PERCOLATIVE PROBLEMS

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Abstract. We sketch elementary results and open problems in the theory of percolation and random-cluster models. The presentation is rather selective, and is intended to stimulate interest rather than to survey the established theory. In the case of the random-cluster model, we include sketch proofs of basic material such as the FKG inequality and the comparison inequalities.

Key words: Percolation, random-cluster model, Potts model, phase transition, FKG inequality.

1. Introduction

This paper falls naturally into two (related) halves. The first of these is concerned with the percolation model, and the second with the random-cluster model. The emphasis throughout is upon unsolved problems which are easy to state; some of these are chestnuts of varying ages, and some are recent and may be relatively tractable.

The percolation model is the subject of Sections 2–4, the last of which contains a selection of open questions. In Section 5 we turn to the random-cluster model of Fortuin and Kasteleyn, and for this process we present and prove several of the basic properties in advance of describing in Section 9 some stimulating problems worthy of resolution.

2. Bond Percolation

Our lattice is the hypercubic lattice $\mathbb{L}^d$, having vertex set $\mathbb{Z}^d$ and edge set $\mathbb{E}^d$ containing all pairs $\langle x, y\rangle$ whose L^1 distance

$$\|y-x\| = \sum_{i=1}^{d} |y_i - x_i|$$

satisfies $\|y-x\| = 1$; for $z \in \mathbb{Z}^d$, we write $z = (z_1, z_2, \dots, z_d)$. Throughout we shall assume that $d \geq 2$.

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Let $0 \le p \le 1$, and call an edge e ($\in \mathbb{E}^d$) *open* with probability p, and *closed* otherwise; different edges are designated open or closed independently of one another. Consider the random subgraph of $\mathbb{L}^d$ containing the vertex set $\mathbb{Z}^d$ and the open edges only. The connected components of this graph are called *open clusters*, and percolation theory is concerned with their sizes and geometry. We write $C(x)$ for the open cluster containing the vertex x, and $C = C(0)$ for the cluster containing the origin 0. The number of vertices in $C(x)$ is denoted by $|C(x)|$.

There is a 'phase transition', in the following sense. Define the *percolation probability*

$$\theta(p) = \mathbf{P}_p(|C| = \infty), \tag{2.1}$$

where $\mathbf{P}_p$ is the associated probability measure, and define the *critical probability* by

$$p_c = \sup\{p : \theta(p) = 0\}. \tag{2.2}$$

It is a fundamental fact that $0 < p_c < 1$. The value p_c marks a transition from a subcritical phase (when $p < p_c$, and all open clusters are a.s. finite) to a supercritical phase (when $p > p_c$, and there exists a.s. an infinite open cluster). The most basic problem is to understand the nature of the singularity of the process at the point of phase transition.

Rather than attempt an accurate bibliography, the reader is referred to [22] for history and references prior to 1989.

3. Some Open Problems for Percolation

3.1. BK Inequality

'Correlation inequalities' play an important role in studying percolation, and the FKG and BK inequalities are fundamental techniques. Whereas the FKG inequality is rather well understood, there are interesting unresolved questions concerning the BK inequality.

Consider the probability space $(\Omega, \mathcal{F}, \mu)$ where $\Omega = \{0,1\}^E$, E being a finite set, $\mathcal{F}$ is the σ-field of all subsets of Ω, and μ is product measure with density p, i.e.,

$$\mu(\omega) = \prod_{e \in E} \{p^{\omega(e)}(1-p)^{1-\omega(e)}\}, \quad \omega = (\omega(e) : e \in E) \in \Omega. \tag{3.1}$$

There is a natural partial order on Ω given by $\omega \le \omega'$ if and only if $\omega(e) \le \omega'(e)$ for all $e \in E$. An event A in $\mathcal{F}$ is called *increasing* if its indicator function I_A is an increasing function on the partially ordered space $(\Omega, \le)$. The FKG inequality (see [21, 28] and Section 7) states that

$$\mu(A \cap B) \ge \mu(A)\mu(B) \quad \text{for increasing events } A, B. \tag{3.2}$$

The BK inequality provides a converse relation, but with $A \cap B$ replaced by another event $A \circ B$ defined as follows. Let A and B be increasing events. Each ω ($\in \Omega$) is specified uniquely by the set $K(\omega) = \{e : \omega(e) = 1\}$ of edges with state 1. We define $A \circ B$ to be the set of all ω for which there exists a subset H of

$K(\omega)$ such that ω', determined by $K(\omega') = H$, belongs to A, and w'', determined by $K(\omega'') = K(\omega) \setminus H$, belongs to B. We speak of $A \circ B$ as the event that A and B occur disjointly. The BK inequality ([9]) states that

$$\mu(A \circ B) \leq \mu(A)\mu(B) \quad \text{for increasing events } A, B. \tag{3.3}$$

It is conjectured that such an inequality is valid *for all* events A and B, so long as $A \circ B$ is interpreted correctly. For general events A and B, we define the event $A \,\square\, B$ as follows. For $\omega = (\omega(e) : e \in E)$ and $K \subseteq E$, we define the cylinder event $C(\omega, K)$ by

$$C(\omega, K) = \{\omega' \in \Omega : \omega'(e) = \omega(e) \text{ for } e \in K\}.$$

We now define $A \,\square\, B$ to be the set of all ω ($\in \Omega$) for which there exists $K \subseteq E$ such that $C(\omega, K) \subseteq A$ and $C(\omega, E \setminus K) \subseteq B$.

Conjecture 3.1. *For all events A and B,*

$$\mu(A \,\square\, B) \leq \mu(A)\mu(B). \tag{3.4}$$

This conjecture has as special cases both the FKG and BK inequalities, since $A \,\square\, B = A \circ B$ and $A \,\square\, B^c = A \cap B^c$ for increasing events A and B.

In the case $p = \frac{1}{2}$, the conjecture reduces to a counting problem: prove that

$$2^{|E|}|A \,\square\, B| \leq |A| \cdot |B| \quad \text{for all events } A, B. \tag{3.5}$$

In a discussion [8] of partial results, it is proved that (3.5) would imply the full conjecture.

Finally we ask *for what* probability measures μ is the BK inequality (3.3) valid? For example, is it valid for the measure which assigns probability $\binom{|E|}{M}^{-1}$ to each of the sequences ω containing exactly M ones and $|E| - M$ zeros, where M is fixed?

3.2. Smoothness of Percolation Probability

It is known that $\theta(p) = 0$ for $p < p_c$ (by definition) and that θ is infinitely differentiable when $p > p_c$. It is a major open problem to prove that $\theta(p_c) = 0$, which is equivalent (via the right continuity of θ) to the statement

$$\theta \text{ is continuous at } p_c. \tag{3.6}$$

This problem has been settled affirmatively when $d = 2$, and for the following discussion we assume that $d \geq 3$.

It is known that the critical probability p_c of $\mathbb{Z}^d$ is the same as the critical probability $p_c(H)$ of the half-space $H = \mathbb{Z}_+ \times \mathbb{Z}^{d-1}$ ([26]). Furthermore, we know (see [6]) that H contains no infinite cluster when $p = p_c$. It is therefore required to rule out the following outlandish possibility: when $p = p_c$ there exists a.s. an infinite open cluster, but this cluster decomposes a.s. into finite clusters whenever $\mathbb{Z}^d$ is sliced into two disjoint half-spaces.

3.3. Uniqueness of Infinite Cluster

Let N be the number of infinite open clusters. Then, for all p,

$$\text{either } \mathbf{P}_p(N=0)=1 \text{ or } \mathbf{P}_p(N=1)=1\,,$$

and the easiest proof of this may be found in [13]. It has been asked by Mathew Penrose whether N has such a property *simultaneously for all values of* p.

In order to make sense of this question, we introduce a family $(X(e) : e \in \mathbb{E}^d)$ of independent random variables each having the uniform distribution on $[0,1]$. For $0 \le p \le 1$ we define the vector η_p by

$$\eta_p(e) = \begin{cases} 1 & \text{if } X(e) < p, \\ 0 & \text{if } X(e) \ge p, \end{cases} \tag{3.7}$$

and note that $\mathbf{P}(\eta_p = 1) = p$. We call an edge e p-*open* if $\eta_p(e) = 1$, and p-*closed* otherwise. Let N_p be the number of infinite p-open clusters. Is it the case that[1]

$$\mathbf{P}\big(N_p \in \{0,1\} \text{ for all } p\big) = 1? \tag{3.8}$$

This is certainly valid when $d = 2$.

3.4. Critical Exponents

There is a wealth of problems concerning critical exponents and scaling theory, and these have received ample attention elsewhere (see [22, Chaps. 7, 8] for example). We confine ourselves here to a few very basic examples of such problems. Those mentioned here are intended primarily to stimulate interest in the major challenge to mathematicians *to make sense of scaling theory*.

It is thought to be the case that $\theta(p)$ behaves in the manner of $|p-p_c|^\beta$ in the limit as $p \downarrow p_c$, where β is a 'critical exponent' whose value depends on the number of dimensions. No proof is known that

$$\beta = \lim_{p \downarrow p_c} \left\{ \frac{\log \theta(p)}{\log |p-p_c|} \right\} \tag{3.9}$$

exists. Possibly

$$a(p)|p-p_c|^\beta \le \theta(p) \le b(p)|p-p_c|^\beta \quad \text{for } p > p_c \tag{3.10}$$

for some functions a and b which are slowly varying as $p \downarrow p_c$. There are corresponding conjectures for other macroscopic functions.

The value of $\beta = \beta(d)$ should depend on d, and it is conjectured that

$$\beta(2) = \tfrac{5}{36}, \quad \beta(d) = 1 \quad \text{for } d \ge 6. \tag{3.11}$$

This is part of a large family of conjectures dealing with the cases $d = 2$ and $d \ge 6$. When $d = 2$, it is thought that all critical exponents are rational. When $d \ge 6$ it is

[1] This question was answered affirmatively by Ken Alexander during the meeting.

thought that any given exponent takes its 'mean-field value', i.e., the value obtained when the lattice is replaced by an infinite regular tree. See [27, 34, 38].

The first 'proof' that $p_c = \frac{1}{2}$ when $d = 2$ utilized the self-duality of the square lattice. Sykes and Essam [42] established the relation

$$\kappa(p) = \kappa(1-p) + 1 - 2p \tag{3.12}$$

where $\kappa(p) = \mathbf{E}_p(|C|^{-1})$ and $\mathbf{E}_p$ is the expectation operator corresponding to $\mathbf{P}_p$. *Assuming that κ has a unique singularity*, and that this is at the point p_c, then it follows from (3.12) that $p_c = 1 - p_c$ and hence $p_c = \frac{1}{2}$. Alternative proofs that $p_c = \frac{1}{2}$ are now available (see [22, 33]). However it is not ruled out that κ is infinitely differentiable on $[0, 1]$. It may be conjectured that κ is twice but not thrice differentiable at p_c.

4.1. Related Problems

4.1. Wind-Tree Problem

Versions of the wind-tree problem have been discussed by Lorenz, Ehrenfest [16], and Hauge and Cohen [29]. The following version is close to percolation theory. We start with the square lattice $\mathbb{L}^2$, a bucket of small double-sided mirrors, and a parameter p taking values in $[0, 1]$. For each vertex x we perform the following experiment. With probability p, we pick a mirror from the bucket and place it at the vertex x, in such a way that a ray of light arriving at x, parallel to any coordinate axis, is reflected through either $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ (measured clockwise), each possibility having probability $\frac{1}{2}$. The remaining probability $1-p$ is the chance that we do nothing at x. We think of the mirrors as being random scatterers of light.

How many vertices can see the origin? More precisely, we suppose that four rays of light are emitted along the coordinate axes from a light source placed at the origin. Let C be the set of vertices which are illuminated by one or more of these light rays, and let $\theta(p)$ be the probability that C is infinite. Clearly $\theta(0) = 1$, and it is straightforward to see that $\theta(1) = 0$, using a standard result for bond percolation on $\mathbb{L}^2$. Let

$$p_c(\mathrm{WT}) = \sup\{p : \theta(p) > 0\}. \tag{4.1}$$

Is it the case that $0 < p_c(\mathrm{WT}) < 1$?

4.2. Random Orientations

Here is a small problem in two dimensions. Each edge of $\mathbb{L}^2$ is oriented in a random direction, horizontal edges being oriented eastwards with probability p and westwards otherwise, and vertical edges being oriented northwards with probability p and southwards otherwise. Let $\eta(p)$ be the probability that there exists an infinite oriented path starting at the origin. It is not hard to see that $\eta(\frac{1}{2}) = 0$, and also that $\eta(p) = \eta(1-p)$. Is it the case that $\eta(p) > 0$ if $p \neq \frac{1}{2}$?

4.3. Uniqueness for Minimal Spanning Trees

The following question concerning 'continuous percolation' has been posed in [5]. Let $\mathbf{X} = (X_i)$ be the set of points of a Poisson process in $\mathbb{R}^d$ with intensity 1, where

$d \geq 2$. We construct a spanning forest on $\mathbf{X}$ in the following way. For each $X \in \mathbf{X}$ we define trees $t_m(X, \mathbf{X})$, $m \geq 0$. Let $\xi_1 = X$ and let t_1 be the single vertex ξ_1. Let t_2 be the tree consisting of the vertex ξ_1 together with the vertex ξ_2 ($\in \mathbf{X} \setminus \{\xi_1\}$) which is closest to ξ_1, these two vertices being joined by an edge. Having constructed t_{m-1}, we define $t_m = t_m(X, \mathbf{X})$ by adding to t_{m-1} a new edge $\langle \xi_{i_m}, \xi_m \rangle$ where $1 \leq i_m < m$ and ξ_m ($\in \mathbf{X} \setminus \{\xi_1, \xi_2, \dots, \xi_{m-1}\}$) is chosen so that the Euclidean distance $|\xi_{i_m} - \xi_m|$ is minimal over all possible edges joining t_{m-1} to $\mathbf{X} \setminus \{\xi_1, \xi_2, \dots, \xi_{m-1}\}$. Finally we set $t(X, \mathbf{X}) = \bigcup_{m=1}^{\infty} t_m(X, \mathbf{X})$.

Each point X gives rise to an infinite tree $t(X, \mathbf{X})$. We now use these trees to make a forest. Let F be the graph with vertex set $\mathbf{X}$, and which has each $\langle X_i, X_j \rangle$ as an edge if and only if it is in either $t(X_i, \mathbf{X})$ or $t(X_j, \mathbf{X})$. It may be seen that F is a forest, every component of which is an infinite tree.

Aldous and Steele conjecture that F is a.s. a tree, which is to say that F is a.s. connected. This tempting conjecture might be related to the uniqueness of the infinite open cluster of percolation.

4.4. Collisions of Random Walks

The following problem arises in the study of collisions of random walks (see [14, 43]). Let k be a positive integer. Let $(X_i, Y_i : i \geq 0)$ be independent random variables, each being equally likely to take any of the values $1, 2, \dots, k$. We declare the point (i, j) of $\mathbb{Z}^2$ *open* if $X_i \neq Y_j$. Let $\theta(k)$ be the probability that there is an infinite open path of $\mathbb{L}^2$ beginning at the origin, each edge of which leads the path either northwards or eastwards away from the origin. It may be shown that $\theta(3) = 0$. Is it true that $\theta(k) > 0$ for large k, perhaps for $k = 4$?

5. The Random-Cluster Model

The random-cluster model is a family of processes which includes percolation, the Ising and Potts models, and related systems. Its discovery was reported by Fortuin and Kasteleyn in a series of papers [17, 18, 19, 20, 32] published around 1970, and it has excited considerable interest recently.

Here are brief descriptions of the Potts and random-cluster models. We start with a finite graph $G = (V, E)$. The Potts model has sample space $\Sigma_V = \{1, 2, \dots, q\}^V$, where q is an integer satisfying $q \geq 2$. A spin vector σ in this sample space has probability

$$\pi(\sigma) = \frac{1}{Z} \exp\left(-J \sum_{e \in E} \big(1 - \delta_\sigma(e)\big) \right), \quad \text{for } \sigma \in \Sigma_V, \tag{5.1}$$

where $\delta_\sigma(e)$ is the indicator function of the event that the endpoints of e have the same spin (see (6.3)), and Z is the normalizing factor. The parameter J describes the strength of pair-interactions. The random-cluster model is a (random) subgraph (V, F) of G, the edge set F being chosen at random according to the probability mass function

$$\phi(F) = \frac{1}{Z} p^{|F|} (1-p)^{|E \setminus F|} q^{k(F)}, \quad \text{for } F \subseteq E, \tag{5.2}$$

where $k(F)$ is the number of components of (V, F). Here p and q satisfy $0 \leq p \leq 1$

and $q > 0$. The main observation is that the structures of π and ϕ are closely related, when the parameters J and p satisfy $e^{-J} = 1 - p$. Since $0 \le p \le 1$, this requires $J \ge 0$, which is to say that the Potts model must be *ferromagnetic*. [If $J < 0$ then $p < 0$, and (5.2) defines a signed measure but not a probability measure.]

The random-cluster measures (5.2) form a richer family than the (ferromagnetic) Potts measures, since they are well defined for all *real* positive q. There are many techniques which bear on the study of the random-cluster model. Some of these are valid for all q, others for $q \ge 1$, others for sufficiently large q, and others for integral values of q. To develop a coherent and cohesive theory of this model is a target of substantial appeal.

We pursue two targets in the rest of this paper. In Sections 6–8, we summarize some basic properties of random-cluster models; this material is well known and has appeared elsewhere (see [4, 15] and the original papers of Fortuin and Kasteleyn). Finally, in Section 9 we highlight open problems.

6. Potts and Random-Cluster Processes

Potts and random-cluster processes may be viewed as the two marginal models obtained in the construction of a certain bivariate model; this was discovered by Edwards and Sokal [15].

Let $G = (V, E)$ be a finite connected graph with no loops or multiple edges. We write $u \sim v$ whenever the two vertices u and v of G are adjacent; in this case the edge joining u to v is denoted by $\langle u, v\rangle$.

Let q be an integer satisfying $q \ge 2$. Potts models have realizations in the set $\Sigma_V = \{1, 2, \dots, q\}^V$ of 'spin vectors'; a typical realization is an assignment $\sigma = (\sigma(u) : u \in V)$ of an integer from $\{1, 2, \dots, q\}$ to each vertex. A Potts model with $q = 2$ is called an Ising model [31]. Random-cluster processes have realizations in the set $\Omega_E = \{0, 1\}^E$ of 'edge-configurations'. A typical realization is a vector $\omega = (\omega(e) : e \in E)$ of 0's and 1's. Instead of working with the vector ω, it is often convenient to work with the set

$$\eta(\omega) = \{e \in E : \omega(e) = 1\} \tag{6.1}$$

of 'open' edges.

The two processes referred to above may be constructed on the same sample space $\Sigma_V \times \Omega_E$ as follows. Let p satisfy $0 \le p \le 1$, and define the probability mass function μ on $\Sigma_V \times \Omega_E$ by

$$\mu(\sigma, \omega) = \frac{1}{Z} \prod_{e \in E} \Big\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_\sigma(e)\Big\}, \tag{6.2}$$

where Z is the appropriate normalizing constant, $\delta_{i,j}$ is the Kronecker delta, and $\delta_\sigma(e)$ is given by

$$\delta_\sigma(e) = \begin{cases} 1 & \text{if } \sigma(u) = \sigma(v), \text{ where } e = \langle u, v\rangle, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Let us calculate the marginal measures of μ. Summing over all $\omega \in \Omega_E$, we obtain the marginal mass function $\pi(\sigma)$ on Σ_V:

$$\begin{aligned}\pi(\sigma) &= \frac{1}{Z}\prod_{e\in E}\Bigg[\sum_{\omega(e)=0,1}\Big\{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_\sigma(e)\Big\}\Bigg] \\ &= \frac{1}{Z}\prod_{e\in E}\{(1-p)+p\delta_\sigma(e)\} = \frac{1}{Z}\prod_{e\in E}\exp\{-J(1-\delta_\sigma(e))\}\end{aligned} \tag{6.4}$$

where J is given by

$$e^{-J} = 1-p \tag{6.5}$$

and satisfies $0 \le J \le \infty$. The mass function π on Σ_V is therefore the Potts measure [41, 46]. The letter π stands for 'Potts'.

In order to calculate the marginal mass function on Ω_E, we rewrite $\mu(\sigma,\omega)$ as

$$\begin{aligned}\mu(\sigma,w) &= \frac{1}{Z}\Bigg\{\prod_{e:\omega(e)=1} p\delta_\sigma(e)\Bigg\}\Bigg\{\prod_{e:\omega(e)=0}(1-p)\Bigg\} \\ &= \frac{1}{Z}\,p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}I(\sigma,\omega)\end{aligned} \tag{6.6}$$

where

$$I(\sigma,\omega) = \prod_{e:\omega(e)=1}\delta_\sigma(e) = \prod_{e\in\eta(\omega)}\delta_\sigma(e)$$

is the indicator function of the event that σ assigns a constant spin to all vertices in any given component of the graph $(V,\eta(\omega))$. Summing (6.6) over all $\sigma \in \Sigma_V$, we obtain the marginal mass function ϕ on Ω_E given by

$$\phi(\omega) = \frac{1}{Z}\,p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}q^{k(\omega)}, \tag{6.7}$$

where $k(\omega)$ is the number of components of $(V,\eta(\omega))$; this holds since there are q admissible spin values for each such component. The letter ϕ in (6.7) stands for 'Fortuin–Kasteleyn'. The form of ϕ is particularly attractive for at least two reasons. First, the formula (6.7) may be used to define a probability measure on Ω_E *for any positive value of* q; thus, random-cluster processes provide an interpolation of Potts models to non-integral values of q. Secondly, setting $q = 1$ we obtain the usual bond percolation model, in which edges are 'open' or 'closed' independently of one another.

Suppose we are studying the Potts model, and are interested in some 'observable' $f : \Sigma_V \to \mathbb{R}$; a particular example of interest is the 'two-point function' $\delta_{\sigma(u),\sigma(v)}$ for given $u, v \in V$. The mean value of $f(\sigma)$ satisfies

$$\begin{aligned}\mathbf{E}_\pi(f) &= \sum_\sigma f(\sigma)\pi(\sigma) = \sum_{\sigma,\omega} f(\sigma)\mu(\sigma,\omega) \\ &= \sum_\omega F(\omega)\phi(\omega) = \mathbf{E}_\phi(F)\end{aligned} \tag{6.8}$$

where $F : \Omega_E \to \mathbb{R}$ is given by

$$F(\omega) = \sum_{\sigma} f(\sigma)\mu(\sigma \mid \omega) \tag{6.9}$$

and $\mathbf{E}_\pi$ and $\mathbf{E}_\phi$ denote expectation with respect to the appropriate measure. This piece of formalism, $\mathbf{E}_\pi(f) = \mathbf{E}_\phi(F)$, has substantial value in practice. To see this, first let us calculate the conditional mass function $\mu(\sigma \mid \omega)$. By (6.6) and (6.7),

$$\mu(\sigma \mid \omega) = q^{-k(\omega)} I(\sigma, \omega), \tag{6.10}$$

which may be expressed as follows. Conditional on ω, we assign a constant spin to all vertices in any given component of $(V, \eta(\omega))$; such spins are equally likely to take any value $1, 2, \dots, q$, and the spins assigned to different components are independent.

As a major application of (6.9), define $f : \Sigma_V \to \mathbb{R}$ by

$$f(\sigma) = \delta_{\sigma(u),\sigma(v)} - \frac{1}{q}\,,$$

where u and v are two fixed vertices; the term q^{-1} is the probability that two independent and equidistributed spins are equal. It follows from (6.9) and (6.10) that

$$\mathbf{E}_\pi(f) = \mathbf{E}_\phi\big((1 - q^{-1}) I_{\{u \leftrightarrow v\}}\big) = (1 - q^{-1})\phi(u \leftrightarrow v), \tag{6.11}$$

where I_F denotes the indicator function of an event F $(\subseteq \Omega_E)$, and we write $\{A \leftrightarrow B\}$ for the event there exist $a \in A$ $(\subseteq V)$ and $b \in B$ $(\subseteq V)$ such that a and b are in the same component of $(V, \eta(\omega))$. Equation (6.11) tells us that the two-point correlation function of the Potts model equals (apart from a constant factor) the probability of a certain connection in the random-cluster process. Thus, questions of correlation structure of Potts models become questions of stochastic geometry of the random-cluster process.

7. Useful Properties

This section contains an account of some of the useful properties of the random-cluster measure. Most useful is the material of Sections 7.2 and 7.3, which appeared in the original work of Fortuin and Kasteleyn as well as in [4].

As before, $G = (V, E)$ is a finite simple graph, $\Omega_E = \{0, 1\}^E$, $0 \le p \le 1$, and $q > 0$. The mass function in question is

$$\phi_{p,q}(\omega) = \frac{1}{Z_{p,q}} \left\{ \prod_{e \in E} p^{\omega(e)}(1 - p)^{1-\omega(e)} \right\} q^{k(\omega)}, \quad \omega \in \Omega_E, \tag{7.1}$$

where

$$Z_{p,q} = \sum_{\omega \in \Omega_E} \left\{ \prod_{e \in E} p^{\omega(e)}(1 - p)^{1-\omega(e)} \right\} q^{k(\omega)} \tag{7.2}$$

is the normalizing factor, or 'partition function'. We write $\phi_{p,q}$ here to emphasize the role of the parameters.

7.1. The Value of q

Whereas the Potts model may be defined for integer values of q only, the random-cluster measure (7.1) is well defined for all non-negative real values of q. Therefore, the random-cluster measures enable an interpolation of Potts models to general values of q ($\in (0,\infty)$). Indeed in the context of signed measures, $\phi_{p,q}$ may be defined even for negative values of q. Henceforth we assume that $q \in (0,\infty)$.

Professor Kasteleyn has pointed out that the random-cluster model is more general than the Potts model in the following additional regard. We saw at (6.8) that, for every function f of the Potts model, there exists a corresponding function F of the associated random-cluster process, and furthermore F does not depend on the value of p (but only on q). The converse is false: in general there may exist functions F with no corresponding f *independent of* J.

7.2. FKG Inequality

The measure $\phi_{p,q}$ satisfies the FKG inequality if and only if $q \geq 1$. This fact is not difficult to prove, and has many applications. Possibly as a result of this fact, there appears to have been no serious study of the case $0 < q < 1$. Before stating this inequality, we recall some notation.

A function $f : \Omega_E \to \mathbb{R}$ is called *increasing* if $f(\omega) \leq f(\omega')$ whenever $\omega \leq \omega'$; f is *decreasing* if $-f$ is increasing. An event F ($\subseteq \Omega_E$) is called *increasing* (respectively *decreasing*) if its indicator function I_F is increasing (respectively decreasing). Finally, we write $\mathbf{E}_{p,q}$ for expectation with respect to $\phi_{p,q}$.

Theorem 7.1 (FKG inequality). *Suppose that $q \geq 1$. If f and g are increasing functions on Ω_E, then*

$$\mathbf{E}_{p,q}(fg) \geq \mathbf{E}_{p,q}(f)\mathbf{E}_{p,q}(g)\,. \tag{7.3}$$

Replacing f and g by $-f$ and $-g$, we deduce that (7.3) holds for decreasing f and g, also. Specializing to indicator functions, we obtain that

$$\phi_{p,q}(A \cap B) \geq \phi_{p,q}(A)\phi_{p,q}(B) \quad \text{for increasing events } A, B, \tag{7.4}$$

whenever $q \geq 1$. It is not difficult to see that the FKG inequality does not generally hold when $0 < q < 1$.

Proof. A mass function μ on Ω_E satisfies the FKG inequality if ([21])

$$\mu(\omega \vee \omega')\mu(\omega \wedge \omega') \geq \mu(\omega)\mu(\omega') \quad \text{for all } \omega, \omega' \in \Omega_E, \tag{7.5}$$

where $\omega \vee \omega'$ and $\omega \wedge \omega'$ are the pointwise maximum and pointwise minimum configurations,

$$\omega \vee \omega'(e) = \max\{\omega(e), \omega'(e)\}, \quad \omega \wedge \omega'(e) = \min\{\omega(e), \omega'(e)\}\,;$$

note that

$$\eta(\omega \vee \omega') = \eta(\omega) \cup \eta(\omega'), \quad \eta(\omega \wedge \omega') = \eta(\omega) \cap \eta(\omega').$$

Substituting $\mu = \phi_{p,q}$, we see that (7.5) is equivalent to

$$k(\omega \vee \omega') + k(\omega \wedge \omega') \geq k(\omega) + k(\omega') \quad \text{for all } \omega, \omega', \tag{7.6}$$

so long as $q \geq 1$. Assume henceforth that $q \geq 1$. Inequality (7.6) is easily proved by induction on $|\eta(\omega) \cup \eta(\omega')|$, and the rest of the proof may be skipped. Inequality (7.6) is trivially true if $\eta(\omega) \cup \eta(\omega') = \varnothing$. Suppose it is valid for $|\eta(\omega) \cup \eta(\omega')| \leq k$. Let ω, ω' satisfy $|\eta(\omega) \cup \eta(\omega')| = k+1$; we may assume $\omega \neq \omega'$, since (7.6) is trivial otherwise. Without loss of generality we may assume that there exists $e \in \eta(\omega) \setminus \eta(\omega')$, and we write ω_e for the configuration ω with e 'switched off', i.e.,

$$\omega_e(f) = \begin{cases} \omega(f) & \text{if } f \neq e \\ 0 & \text{if } f = e. \end{cases} \tag{7.7}$$

From the induction hypothesis,

$$k(\omega_e \vee \omega') + k(\omega_e \wedge \omega') \geq k(\omega_e) + k(\omega'). \tag{7.8}$$

Write C_e for the indicator function of the event that the endpoints of e are in the same component. Trivially,

$$C_e(\omega_e \vee \omega') \geq C_e(\omega_e), \tag{7.9}$$

since $\omega_e \leq (\omega_e \vee \omega')$. Adding (7.8) and (7.9), we obtain (7.6), on noting that

$$k(\nu_e) + C_e(\nu_e) = k(\nu) + 1 \quad \text{for } \nu \ (\in \Omega_E) \text{ satisfying } \nu(e) = 1,$$

and $\omega_e \wedge \omega' = \omega \wedge \omega'$. □

7.3. Comparison Inequalities

Given two mass functions μ_1 and μ_2 on Ω_E, we say that μ_2 *dominates* μ_1, and write $\mu_1 \leq \mu_2$, if

$$\sum_{\omega \in \Omega_E} f(\omega)\mu_1(\omega) \leq \sum_{\omega \in \Omega_E} f(\omega)\mu_2(\omega)$$

for all increasing functions $f : \Omega_E \to \mathbb{R}$. One may establish certain domination inequalities involving the measures $\phi_{p,q}$ for different values of the parameters p and q. A principal application of such inequalities is to prove the existence of phase transition for different values of p and q, for the infinite-volume random-cluster process (see [4]).

Theorem 7.2 (Comparison inequalities). *It is the case that*

$$\phi_{p',q'} \leq \phi_{p,q} \qquad \text{if} \quad q' \geq q,\ q' \geq 1,\ p' \leq p, \tag{7.10}$$

$$\phi_{p',q'} \geq \phi_{p,q} \qquad \text{if} \quad q' \geq q,\ q' \geq 1,\ \frac{p'}{q'(1-p')} \geq \frac{p}{q(1-p)}. \tag{7.11}$$

Proof. Since $q' \geq 1$, the measure $\phi_{p',q'}$ satisfies the FKG inequality. The theorem will follow by applying this inequality with suitable choices of increasing functions. Note that

$$\phi_{p,q}(\omega) = \frac{\pi_{p',q'}(\omega)g(\omega)}{\sum_\omega \phi_{p',q'}(\omega)g(\omega)}$$

where g satisfies

$$\begin{aligned} g(\omega) &= \left(\frac{q}{q'}\right)^{k(\omega)} \prod_{e\in E} \left(\frac{p}{1-p}\Big/\frac{p'}{1-p'}\right)^{\omega(e)} \\ &= \left(\frac{q}{q'}\right)^{k(\omega)+|\eta(\omega)|} \prod_{e\in E} \left(\frac{p}{q(1-p)}\Big/\frac{p'}{q'(1-p')}\right)^{\omega(e)}. \end{aligned}$$

Now $k(\omega)$ is a *decreasing* function of ω, and $k(\omega)+|\eta(\omega)|$ is an *increasing* function of ω. Therefore

(a) if $q \le q'$ and $p \ge p'$, then g is increasing,

(b) if $q \le q'$ and $p/[q(1-p)] \le p'/[q'(1-p')]$, then g is decreasing.

Under part (a), if f is increasing, then

$$\mathbf{E}_{p,q}(f) = \frac{\mathbf{E}_{p',q'}(fg)}{\mathbf{E}_{p',q'}(g)} \ge \mathbf{E}_{p',q'}(f)$$

by the FKG inequality. Under part (b) the inequality is reversed, since f is increasing and g is decreasing. □

7.4. Rank-Generating Function

The *rank-generating function* of the simple graph $G = (V, E)$ is the function

$$W_G(x, y) = \sum_{E' \subseteq E} x^{r(G')} y^{c(G')}, \quad x, y \in \mathbb{R},$$

where $r(E') = |V| - k(G')$ is the *rank* of the graph $G' = (V, E')$, and $c(G') = |E'| - |V| + k(G')$ is the *co-rank*; as usual, $k(G')$ denotes the number of components of the graph G'. The rank-generating function has various useful properties, and occurs in several contexts in graph theory; see [11, 44]. The rank-generating function sometimes crops up in other forms. For example, the function

$$T_G(x, y) = (x-1)^{|V|-1} W_G\big((x-1)^{-1}, y-1\big)$$

is known as the *dichromatic* (or *Tutte*) *polynomial*. The partition function $Z_{p,q} = Z_{p,q}(G)$, given in (7.2), is easily seen to satisfy

$$Z_{p,q}(G) = q^{|V|}(1-p)^{|E|} W_G\left(\frac{p}{q(1-p)}, \frac{p}{1-p}\right), \tag{7.12}$$

a relationship which provides a link with other classical quantities associated with a graph. See [18] also.

7.5. Hypergraphs

Whereas the random-cluster model above is defined on a *graph*, and corresponding Potts models have *pair interactions*, the theory may be extended easily to *hypergraphs* and *many-body interactions*. We shall not pursue this natural extension here, but refer the reader to [23] and the references therein.

8. Infinite-Volume Limits and Phase Transition

In studying random-cluster measures on lattices, we restrict ourselves to the case of the hypercubic lattice in d dimensions, where $d \geq 2$; similar observations are valid in greater generality. For any subset S of $\mathbb{Z}^d$, we write ∂S for its boundary, i.e.,

$$\partial S = \{s \in S : \langle s, t\rangle \in \mathbb{E}^d \text{ for some } t \notin S\}.$$

Let Λ be a finite box of $\mathbb{L}^d$, which is to say that

$$\Lambda = \prod_{i=1}^{d} [x_i, y_i]$$

for some $x, y \in \mathbb{Z}^d$; we interpret $[x_i, y_i]$ as the set $\{x_i, x_i + 1, \dots, y_i\}$. The set Λ generates a subgraph of $\mathbb{L}^d$ having vertex set Λ and edge set $\mathbb{E}_\Lambda$ containing all $\langle x, y\rangle$ with $x, y \in \Lambda$.

We are interested in the thermodynamic limit (as $\Lambda \uparrow \mathbb{Z}^d$) of the random-cluster measure on the finite box Λ. Let $\Omega = \{0,1\}^{\mathbb{E}^d}$ be the set of 'edge-configurations' of $\mathbb{L}^d$. Let Ω^1_Λ be the subset of Ω containing all $\omega \in \Omega$ for which $\omega(e) = 1$ for $e \notin \mathbb{E}_\Lambda$; similarly define Ω^0_Λ as the subset of Ω containing all ω with $\omega(e) = 0$ for $e \notin \mathbb{E}_\Lambda$. One speaks of configurations in Ω^1_Λ as having 'wired' boundary conditions, and configurations in Ω^0_Λ as having 'free' boundary conditions. We now define two random-cluster measures on $\mathbb{L}^d$. Let $0 < p < 1$ and $q > 0$. For $b = 0, 1$, define

$$\phi^b_{\Lambda,p,q}(\omega) = \frac{1}{Z^b_\Lambda}\left\{\prod_{e\in\mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\} q^{k(\omega,\Lambda)}, \quad \omega \in \Omega^b_\Lambda, \tag{8.1}$$

where

$$Z^b_\Lambda = \sum_{\omega\in\Omega^b_\Lambda}\left\{\prod_{e\in\mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\} q^{k(\omega,\Lambda)} \tag{8.2}$$

is the appropriate normalizing constant, and $k(\omega, \Lambda)$ is the number of clusters of $(\mathbb{Z}^d, \eta(\omega))$ which intersect Λ.

Theorem 8.1 (Thermodynamic limit). *Suppose $q \geq 1$. The weak limits*

$$\phi^b_{p,q} = \lim_{\Lambda\uparrow\mathbb{Z}^d} \phi^b_{\Lambda,p,q}, \quad \textit{for } b = 0, 1, \tag{8.3}$$

exist and satisfy $\phi^0_{p,q} \leq \phi^1_{p,q}$.

The limits in (8.3) are to be interpreted along any increasing sequence of finite boxes, and the weak convergence is in the sense that $\phi^b_{\Lambda,p,q}(A) \to \phi^b_{p,q}(A)$ for all finite-dimensional cylinders A. The assumption that $q \geq 1$ is necessary for the proof, which relies on the validity of the FKG inequality.

One may discuss other boundary conditions, 'mixed' conditions which are more complicated than either wired or free; such conditions are relevant to random-cluster models arising from Potts models with mixed boundary conditions. We omit a detailed discussion here, but note that $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are the most 'extreme' measures obtainable in the infinite-volume limit, amongst an important subclass of boundary conditions. Define a *random-cluster measure* ϕ on $\mathbb{E}^d$ to be a measure with the property that, conditional on the states of edges lying outside any given finite set E ($\subseteq \mathbb{E}^d$), the states of edges within E satisfy (7.1) on the graph induced by E with the appropriate boundary condition specifying which endpoints of edges in E are joined by edges outside E. We write $\mathcal{R}_{p,q}$ for the class of such measures, and note that every $\phi \in \mathcal{R}_{p,q}$ satisfies $\phi^0_{p,q} \le \phi \le \phi^1_{p,q}$. See [24].

Sketch proof of Theorem 8.1. Let Λ and Λ' be two finite boxes satisfying $\Lambda \subseteq \Lambda'$, and let A be the event that all edges $e \in \mathbb{E}_{\Lambda'} \setminus \mathbb{E}_\Lambda$ have state 0. Now $\phi^0_{\Lambda,p,q}$ may be thought of as the measure $\phi^0_{\Lambda',p,q}$ *conditioned* on the event A. Since A is a decreasing event, we have by the FKG inequality that

$$\phi^0_{\Lambda,p,q}(\cdot) = \phi^0_{\Lambda',p,q}(\cdot \mid A) \le \phi^0_{\Lambda',p,q}(\cdot); \tag{8.4}$$

a similar argument yields

$$\phi^1_{\Lambda,p,q} \ge \phi^1_{\Lambda',p,q} \,. \tag{8.5}$$

By monotonicity, the limits exist in (8.3). [By (8.4) and (8.5), $\lim_{\Lambda\uparrow\mathbb{Z}^d} \phi^b_{\Lambda,p,q}(B)$ exists for any increasing finite-dimensional cylinder B, and for $b = 0, 1$; such cylinders generate the appropriate σ-field.] To show that $\phi^0_{p,q} \le \phi^1_{p,q}$, it suffices that

$$\phi^0_{\Lambda',p,q} \le \phi^1_{\Lambda',p,q} \,. \tag{8.6}$$

This too follows by the FKG inequality, since $\phi^b_{\Lambda',p,q}$ may be thought of as the random-cluster measure on a larger region Λ'' *conditioned* on the extra edges having state b; the conditional measure with $b = 0$ must lie underneath the conditional measure with $b = 1$. □

An indicator of phase transition in the Potts model is the 'magnetization', defined as follows. Consider a Potts measure π^1_Λ on Λ having '1' boundary conditions, which is to say that all vertices in the boundary $\partial\Lambda$ are constrained to have spin value 1. Let $\tau_\Lambda = \pi^1_\Lambda(\sigma(0) = 1) - q^{-1}$, the 'effect' of these boundary conditions on the spin at the origin. Passing to the corresponding random-cluster measure ϕ^1_Λ, as in Section 6, we see as in (6.11) that

$$\tau_\Lambda = (1 - q^{-1})\phi^1_\Lambda(0 \leftrightarrow \partial\Lambda) \,. \tag{8.7}$$

In reaching this conclusion, we have suppressed the reference to parameter values, and applied (6.11) to the graph obtained from Λ by identifying all vertices in $\partial\Lambda$.

We say that phase transition takes place in the Potts model if $\tau = \lim_{\Lambda\uparrow\mathbb{Z}^d} \tau_\Lambda$ satisfies $\tau > 0$. In studying the random-cluster process, we shall work with the analogous quantity

$$\theta_\Lambda(p, q) = \phi^1_\Lambda(0 \leftrightarrow \partial\Lambda) \tag{8.8}$$

and with the infinite-volume limit

$$\theta(p, q) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \theta_\Lambda(p, q) \,; \tag{8.9}$$

this limit exists if $q \geq 1$ (see [4, p. 22]). We have that

$$\theta(p,q) = \phi^1(0 \leftrightarrow \infty)\,,$$

the ϕ^1-probability that the origin is in an infinite cluster; in the case $q = 1$, this coincides with the 'percolation probability' of the percolation model. Using the comparison inequality (7.10), $\theta(p,q)$ is a non-decreasing function of p, and we may therefore define the critical value

$$p_c(q) = \sup\{p : \theta(p,q) = 0\}, \quad \text{for } q \geq 1. \tag{8.10}$$

How does $p_c(q)$ depend on the choice of q? The comparison inequalities imply that

$$\frac{1}{p_c(q')} \leq \frac{1}{p_c(q)} \leq \frac{q'/q}{p_c(q')} - \frac{q'}{q} + 1 \quad \text{if } 1 \leq q \leq q'. \tag{8.11}$$

In particular, since $0 < p_c(1) < 1$ ([22, p. 14]), we have that $0 < p_c(q) < 1$ for all $q \geq 1$, implying the existence of a phase transition for all values of q (≥ 1). It follows that $p_c(q)$ is a Lipschitz-continuous and nondecreasing function of q; strict monotonicity may be shown using the method of [10].

9. Open Problems for Random-cluster Processes

9.1. Value of Critical Point

It is unreasonable to expect an exact calculation of the critical point $p_c(q)$ in general. For certain two-dimensional lattices however, the method of planar duality is applicable and leads to conjectured values.

Conjecture 9.1. *The critical value for the random-cluster process on the square lattice is*

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}, \quad q \geq 1.$$

This has been proved for $q = 1$ (percolation), for $q = 2$ (Ising model), and for all sufficiently large values of q ([36, 37]). The argument of [30] may possibly be adapted to prove the conjecture when $q \geq 4$. See [7, 45] also.

Corresponding conjectures may be made for certain other two-dimensional lattices, such as the triangular and hexagonal lattices, and also for certain processes in which the value of p may depend on the inclination of the edge in question. In making such conjectures, one uses the method of duality together with the star-triangle transformation.

9.2. Continuity of Percolation Probability

It is thought that $\theta(p,q)$ is continuous at the critical value p if and only if q is sufficiently small. Since θ is right-continuous, this amounts to deciding whether $\theta(p_c(q),q) = 0$ ('second order transition') or $\theta(p_c(q),q) > 0$ ('first order transition') for a given value of q.

Conjecture 9.2. *There exists a real* $Q = Q(d)$ *such that*

$$\theta(p_c(q), q) \begin{cases} = 0 & \text{if } 1 \le q < Q(d) \\ > 0 & \text{if } q > Q(d). \end{cases}$$

Furthermore $Q(2) = 4$*, and* $Q(d) = 2$ *for* $d \ge 6$*.*

That $\theta(p_c(q), q) > 0$ when q is large has been proved in [37]. As remarked in Section 3.2, it is not even known that $\theta(p_c(1), 1) = 0$.

9.3. Exponential Decay

Suppose $q \ge 1$. Let $\tau_{p,q}(x, y)$ be the $\phi^1_{p,q}$-probability of a path joining the vertices x and y, and denote by e_n the vertex $(n, 0, 0, \dots, 0)$. We have by the FKG inequality that

$$\tau_{p,q}(0, e_{m+n}) \ge \tau_{p,q}(0, e_m)\tau_{p,q}(e_m, e_{m+n}),$$

whence the *correlation length* $\xi(p, q)$, defined by

$$\xi(p, q)^{-1} = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log \tau_{p,q}(0, e_n) \right\} \tag{9.1}$$

exists. Presumably

$$0 < \xi(p, q) < \infty \quad \text{if } 0 < p < p_c(q), \tag{9.2}$$

but the finiteness of $\xi(p, q)$ near $p_c(q)$ is unproven in general. It is known to hold for $q = 1, 2$, and for large values of q ([1, 2, 3, 22, 35, 37, 40]).

By monotonicity, the quantity

$$\mu(q) = \lim_{p\uparrow p_c(q)} \xi(p, q)^{-1}$$

exists, and it is thought that

$$\mu(q) \begin{cases} = 0 & \text{if } q < Q(d) \\ > 0 & \text{if } q > Q(d)\,; \end{cases} \tag{9.3}$$

once again, the existence of the mass gap (i.e., the fact that $\mu(q) > 0$) has been proved in [37] for sufficiently large q.

9.4. Uniqueness of Random-Cluster Measures

For given values of p and q, how large is the class $\mathcal{R}_{p,q}$ of random-cluster measures? It is presumably the case that $|\mathcal{R}_{p,q}| = 1$ whenever $q \ge 1$ and $p \ne p_c(q)$, but this is not proved in general. Furthermore, when $p = p_c(q)$, it is presumably the case that $|\mathcal{R}_{p,q}| = 1$ if $\theta(p_c(q), q) = 0$, and otherwise $\mathcal{R}_{p,q}$ has exactly two extremal measures $\{\phi^0_{p,q}, \phi^1_{p,q}\}$.

Partial results are known in the direction of the uniqueness of random-cluster measures. First, $|\mathcal{R}_{p,q}| = 1$ if and only if $\phi^0_{p,q} = \phi^1_{p,q}$; furthermore ([4])

$$\phi^0_{p,q} = \phi^1_{p,q} \quad \text{if } \theta(p, q) = 0\,, \tag{9.4}$$

so that there is a unique such measure throughout the subcritical phase.

If $d = 2$ and $p \ne p_c(q)$, then the uniqueness follows by exploiting self-duality (see [24] for a discussion). If $d \ge 3$ and q is sufficiently large, then the uniqueness is a consequence of Pirogov–Sinai theory ([37, 39]).

One may use a general argument based on the convexity of free energy to prove that $|\mathcal{R}_{p,q}| = 1$ for all values of p except (at most) countably many ([24, 25]).

9.5. The Case $q < 1$

If $0 < q < 1$, then the FKG inequality is not valid. In the absence of the consequent monotonicity, it is no longer clear whether or not there is a phase transition, and what should be the form of such a transition.

Using an argument based on convexity of free energy (see [24]), one may show that the edge-density $\phi_p(\omega(e) = 1)$ is non-decreasing in p, where ϕ_p is any translation-invariant random-cluster measure with parameters p and q. Increasing events other than $\{\omega(e) = 1\}$ may not generally have probabilities which are monotonic in p.

The mean-field random-cluster model, when the underlying graph is the complete graph on n vertices, and $p = \lambda/n$, may be solved exactly for all positive values of q, even $q \in (0, 1)$; see [12].

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