words: interacting particle systems, tagged particle, velocity autocorrelation function, long time tail, random motion in random environment

1. Introduction

For the early 20th century physicist matter had three phases: gas, solid or liquid. Whereas the earliest successes of statistical physics were achieved in explaining the fundamental laws of gases and solids on the basis of simple microscopic laws of interaction, it took much longer to come to a satisfactory understanding of the basic properties of liquids. The reason for this is, quite simply, that the particles in a liquid are neither (more or less) freely moving about, like in a gas, nor are they (more or less) localized, like in a solid. Consequently, at the microscopic level the interactions in a liquid are more complex[1].

In this paper we shall address one particular question about the fluid, of interest to the physicist: 'What is the motion of a *tagged particle* like?' Imagine that we pick a particle at random, give it a tag, and follow its erratic motion as time evolves. Suppose that this tagged particle is 'in equilibrium' with its environment. Then what is the law of its motion as a result of its repeated collisions with the other particles[2]? (See the figure.)

Let

$$\begin{aligned} X(t) &= \text{position of tagged particle at time } t \\ V(t) &= \text{velocity of tagged particle at time } t \end{aligned} \tag{1}$$

(put $X(0) = 0$). Two important quantities characterizing these random processes

[1] For simplicity we consider only classical simple fluids, i.e., particles are monatomic with no internal degrees of freedom and are subject to the Newtonian laws of mechanics. Fluid is the collective name for dense gas or liquid.

[2] A collision should be thought of as a fast but smooth deflection of trajectories. At short distances particles repel each other according to the Lennard–Jones potential.

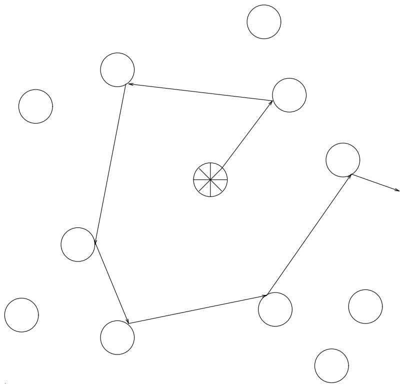

Fig. 1. Tagged particle in $\mathbb{R}^d$.

are

$$\begin{aligned} \langle X^2(t)\rangle &= \text{mean-square displacement at time } t \\ \langle V(0)V(t)\rangle &= \text{velocity autocorrelation at time } t \end{aligned} \tag{2}$$

where $\langle\cdot\rangle$ denotes expectation with respect to the 'equilibrium ensemble' describing the system, i.e., the law of the motion of all the particles. Of course, this law will not be known in general and it is here that the difficulty lies in analyzing the quantities in (2)[3].

Lemma 1. *If $\{V(t) : t \geq 0\}$ is a stationary process, then*

$$\frac{1}{2}\frac{d^2}{dt^2}\langle X^2(t)\rangle = \langle V(0)V(t)\rangle \quad (t \geq 0). \tag{3}$$

Proof. Since $X(t) = \int_0^t V(s)ds$ it follows that $\frac{1}{2}\frac{d}{dt}\langle X^2(t)\rangle = \langle X(t)V(t)\rangle = \int_0^t \langle V(s) V(t)\rangle ds$. By stationarity, $\langle V(s)V(t)\rangle = \langle V(0)V(t-s)\rangle$ and therefore the last integral can be rewritten as $\int_0^t \langle V(0)V(s)\rangle ds$. Differentiate once more with respect to t to get (3). □

Lemma 1 shows that the two quantities in (2) have a simple relation when $\{X(t) : t \geq 0\}$ has stationary increments, which is the mathematical way of saying that the tagged particle is 'in equilibrium' with its environment. If $X(t)$ were a pure diffusion (like a Wiener process), then we would have $\langle X^2(t)\rangle = 2Dt$ for all $t \geq 0$, where D is the so-called *diffusion constant*. However, the motion of the tagged particle is *not* purely diffusive because it is carrying momentum and because it is not colliding with the other particles all the time. Equation (3) shows that the correction to the linear growth of $\langle X^2(t)\rangle$ will manifest itself as $\langle V(0)V(t)\rangle \neq 0$, typical for this situation.

Lemma 2. *If $D = \frac{1}{2}\lim_{t\to\infty} \frac{d}{dt}\langle X^2(t)\rangle$ exists, then*

$$D = \int_0^\infty \langle V(0)V(s)\rangle ds. \tag{4}$$

[3] $X^2(t)$ denotes the inner product of the vector $X(t)$ with itself, and similarly for $V(0)V(t)$.

Proof. See the proof of Lemma 1. □

Lemma 2 relates the diffusion constant D to the velocity autocorrelation function $t \to \langle V(0)V(t)\rangle$. Equation (4) is known as the *Green–Kubo formula*[4].

For over half a century it was believed by physicists that

$$\langle V(0)V(t)\rangle = e^{-At(1+o(1))} \quad (t \to \infty) \tag{5}$$

with $A > 0$ some rate that depends on the parameters of the fluid (such as temperature and density). Intuitively (5) was quite a reasonable guess. The tagged particle gets repeatedly 'kicked around' by the other particles and therefore it should lose memory of its initial velocity at some positive rate. Moreover, the exponential decay was predicted by the semi-phenomenological theories based on the Boltzmann equation and the Langevin equation, which were the main descriptive tools at the time. Furthermore, the asymptotic linear growth of the mean-square displacement (implied by (5) via Lemma 2) had been observed in numerous physical experiments (e.g., the famous experiments by Perrin 1909 on Brownian particles). So there was no reason to doubt (5).

Clouds appeared on the sky in the late 1960's. Namely, computer simulations by Alder and Wainwright (1967, 1969, 1970), carried out with models of elastically colliding hard disks and spheres, indicated that

$$\langle V(0)V(t)\rangle = At^{-d/2}(1 + o(1)) \quad (t \to \infty) \tag{6}$$

with $d = 2$ or 3 the dimension and $A > 0$ some system-dependent amplitude. The slow decay in (6), as opposed to the fast decay in (5), came as a *complete surprise*. How is such behavior possible? How can it be reconciled with the idea of the tagged particle being repeatedly kicked around? How does the motion manage to carry the correlation over a long time interval?

The behavior in (6) has been confirmed by further computer simulations with various types of models with short range interaction (Wood 1973, 1975, Levesque and Ashurst 1974, van der Hoef et al. 1991). It has also been observed in actual experiments (Kim and Matta 1973, Carneiro 1976, Boon and Bouiller 1976, Paul and Pusey 1981, Morkel et al. 1987), leaving no doubt as to the reality of the claim[5]. Ever since these developments the slow decay in (6) has become known as a 'long time tail', a terminology introduced in the physics literature. (The fast decay in (5) is called a 'short time tail'.)

A remarkable consequence of (6) is that $D = \infty$ in $d = 1$ and 2 (by Lemma 2). Thus the motion in low dimension appears to be superdiffusive, which means that the hydrodynamic diffusion equation is not valid. But the impact of (6) goes further. Namely, the foundation of hydrodynamics is based on the assumption that non-equilibrium behavior has two time regimes: (i) short (a few mean collision times), where microscopic relaxation takes place; (ii) long (many mean collision times),

[4] Note that Lemmas 1 and 2 do not apply to pure diffusion, because for such motion $V(t)$ does not exist. One is tempted to write $\langle V(0)V(t)\rangle \equiv 0$ for $t > 0$. This is OK with (3) but not with (4) (see also footnote 2).

[5] The interpretation of these experiments (based on hydrodynamic measurement, incoherent neutron scattering and light scattering) is a subtle matter, as it is not always clear to what extent the results reflect the long time tail of a single particle (see Boon and Yip 1980).

where only macroscopic relaxation is important (and the system is in so-called 'local equilibrium'). The fact that the velocity autocorrelation persists from the short into the long time regime indicates that the above separation of time regimes is somewhat arbitrary and needs clarification.

To close this introduction, let us note that there is of course no complete theoretical foundation for (6). A real fluid, or even a caricature model like n elastically colliding hard spheres, is much too hard for a rigorous mathematical analysis. Nevertheless, over the years a good heuristic explanation has crystallized out. The most convincing argument is due to van Beijeren (1982): "If one assumes that the motion of the tagged particle is roughly diffusion-like, then after time t it will have travelled a distance of order $t^{\frac{1}{2}}$ in each of the d directions. This means that the tagged particle has been in direct or indirect contact with all the particles in a volume of order $t^{d/2}$. With each of these it has shared some of its momentum, and so the momentum autocorrelation must be of order $1/t^{d/2}$."

In the next section we shall see how this argument can be made more precise and what it predicts for the amplitude of the long time tail.

2. Semi-Phenomenological Theories

Several semi-phenomenological theories have been proposed to explain the power $\frac{1}{2}d$ and identify the amplitude A. We mention two of these that have been particularly successful. For further references see van Beijeren (1982) and Cohen (1993a, b).

2.1. Hydrodynamics

The following argument is taken from van Beijeren (1982). Consider a system of particles in equilibrium, *conditioned* on having a particle at the origin with velocity v at time $t = 0$. Give this particle a tag. Define

$$\begin{aligned} \rho(x,t) &= \text{probability density for tagged particle to be at } x \text{ at time } t \\ u(x,t) &= \text{average velocity density of particles at } x \text{ at time } t. \end{aligned} \tag{7}$$

Due to the condition $V(0) = v$ the system starts from a *non-equilibrium* situation with initial conditions

$$\begin{aligned} \rho(x,0) &= \delta(x) \\ u(x,0) &= v\delta(x) \end{aligned} \tag{8}$$

where $\delta(x)$ is the Dirac function. Since this is only a *tiny deviation* from equilibrium it is reasonable to assume that after a long time (i.e., much longer than the mean collision time of a typical particle in the system) the evolution of $\rho(x,t)$ and $u(x,t)$ is described to first approximation by *linearized* hydrodynamic equations, namely

$$\begin{aligned} \frac{\partial}{\partial t}\rho(x,t) &= D \bigtriangledown^2 \rho(x,t) \\ \frac{\partial}{\partial t}u_\perp(x,t) &= -\nu \bigtriangledown \times (\bigtriangledown \times u_\perp(x,t)). \end{aligned} \tag{9}$$

Here $u_\perp$ is the transversal part of the velocity field ($\bigtriangledown \cdot u_\perp = 0$), D is the diffusion constant, and ν is the kinematic viscosity. The equation for the longitudinal part of the velocity field is slightly more involved but is omitted, because its contribution to the velocity autocorrelation function turns out to decay exponentially.

The solution of (8) and (9) is given most easily in terms of the Fourier transforms $\hat{z}(k,t) = \int dx \exp[-i(k \cdot x)] z(x,t)$ $(z = \rho, u_\perp)$ and reads

$$\begin{aligned} \hat{\rho}(k,t) &= e^{-Dk^2 t}, \\ \hat{u}_\perp(k,t) &= \Big[v - \tfrac{(v \cdot k)k}{k^2}\Big] e^{-\nu k^2 t}. \end{aligned} \tag{10}$$

Now let us assume that after a long time the tagged particle has the same average velocity as the other particles in its neighborhood. Then we have

$$\begin{aligned} \langle V(t) | V(0) = v \rangle &\sim \int dx\ \rho(x,t) \frac{1}{n} u(x,t) \\ &\sim \frac{1}{n} \frac{1}{(2\pi)^d} \int dk\ \hat{\rho}(k,t) \hat{u}_\perp(-k,t), \end{aligned} \tag{11}$$

where n is the particle density. Substitution of (10) into (11) yields

$$\begin{aligned} \langle V(t) | V(0) = v \rangle &\sim \frac{1}{n} \frac{1}{(2\pi)^d} \int dk \Big[v - \frac{(v \cdot k)k}{k^2}\Big] e^{-(D+\nu)k^2 t} \\ &= \frac{1}{n} \frac{d-1}{d} \frac{1}{[4\pi(D+\nu)]^{d/2}}\ v. \end{aligned} \tag{12}$$

The contribution of the longitudinal part of the velocity field is of higher order. Only the transversal part contributes to the leading order, hence the factor $(d-1)/d$.

Equation (12) says that $\langle V(t) | V(0) = v \rangle$ points in the *same* direction as the initial velocity $V(0) = v$ but is reduced in length by an appropriate time-dependent factor due to the diffusive spreading out of the density and the momentum. Finally, we must average over v with respect to the equilibrium ensemble of the system. Since v has the Maxwell–Boltzmann distribution, which is an isotropic Gaussian with mean zero and variance $(\beta m)^{-1}$, where β is the inverse temperature and m is the mass of a particle, it follows that

$$\begin{aligned} &\langle V(0) V(t) \rangle \sim A t^{-d/2} \\ &A = \frac{1}{\beta m n} \frac{d-1}{d} \frac{1}{[4\pi(D+\nu)]^{d/2}}. \end{aligned} \tag{13}$$

Thus we have arrived at (6) with an explicit identification of the amplitude A in terms of system parameters.

The result in (13) turns out to compare very favorably with numerical simulations and with actual experiments. Even though the above derivation clearly is not on a microscopic basis, (13) is nevertheless believed to be *exact*, at least for $d \geq 3$. There is an obvious inconsistency in $d = 1$ and 2 because (13) implies $D = \infty$, as was already noted before. Apparently, hydrodynamic equations like (9) do not hold in low dimensions. There are some speculations that in $d = 2$ (where D diverges only marginally) the result in (13) should be corrected by a logarithmic factor (van der Hoef and Frenkel 1991).

2.2. Kinetic Theory

At low densities the main source of interaction between the particles comes from uncorrelated binary collisions, i.e., no two particles collide more than once. The behavior of the system in such a situation can be described by the classical Boltzmann equation, which lies halfway between a microscopic and a macroscopic description. At higher densities, however, more complex collision patterns come into play, namely, correlated binary collisions as well as collisions involving three or more particles at the same time. Kinetic theory aims at systematically including the effects of some of these collision patterns, via so-called cluster expansion techniques. For a recent historic account of this fascinating area we refer the reader to Cohen (1993a, b).

Dorfman and Cohen (1970, 1972, 1975) derive (13) on the basis of kinetic theory. They show that the main collisions responsible for the long time tail are the so-called 'ring collisions'. These are sequences of correlated binary collisions where the initial momentum of the tagged particle is transferred to the surrounding particles in a ring-like motion. After about 25 mean collision times this ring-like motion 'kicks the tagged particle in the back'. In other words, the tagged particle causes a *vortex type* perturbation of the velocity field around itself and it picks up part of this perturbation some time later.

The vortex has also been seen in the computer simulations. Moreover, the vortex structure is compatible with the fact that in (12) only the transversal (= divergence free) part of the velocity field contributes to the long time tail. The 'kick in the back' coincides with $\langle V(t)|V(0)=v\rangle$ and v pointing in the same direction.

The difference between (13) and the result obtained from kinetic theory is that not D,ν appear in the amplitude but the so-called 'bare' transport coefficients D_0,ν_0. These are the ones predicted by kinetic theory (to lowest order in the density). It seems that the simulations predict a long time tail with a 'bare' amplitude for times between 10 and 50 mean collision times, and that only for longer times the full amplitude is seen.

For a class of short range interactions with hard core repulsion kinetic theory identifies D_0 and ν_0 explicitly in terms of system parameters. For instance, for hard spheres the result is

$$\begin{aligned} D_0 \;=\; \nu_0 \quad &= \frac{1}{2an}\frac{1}{(\pi\beta m)^{\frac{1}{2}}}, \quad (d=2),\\ D_0 \;=\; \tfrac{6}{5}\nu_0 \;&= \frac{3}{8a^2n}\frac{1}{(\pi\beta m)^{\frac{1}{2}}}, \quad (d=3), \end{aligned} \tag{14}$$

where a is the radius of a particle.

3. Lorentz Models

In order to make possible a more mathematical approach to the study of the tagged particle motion, the following two simplifications are introduced into the modelling:

(I) The *environment is frozen*, i.e., only the tagged particle moves. The other particles are kept fixed, their positions being chosen either periodically or randomly.

(II) The *collision rules are random*, i.e., the way in which the tagged particle is scattered when colliding with one of the other particles has a stochastic component.

Models with property (I) are called *deterministic Lorentz models.* Examples are:

(1) A hard disk or sphere collides elastically with scatterers. The scatterers are hard disks or spheres too, whose centers are placed at the sites of $\mathbb{Z}^2$ resp. $\mathbb{Z}^3$ (Bunimovich and Sinai 1981).

(2) A light beam moves along the bonds of $\mathbb{Z}^2$ and gets deflected by mirrors placed at every site. The axes of the mirrors make an angle of 45 degrees with the lattice axes and are chosen randomly from the two possible orientations (Cohen 1992).

Models with properties (I) and (II) are called *stochastic Lorentz models.* Examples are:

(3) A particle moves in a straight line along the bonds of $\mathbb{Z}^d$ except when hitting a scatterer. At a scatterer it changes the direction of its motion randomly to one of the $2d$ possible directions. The scatterers are placed at the lattice sites randomly with density lower than one (van Beijeren and Spohn 1983, den Hollander et al. 1992b).

(4) A particle performs a random walk on an infinite percolation cluster in $\mathbb{Z}^d$. At each step it chooses with equal probability from the nearest-neighbor sites that are in the cluster (Kesten 1986).

Three types of questions have been studied for Lorentz models:

(a) Is the motion asymptotically diffusive? (Is the limit law of $\{t^{-\frac{1}{2}}X(\lfloor ut \rfloor) : u \in [0,1]\}$ as $t \to \infty$ that of a Wiener process?)

(b) Is the motion recurrent or transient? (Does $X(t)$ return to every finite neighborhood of 0 with probability $= 1$ or < 1?)

(c) Is there a long time tail? (Does (6) hold?)

Although in this paper we are mainly concerned with question (c), we want to say a few words about questions (a) and (b) as well.

For deterministic Lorentz models (a) and (b) have been settled partially in a few cases. For instance, in example (1) with $d = 2$ we know that (under some conditions) the motion is asymptotically diffusive, but we do not know whether it is recurrent or not[6]. In example (2) we know that the motion is recurrent and not asymptotically diffusive, but we do not know its scaling behavior. Unfortunately there are, to date, no examples proving or disproving (c)[7].

For stochastic Lorentz models the situation is better. If the motion is *Markov* and *reversible* (in an appropriate sense), then a general theorem due to DeMasi et al. (1989) gives the answer to (a): The path scales to a Wiener process with a diffusion constant $D \in [0, \infty)$, and in many examples it can be proved that actually $D > 0$ so that the limit law is non-degenerate (e.g., in example (4) strictly above criticality). For *non-reversible* Markov motion there is no comparible result of such generality, although in some examples (a) can still be settled in the affirmative (e.g., in example (3); den Hollander et al. 1992b). Question (b) seems to be hard, both for reversible

[6] Weaker versions of (b) can sometimes be answered, like $X(t)$ returns to $B \log^{\beta} t$ infinitely often with probability $= 1$, where B is some ball in $\mathbb{R}^2$ and $\beta > 0$ some power.

[7] In example (1) the velocity autocorrelation function has been shown to decay faster than any power, but this is a consequence of the periodic configuration of the scatterers. For random configurations one expects a long time tail.

and for non-reversible Markov motion. One can prove recurrence in $d = 1$ and 2 for reversible systems (Durrett 1986), but typically transience in $d \geq 3$ remains open. There are, however, examples where recurrence holds in any $d \geq 1$ (Durrett 1986, Bramson and Durrett 1988). Question (c) has so far been answered only for a few very simple models. The first results in this direction are due to van Beijeren and Spohn (1983), who proved a long time tail for a class of models in $d = 1$. Recently, den Hollander et al. (1992a, c) have obtained the first rigorous proofs of long time tails in $d \geq 2$. These will be discussed in the next section.

4. Two Models

In stochastic Lorentz models one expects a long time tail result of the type as in (6) on the basis of the following heuristic argument. Assume that the motion is asymptotically diffusive (so the answer to (a) is affirmative). The slow decay of the velocity autocorrelation function arises because the particle may return to the origin and recognize the environment. This induces a memory effect, which is governed by slowly decaying return probabilities typical for diffusive motion. However, such an explanation clearly is rather vague, and we are left with the task to get mathematically precise results in concrete situations.

A problem that comes up in stochastic Lorentz models is that, because the motion is random, it is not always obvious what is meant by the velocity $V(t)$. In some cases there is a natural choice for defining this quantity (as e.g., in examples (3) and (4) of Section 3), but in other cases one needs to introduce the appropriate notion. We shall see more of this in the examples below[8].

4.1. Random Waiting Times

In this model, which was first studied by Denteneer and Ernst (1984), the role of the random environment is played by an i.i.d. collection of random variables

$$W = \{w(x) : x \in \mathbb{Z}^d\}, \tag{15}$$

whose law $\mu = \gamma^{\mathbb{Z}^d}$ is assumed to satisfy $\gamma(0,\infty) = 1$ and

$$\begin{array}{ll} \int w^{-1}\gamma(dw) & < \infty \\ \int w^2\gamma(dw) & < \infty. \end{array} \tag{16}$$

Given W, let $\{X(t) : t \geq 0\}$ be simple random walk with jump rate $w^{-1}(X(t))$, $(X(0) = 0)$. That is, after $X(t)$ jumps to a site x it waits there during a random time which is exponentially distributed with mean $w(x)$. After this time it jumps with equal probability to one of the nearest neighbors of x, waits there again, etc. We shall write P_W to denote the law of this walk in the fixed environment W, and $P_\mu = \int P_W \mu(dW)$ to denote the law after averaging over W with respect to μ. The respective expectations will be written E_W and E_μ.

It is not hard to prove that

$$E_\mu X^2(t) \sim M^{-1}t, \quad (t \to \infty), \tag{17}$$

[8] In the physics literature $\langle V(0)V(t)\rangle$ is often *defined* by relation (3), in order to avoid the definition of $V(t)$ itself.

with $M = \int w\gamma(dw)$ the mean of γ. Indeed, over a large time interval of length t the walk will make $\sim M^{-1}t$ steps, because it will visit many different sites and the average delay per visit is M. Since the mean-square displacement of simple random walk after n steps is exactly n, the latter implies (17).

Theorem 1 below is a long time tail for the quantity

$$\Delta(t) = \frac{d}{dt}E_\mu X^2(t) - M^{-1}, \tag{18}$$

which measures the correction to (17).

Theorem 1. *Let M and V^2 be the mean and variance of γ. Then*

$$\lim_{t\to\infty} t^{d/2}\Delta(t) = V^2 M^{(d/2)-3}\Big(\frac{d}{2\pi}\Big)^{d/2}. \tag{19}$$

Theorem 1 says that $\frac{1}{2}\frac{d^2}{dt^2}E_\mu X^2(t) \sim At^{-(d/2)-1}$ as $t\to\infty$, with the constant A given by with $A = -\frac{1}{4}dV^2M^{(d/2)-3}\ (d/(2\pi))^{d/2}$. Thus (19) is of the same type as (6) (recall (3) in Lemma 1), except that the exponent is $\frac{1}{2}d+1$ instead of $\frac{1}{2}d$.

However, we have to be careful with concluding right away that (19) and (6) are equivalent. Namely, the problem is that $\{X(t) : t \geq 0\}$ does *not* have stationary increments under the law P_μ (which was needed in Lemma 1). To see why, let us introduce the *environment process* $\{W(t) : t \geq 0\}$ defined by

$$W(t) = \{w(x + X(t)) : x \in \mathbb{Z}^d\}, \tag{20}$$

i.e., the environment of waiting times as seen relative to the position of the walk. Now, $W(0)$ has law $\mu = \gamma^{\mathbb{Z}^d}$ but $W(t)$ for $t > 0$ has not, simply because $X(t)$ stays longer on sites with large $w(x)$ than on sites with small $w(x)$. Consequently, the environment process is not stationary under the law P_μ and so neither is the process of increments of the random walk. The way out of this dilemma is the following.

Lemma 3. *Let μ_0 be the law defined by*

$$\frac{d\mu_0}{d\mu}(\{w(x) : x \in \mathbb{Z}^d\}) = \frac{w(0)}{M}. \tag{21}$$

Then the environment process is stationary, ergodic and reversible under the law P_{μ_0}. Moreover, $E_{\mu_0}w^{-1}(X(t)) = M^{-1}$ and

$$M^{-1}\Delta(t) = E_{\mu_0}\Big([w^{-1}(X(0)) - M^{-1}][w^{-1}(X(t)) - M^{-1}]\Big). \tag{22}$$

The proof of (22) is based on the observation that $X^2(t) - \int_0^t w^{-1}(X(s))ds$ is a martingale under the law P_W for μ-a.s. all W. The reversibility property can be shown to imply that $t \to \Delta(t)$ is completely monotone (i.e., $\Delta(t)$ has a spectral representation of the form $\int_0^\infty e^{-\gamma t}\alpha(d\gamma)$ with α a positive measure).

The relation expressed by (22) should be viewed as the analogue of (3) in Lemma 1. The right-hand side of (22) is the autocorrelation function of $w^{-1}(X(t)) =$

$(W(t))^{-1}(0)$, which is stationary under the law P_{μ_0}. Therefore Theorem 1 can now be viewed as a long time tail in the proper sense of the word. The jump intensity $w^{-1}(X(t))$ for the random walk in our model is the analogue of the velocity $V(t)$ for the mechanical tagged particle in Section 1.

In the remaining part of this section we shall give a *sketch* of how Theorem 1 comes about, without going into the technical details of the proof.

The first step in the argument is a Feynman–Kac formula expressing the Laplace transform of $\Delta(t)$ in terms of simpler quantities. To write down this representation, let us define $\{\tilde{X}(t) : t \geq 0\}$ to be our process when $w(x) = 1$ for all x, i.e., simple random walk with jump rate 1 everywhere. Let $\tilde{E}$ denote expectation with respect to the law of this random walk and let

$$\ell_t(x) = \int_0^t 1_{\{\tilde{X}(s)=x\}} ds, \quad (t \geq 0, x \in \mathbb{Z}^d), \tag{23}$$

be the *local times*. Then one can show that

$$\int_0^\infty e^{-\lambda t} dE_W X^2(t) = \int_0^\infty dt \ \tilde{E}\Big(\exp[-\lambda \sum_x \ell_t(x) w(x)] \Big). \tag{24}$$

After integration over W (recall (18)) and insertion of the identity $\sum_x \ell_t(x) = t$, one arrives at the following expression.

Lemma 4. *For $\lambda > 0$*

$$\int_0^\infty e^{-\lambda t} \Delta(t) dt = -\frac{1}{\lambda M} + \int_0^\infty e^{-\lambda M t} \tilde{E}\Big(\prod_x \Gamma(\lambda \ell_t(x)) \Big) dt \tag{25}$$

with

$$\Gamma(\xi) = \int e^{-\xi[w-M]} \gamma(dw), \quad (\xi \geq 0). \tag{26}$$

The next step in the argument is to use Lemma 4 as the starting point for a Tauberian–Abelian analysis, i.e., to study the $\lambda \downarrow 0$ behavior of the right-hand side of (25) and from this deduce the $t \to \infty$ behavior of $\Delta(t)$ in the left-hand side of (25). To carry through this analysis we need a large deviation estimate for the local times of the following type: '$\sup_x \lambda \ell_t(x)$ is small for all t that make up the dominant contribution to the integral as $\lambda \downarrow 0$'. Namely, it is precisely under this condition that we can expand the integrand in the right-hand side of (25) for small λ. Proceeding *naively* with such a computation we have, by (26),

$$\Gamma(\lambda \ell_t(x)) = 1 + \tfrac{1}{2}\lambda^2 V^2 \ell_t^2(x) + \cdots \tag{27}$$

and hence

$$\begin{aligned} \tilde{E}\Big(\textstyle\prod_x \Gamma(\lambda \ell_t(x)) \Big) &= \tilde{E}\Big(1 + \tfrac{1}{2}\lambda^2 V^2 \textstyle\sum_x \ell_t^2(x) + \cdots \Big) \\ &= 1 + \tfrac{1}{2}\lambda^2 V^2 \textstyle\sum_x \tilde{E}\ell_t^2(x) + \cdots . \end{aligned} \tag{28}$$

Next we write, recalling (23),

$$\sum_x \tilde{E}\ell_t^2(x) = 2\int_0^t ds\ (t-s)\tilde{p}_s(0,0) \tag{29}$$

where $t \to \tilde{p}_t(\cdot,\cdot)$ is the transition kernel of $\{\tilde{X}(t) : t \geq 0\}$. Substituting (28) and (29) into (25) we get

$$\int_0^\infty dt\ e^{-\lambda t}\Delta(t) = \lambda^2 V^2 \int_0^\infty dt\ e^{-\lambda M t}\int_0^t ds\ (t-s)\tilde{p}_s(0,0) + \cdots. \tag{30}$$

The last step in the argument is to rewrite the right-hand side of (30) as

$$\begin{aligned} &\textstyle\int_0^\infty dt\ e^{-\lambda t}\zeta(t) + \cdots \\ &\zeta(t) = V^2 M^{-3}\tilde{p}_{t/M}(0,0). \end{aligned} \tag{31}$$

The latter form suggests that $\Delta(t) \sim \zeta(t)$ as $t \to \infty$. This can indeed be shown to be correct by appealing to the complete monotonicity of $t \to \Delta(t)$. Finally we substitute the standard local limit theorem

$$\tilde{p}_t(0,0) \sim \left(\frac{d}{2\pi t}\right)^{d/2}, \quad (t \to \infty), \tag{32}$$

to get the claim in (19).

4.2. Random Color Scenery

The model discussed in this section is of a somewhat different nature than the previous example, but we shall soon see why it blends in.

We begin by associating with each site of $\mathbb{Z}^d$ a color, drawn black or white. Namely, the random environment is an i.i.d. collection of random colors

$$C = \{c(x) : x \in \mathbb{Z}^d\}, \tag{33}$$

whose law $\mu = \gamma^{\mathbb{Z}^d}$ is parametrized by $q = \gamma(B) = 1 - \gamma(W) \in (0,1)$, the density of black sites. Next, we define $\{X(n) : n \geq 0\}$ to be discrete time simple random walk on $\mathbb{Z}^d$ $(X(0) = 0)$. We shall assume that walk and coloring are *independent*. However, they will be linked in an interesting way through the type of question that we shall ask.

Let T_k $(k \geq 0)$ be the successive random times at which the walk hits a black site, defined as

$$\begin{aligned} c(X(n)) &= B \quad n = T_0, T_1, \cdots \\ &= W \quad \text{otherwise.} \end{aligned} \tag{34}$$

Let n_k $(k \geq 0)$ be the interarrival times

$$\begin{aligned} n_0 &= T_0 \\ n_k &= T_k - T_{k-1}, \quad (k \geq 1). \end{aligned} \tag{35}$$

We shall write P_μ, E_μ to denote the joint law and expectation of walk and coloring.

It is not hard to prove that

$$E_\mu T_k \sim q^{-1}k, \quad (k \to \infty). \tag{36}$$

Indeed, over a large time interval of length n the walk will hit $\sim qn$ times a black site, so an average of q^{-1} steps is needed between black visits. With a little effort (36) can be turned into the stronger statement $E_\mu n_k \sim q^{-1}$, $(k \to \infty)$.

Theorem 2 below is a long time tail for the quantity

$$\Delta_k = E_\mu n_k - q^{-1}. \tag{37}$$

Theorem 2. *For any* $q \in (0,1)$

$$\lim_{k\to\infty} k^{d/2}\Delta_k = (1-q)q^{(d/2)-2}\Big(\frac{d}{2\pi}\Big)^{d/2}. \tag{38}$$

To see the link between Theorems 1 and 2, note that $X^2(n) - n$ is a martingale, so that

$$E_\mu X^2(T_k) = E_\mu T_k. \tag{39}$$

Since, by (35) and (37), we have $\delta_k\Delta_k = \delta_k^2 E_\mu T_k$ with δ_k the forward difference operator, Theorem 2 says that $\frac{1}{2}\delta_k^2 E_\mu X^2(T_k) \sim Ak^{-(d/2)-1}(k \to \infty)$ with $A = -\frac{1}{4}d(1-q)q^{(d/2)-2}\ (d/(2\pi))^{d/2}$. Thus (38) is of the same type as (6) (again recall (3) in Lemma 1), except that the exponent is $\frac{1}{2}d+1$ instead of $\frac{1}{2}d$. The difference with the example in Section 4.1 is that we are observing the mean-square displacement not along the full time scale but *along the random time scale* $\{T_k : k \geq 0\}$.

Again, we have to be careful calling (38) and (6) equivalent. The point this time is that $\{X(T_k) : k \geq 1\}$ does *not* have stationary increments under the law P_μ. This is due to a renewal effect. If 0 happens to lie in a big white hole, then the T_k's will be larger than on average, while just the opposite will be true when 0 is packed between black colors. The net result is that the increments $n_k = T_k - T_{k-1}$, $(k \geq 1)$, are not stationary.

The way out is similar to Lemma 3. Define the environment process $\{C(n) : n \geq 0\}$ by putting

$$C(n) = \{c(x + X(n)) : x \in \mathbb{Z}^d\}, \tag{40}$$

which is the color scenery seen by the walk.

Lemma 5. *Let* μ_0 *be the law defined by*

$$\mu_0(\cdot) = \mu(\cdot \mid c(0) = B). \tag{41}$$

Then the environment process is stationary, ergodic and reversible under the law P_{μ_0}[9]. *Moreover,* $E_{\mu_0} n_k = q^{-1}$ *and*

$$q^{-1}\Delta_k = E_{\mu_0}\Big([n_1 - q^{-1}][n_{k+1} - q^{-1}]\Big). \tag{42}$$

[9] The idea of conditioning on a black origin can be traced back to Kac (1947) and $E_{\mu_0} n_k = q^{-1}$ is a version of the Kac recurrence theorem (see Kasteleyn 1987).

The relation expressed by (42) should again be viewed as the analogue of (3) in Lemma 1. The right-hand side of (42) is the autocorrelation function of n_k, which is stationary under the law P_{μ_0}. Therefore Theorem 2 is an authentic long time tail.

Apparently the analogue of the velocity is n_k. One can indeed make sense out of this by observing that the n_k's are the lengths of the successive pieces of the walk measured along the random time scale $\{T_k : k \geq 1\}$. Admittedly, this analogue is somewhat artificial, but the connection is nevertheless legitimate.

We close this section by writing down the analogue of Lemma 4. Recalling (35), (37) and Lemma 5, we have

$$\begin{aligned} E_\mu n_0 + \textstyle\sum_{k\geq 1} z^k \Delta_k &= \textstyle\sum_{k\geq 0} z^k [E_\mu T_k - E_{\mu_0} T_k] \\ &= (1-q) \textstyle\sum_{k\geq 0} z^k [E_{\mu_1} T_k - E_{\mu_0} T_k] \end{aligned} \tag{43}$$

where $\mu_1(\cdot) = \mu(\cdot \,|\, c(0) = W)$. Now let

$$\ell_n(x) = \sum_{m=0}^{n} 1_{\{X(m)=x\}}, \quad (n \geq 0, x \in \mathbb{Z}^d), \tag{44}$$

and let $\tilde{E}$ denote expectation with respect to the law of the random walk. Then, thinking of z as a counting factor for each time the walk visits a black site, one can derive the following expression.

Lemma 6. *For* $z \in (-1, 1)$

$$E_\mu n_0 + \sum_{k\geq 1} z^k \Delta_k = (1-q)\tilde{E}\Big([1 - z^{\ell_n(0)}] \prod_{x\neq 0} [1 - q + q z^{\ell_n(x)}]\Big). \tag{45}$$

Equation (45) serves as the starting point for a Tauberian–Abelian analysis, just as in Section 4.1. Again the argument relies on the Δ_k having a spectral integral representation. The reader is invited to try and check how Theorem 2 comes out of a naive expansion of the right-hand side of (45) for $z \uparrow 1$.

5. Concluding Remarks

It is a somewhat frustrating state of affairs that long time tails are so common and yet are so hard to prove mathematically. Even for stochastic Lorentz models, which are caricatures of reality, there is no satisfactory theory. The reason is that long time tails are closely related to local limit theorems, which are known to be much more difficult to get at than global limit theorems.

We conclude this paper with a few remarks.

(1) For the mechanical tagged particle in Sections 1 and 2 the long time tail comes from diffusion of *density and momentum* (see (11)). The exponent is $\frac{1}{2}d$ and the amplitude is *positive*. For the stochastic tagged particle in Sections 4.1 and 4.2, on the other hand, the long time tail comes from diffusion of *density only* (the stochastic motion does not conserve momentum). The exponent is $\frac{1}{2}d + 1$, showing that the tail is weaker, and the amplitude is *negative*. Unlike the 'kick in the back' felt by the mechanical tagged particle from the vortex flow it creates around itself, the stochastic

tagged particle remembers its initial velocity because it is more likely to return to the origin from the same side it left the origin, so moving in the opposite direction. (This statement must be read with the appropriate interpretation of 'velocity'.)
(2) Bricmont and Kupiainen (1991a, b) have recently studied the asymptotics of a nearest-neighbor random walk on $\mathbb{Z}^d$ where the transition probabilities are a *small random perturbation* of those of simple random walk. They show, using a rigorous renormalization technique, that for $d > 2$ (and under suitable conditions on the perturbation) the motion is asymptotically diffusive with diffusion constant $D \in (0, \infty)$. Their analysis indicates that the velocity autocorrelation function falls off with an exponent $\frac{1}{2}d$, but the proof has not been worked out. This speculation shows that this model is closer to that of a real fluid. The reason is not understood.
(3) The derivation of the long time tails in Theorems 1 and 2 makes heavy use of reversibility, via a spectral integral representation used in the Tauberian–Abelian analysis. Without this representation it seems hard to get the precise asymptotics.
(4) What does the long time tail look like *without* taking the expectation over the law of the random medium? This question, which was first raised by Sinai (see van Beijeren 1982), is addressed in den Hollander et al. (1992c) for the model in Section 4.1. It is shown that $\frac{d}{dt}E_W X^2(t) - M^{-1}$ falls off like $Zt^{-d/4}$ with a *random amplitude*, namely, Z is a Gaussian random variable with mean zero and variance $V^2 M^{(d/2)-4}(d/(4\pi))^{d/2}$. Thus the medium causes fluctuations in the long time tail which dominate the mean behavior. Such fluctuations can cause trouble in simulations and experiments. Namely, although it is true that one often effectively measures some space-average of the tail, the fluctuations may lead to significant errors, particularly because the amplitudes are typically small.

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