DROPLET CONDENSATION IN THE ISING MODEL: MODERATE DEVIATIONS POINT OF VIEW

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Abstract. The threshold for the condensation of the vapour of microscopic droplets into a macroscopic one is studied for the case of the ν-dimensional Ising model. The parameter which drives the condensation is the amount of the condensing phase, and the critical value of it, which turns the condensation on, is found to be of the order of $|V_N|^{(\nu-1)/\nu}$, where $|V_N|$ is the volume of the system. The corresponding behaviour of large and moderate deviations is studied. The principle of large deviations in the strong form is obtained in the region where condensation does not take place.

Key words: Large deviations, moderate deviations, Ising model, droplet.

1. Introduction

In this talk we would like to address the following question: to what extent can the usual picture of the droplet condensation be observed in the most well understood model of the statistical physics, the Ising model. By the droplet condensation picture we mean the process which takes place when one changes the concentration of the solution of one species in another. At a certain concentration threshold the solution reaches the saturation point, after which the solution cannot absorb the extra amount of the solvent, and the formation of a crystal takes place.

It turns out that the qualitative picture described above can indeed be observed in the low temperature Ising model, where the role of two substances is played by the two different phases. It turns out that the condensation takes place when the amount of the solvent exceeds the threshold which is equal to the volume of the system raised to a certain power smaller than one. From the probability theory point of view that threshold belongs to the so called region of moderate deviations (see below), which explains the title of the paper.

From the mathematical point of view, the phenomenon of condensation is a question about the behavior of the probabilities of large and moderate deviations in the phase transition regime. It turns out that the condensation regime is precisely the regime where this behavior is different from the one given by the classical schemes

[1]Partially supported by the Russian fund for fundamental research under grant 93-011-1470.
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of the probability theory of independent or weakly dependent random variables.

So below we will give the overview of the results of the probability theory mentioned, and will compare them with the corresponding results for the Ising model. The complete results will appear under the title *Large and Moderate Deviations in the Ising Model* in the forthcoming book *Probability Contributions to Statistical Mechanics*, edited by R. Dobrushin and published by the American Mathematical Society.

2. Ising Model

In this section we will fix some notation and recall results about the Ising model. Let $\mathbb{Z}^\nu$ be a ν-dimensional integer lattice with sites $t = (t^1, \dots, t^\nu)$, $t^i \in \mathbb{Z}$, $i = 1, \dots, \nu$ and the norm

$$|t| = |t^1| + \cdots + |t^\nu|, \quad t \in \mathbf{Z}^\nu, \tag{2.1}$$

which defines a metric on $\mathbb{Z}^\nu$. For any $V \subseteq \mathbb{Z}^\nu$ we denote by Ω_V the set of all configurations $\{\sigma_t, t \in V\}$, $\sigma_t = \pm 1$. We reserve the notation Ω for the case when $V = \mathbb{Z}^\nu$. If $V' \subset V$, we use $\sigma_{V'} = \{\sigma_t, t \in V'\}$ to denote the restriction of the configuration $\sigma \in \Omega_V$ to V'. Let $|V|$ be the number of points in V, and $|\partial V|$ be the number of points in V lying at the boundary of this volume.

Let $V \subset \mathbb{Z}^\nu$ be a finite subset of $\mathbb{Z}^\nu$, $\bar\sigma = \{\sigma_t, t \in \mathbb{Z}^\nu\}$ be a fixed configuration, and $\beta > 0$ be real constant. In a usual way we define the energy function

$$H_V(\sigma \,|\, \bar\sigma) = -\tfrac{1}{2} \sum_{\substack{s \neq t: \\ s,t \in V, |s-t|=1}} \sigma_s \sigma_t - \sum_{\substack{s \in V, t \in V^c: \\ |s-t|=1}} \sigma_s \sigma_t, \quad \sigma \in \Omega_V, \tag{2.2}$$

and the Gibbs probabilities

$$P_V^\beta(\sigma \,|\, \bar\sigma) = \frac{1}{Z_V(\beta, \bar\sigma)} \exp\{-\beta H_V(\sigma \,|\, \bar\sigma)\}, \quad \sigma \in \Omega_V, \tag{2.3}$$

where the partition function satisfies

$$Z_V(\beta, \bar\sigma) = \sum_{\sigma \in \Omega_V} \exp\{-\beta H_V(\sigma \,|\, \bar\sigma)\}. \tag{2.4}$$

The probability distribution (2.3) is called *the Ising model distribution in the volume V with the boundary condition $\bar\sigma$ at inverse temperature β*. Two particular boundary conditions will be mainly used:

$$\bar\sigma^+ = (\sigma_t \equiv 1, t \in \mathbb{Z}^\nu) \quad \text{and} \quad \bar\sigma^- = (\sigma_t \equiv -1, t \in \mathbb{Z}^\nu). \tag{2.5}$$

The boundary conditions $\bar\sigma^\pm$ are called $\pm$-*boundary conditions.* The corresponding Gibbs probabilities and the partition functions will be denoted by $P_V^{\pm,\beta}(\sigma)$ and $Z_V^\pm(\beta)$.

For any function $f(\sigma)$, $\sigma \in \Omega_W$, where $W \subseteq V$, we denote by

$$\langle f \rangle_V^{+,\beta} = \sum_{\sigma \in \Omega_V} f(\sigma_W) P_V^{+,\beta}(\sigma), \tag{2.6}$$

its Ising model mean value in the volume V with + boundary conditions.

In what follows we will suppose that the dimension ν satisfies the bound $\nu \geq 2$ since otherwise there is no phase transition.

It is well known (see, e.g., [R]), that for any function $f(\sigma)$, $\sigma \in \Omega_W$, where $|W| < \infty$ and any sequence of volumes $V_1 \subset V_2 \dots$, such that $\bigcup_N V_N = \mathbb{Z}^\nu$, the limits

$$\langle f \rangle^{\pm} = \langle f \rangle^{\pm,\beta,h} = \lim_{N \to \infty} \langle f \rangle_{V_N}^{\pm} \tag{2.7}$$

exist and do not depend on a choice of the sequence V_N. These collections of mean values define probability measures $P^{\pm} = P^{\pm,\beta}$ on $\Omega_{\mathbb{Z}^\nu}$ which describe stationary ergodic random fields and are called *the limit pure $\pm$-Gibbs states with parameter β*. Another well known fact is the existence of the critical values β_ν^{crit}, such that $0 < \beta_\nu^{\text{crit}} < \infty$ and the limit Gibbs states $P^{+,\beta,h}$ and $P^{-,\beta,h}$ coincide if $\beta \leq \beta_\nu^{\text{crit}}$. In this case for any function f

$$\langle f \rangle^{+,\beta} = \langle f \rangle^{-,\beta}. \tag{2.8}$$

This case corresponds to the physical situation of the absence of a phase transition. In case $\beta > \beta_\nu^{\text{crit}}$, when the phase transition does take place, the limit states $P^{\pm,\beta}$ are different. The corresponding mean values are connected via symmetry relation:

$$\langle f(*) \rangle^{+,\beta} = \langle f(-*) \rangle^{-,\beta}. \tag{2.9}$$

We define finally magnetization to be

$$m(\beta) = \langle \sigma_t \rangle^{+,\beta}. \tag{2.10}$$

For $\beta > \beta_\nu^{\text{crit}}$ it is positive.

Another example of boundary conditions are periodic boundary conditions. They correspond to the case when instead of a box V we consider a discrete torus $\mathbf{T}_N = \mathbb{Z}^\nu / N\mathbb{Z}^\nu$ with a side N and an obvious norm $|t|$. We define

$$H_{\mathbf{T}_N}^{\text{per}}(\sigma) = -\tfrac{1}{2} \sum_{\substack{s \neq t: \\ s,t \in \mathbf{T}_N, |s-t|=1}} \sigma_s \sigma_t, \quad \sigma \in \Omega_{\mathbf{T}_N}. \tag{2.11}$$

We define the probabilities $P_{\mathbf{T}_N}^{\text{per},\beta}(\sigma)$ and partition functions $Z_{\mathbf{T}_N}^{\text{per}}(\beta)$ by analogs of the relations (2.3), (2.4).

Similarly to (2.6) we introduce the mean value $\langle f \rangle_{\mathbf{T}_N}^{\text{per},\beta,h}$.

3. Droplet Condensation

To explain the condensation phenomenon we have to use the contour technique which is the basis of the mathematical theory of phase transitions of the first order.

In the case of small β the typical configuration of the Ising model is a chaotic mixture of pluses and minuses. In the limit case $\beta = 0$ one gets just the realization of a Bernoulli field. With the growth of β a clustering of pluses and minuses begins. For β large, when we have distinct plus and minus phases, a typical configuration of a realization of plus phase forms a sea of pluses with islands of minuses floating inside it (some of these islands have lakes of pluses inside and so on). In the opposite phase a typical realization is a continent of minuses with lakes of pluses inside it (and again with islands of minuses inside some of these lakes and so on). To formalize this visual picture we shall use the notion of contours of a configuration. To be specific, we consider configurations in a finite volume with + boundary conditions.

Let a finite volume $V \subset \mathbb{Z}^\nu$ be given. We define the boundary of a configuration $\sigma \in \Omega_V$ to be the set of all unit plaquettes of the dual lattice $\mathbb{Z}^{\nu *} = \mathbb{Z}^\nu + (\frac{1}{2}, \dots, \frac{1}{2}) \subset \mathbb{R}^\nu$, which cross any of the bonds (t_1, t_2) of the initial lattice such that $\sigma_{t_1} \neq \sigma_{t_2}$. (If t lies outside V we suppose that $\sigma_t = +1$.) The boundary of a configuration σ splits up into connected components which are non-intersecting polyhedra and which will be called *the contours of the configuration* σ. The family of all the contours of a configuration σ will be denoted by $\Gamma(\sigma)$. For any contour $\Gamma \in \Gamma(\sigma)$ we denote by $|\Gamma|$ the number of its $(\nu - 1)$–dimensional faces and by $\mathrm{Int}\,\Gamma$ the set of lattice sites inside the corresponding polyhedron, and finally by $|\mathrm{Int}\,\Gamma|$ the number of points in this set. The interiors of external contours (i.e., contours which do not lie inside another ones) can be interpreted as droplets of minus phase inside the plus phase.

We shall describe a typical large deviation configuration in the volume V_N by using the conditional probabilities

$$P_N^{+,b_N}(\sigma) = \frac{P_N^+(\sigma)}{\sum_{\sigma \in \Omega_V : S_N(\sigma) = b_N} P_N^+(\sigma)}, \tag{3.1}$$

induced by the condition $S_N(\sigma) = \sum_{t \in V_N} \sigma_t = b_N$. We shall assume that $b_N - |V_N| \equiv 0 \pmod 2$. (In the opposite case the denominator in (3.1) vanishes.) The following theorem describes some typical properties of such configurations.

Theorem 3.1. *Suppose that some sequences of volumes V_N, satisfying the Van Hove condition and integers b_N are given. Suppose that either, for some $D > 0$ and $\beta > \beta_1(D)$,*

$$|V_N| \tanh D_\beta \geq b_N \geq E_N^+, \tag{3.2}$$

or, for $\beta \geq \beta_2$ with β_2 large enough,

$$\lim_{N \to \infty} \frac{|b_N - E_N^+|}{|V_N|^{\nu/\nu+1}} = 0, \tag{3.3}$$

where $E_N^+ = \langle S_N \rangle_{V_N}^{+,\beta}$ is the mean value of S_N with respect to the considered Gibbs distribution. Then there exists a constant $C < \infty$ such that for the events

$$A_N = \{\sigma \in \Omega_{V_N} : \max_{\Gamma \in \Gamma(\sigma)} |\Gamma| \leq C \ln |V_N|\} \tag{3.4}$$

the following limit of probabilities is valid:

$$\lim_{N\to\infty} \sum_{\sigma\in A_N} P_N^{+,b_N}(\sigma) = 1. \tag{3.5}$$

On the other hand suppose that $\beta > \beta_1$, the box V_N is contractible, with

$$|\partial V_N| \le L|V_N|^{\nu-1/\nu} \tag{3.6}$$

for some $L < \infty$,

$$\limsup_{N\to\infty} \frac{b_N - E_N^+}{|V_N|^{\nu/\nu+1}} < 0, \tag{3.7}$$

and for some constant $Q > 0$

$$b_N \ge -E_N^+ - Q|V_N|^{(2\nu-1)/2\nu}. \tag{3.8}$$

Then there exist constants $C < \infty$, $c > 0$ such that for the event

$$\begin{aligned} B_N = \{\sigma \in \Omega_{V_N} : \ & \text{for some contour } \Gamma_0 \in \Gamma(\sigma) \\ & \text{its surface } |\Gamma_0| > c\left(E_N^+ - b_N\right)^{(\nu-1)/\nu}, \\ & \text{its volume } |\mathrm{Int}(\Gamma_0)| > c\left(E_N^+ - b_N\right), \\ & \text{while for all other contours } \Gamma \in \Gamma(\sigma), \Gamma \ne \Gamma_0, \\ & \text{their surfaces } |\Gamma| \le C \ln |V_N|\} \end{aligned} \tag{3.9}$$

the limit of the probabilities satisfies

$$\lim_{N\to\infty} \sum_{\sigma\in B_N} P_N^{+,b_N}(\sigma) = 1. \tag{3.10}$$

Finally suppose that $\beta > \beta_1$, that condition (3.6) holds, that

$$\limsup_{N\to\infty} \frac{b_N + E_N^+}{|V_N|^{(2\nu-1)/2\nu}} = -\infty, \tag{3.11}$$

and that

$$\liminf_{N\to\infty} \frac{b_N}{|V_N|} > -1. \tag{3.12}$$

Then there exist constants $C < \infty$, $c > 0$, such that for the event

$$\begin{aligned} B'_N = \big\{\sigma \in \Omega_{V_N} : \ & \text{for some contour } \Gamma_0 \in \Gamma(\sigma) \\ & \text{its volume } |\mathrm{Int}\ \Gamma_0| > |V_N| - c|\partial V_N| \\ & \text{while for all other contours } \Gamma \in \Gamma(\sigma), \Gamma \ne \Gamma_0, \\ & \text{their surfaces } |\Gamma| \le C \ln |V_N|\big\} \end{aligned} \tag{3.13}$$

the limit of the probabilities satisfies

$$\lim_{N\to\infty} \sum_{\sigma\in B'_N} P_N^{+,b_N}(\sigma) = 1. \tag{3.14}$$

This theorem can be interpreted in the following way. If a value of S_N is close to its mean value E_N^+, then all the droplets of the opposite minus phase have microscopic sizes (they are smaller than $C \ln |V_N|$). If we reduce the number of minus-particles (by considering the values $b_N > E_N^+$), these droplets would further shrink in size and diminish in number, but there will be no qualitative change in a picture. The total deviation of S_N is caused by deviations of the sizes of separate droplets, which are almost independent, and so its probability behaves in the same way as in the classical case of independent variables. If we increase the number of minus-particles (the case $b_N < E_N^+$), the same situation holds, provided the order of the number of additional minus-particles is smaller than $|V_N|^{(\nu-1)/\nu}$. But after we pass this threshold, the microscopic droplets can no longer absorb all the additional minus-particles, so a large minus-droplet of a macroscopic size is formed, which contains the main part of these particles. It is the moment of the condensation of the minus-phase. At that level of speculation it is possible to explain why the scale of the order of $|V_N|^{(\nu-1)/\nu}$ is critical. If the number of additional minus-particles $E_N^+ - b_N$ is approximately equal to $|V_N|^\kappa$, then the classical mechanism, connected with the collective behavior of many small droplets, predicts the deviation probability to be of the order of

$$\exp\left\{-\frac{(E_N^+ - b_N)^2}{2D_N^+}\right\} \approx \exp\{-\bar{c}|V_N|^{2\kappa-1}\}, \tag{3.15}$$

where D_N^+ is the variance of the random variable S_N. On the other hand, a large droplet, containing approximately $|V_N|^\kappa$ particles, has the surface of the order $|V_N|^{\kappa(\nu-1)/\nu}$, and so its probability has the order

$$\exp\{-\tilde{c}|V_N|^{\kappa(\nu-1)/\nu}\}. \tag{3.16}$$

The exponents in (3.15) and (3.16) are equal if $\kappa = \nu/(\nu+1)$.

At the level $b_N \approx -E_N^+$ we have a new change of a picture. The volume of the large contour containing the minus-phase arrives to its maximal possible value $|V_N|$ and so can not grow further. So the droplets of the plus-phase inside the minus-phase begin to shrink. Typical configurations inside the minus-phase become symmetrical to typical configurations of plus-phase via the transformation $b_N \to -b_N$.

4. Classical Behavior of Large and Moderate Deviations

In this section we will describe the classical behavior of the large and moderate deviations probabilities for the sums of random variables. The description we give differs a little from the standard one, and contains more information, so we name it the 'classical large deviation principle in the strong form'. Since we are Ising model oriented, we will consider random variables taking values $+1$ and -1 only.

Consider a random variable

$$S_N = \xi_1^N + \cdots + \xi_N^N, \tag{4.1}$$

where the random variables ξ_i^N take values $+1$ and -1. Let

$$P_N(b) = \mathbf{P}\{S_N = b\}, \tag{4.2}$$

where b are integers such that

$$b - N \equiv 0 \,(\mathrm{mod}\ 2). \tag{4.3}$$

We consider the logarithmic generating function

$$H_N(z) = \ln \sum_b P_N(b) e^{zb} \tag{4.4}$$

and suppose that for each real $\lambda \in \mathbb{R}^1$ there exists a complex neighborhood $\mathcal{O}_\lambda$ of λ in $\mathbb{C}^1$ and a constant C_λ such that for all large enough N

$$|H_N(z)| \le NC_\lambda \qquad \text{if } z \in \mathcal{O}_\lambda. \tag{4.5}$$

We suppose also that the limit

$$\lim_{N\to\infty} \frac{1}{N} H_N(z) = H(z) \tag{4.6}$$

exists in the sense of the uniform convergence in $\mathcal{O}_\lambda$ for all λ. This limit is a holomorphic function of $z \in \mathcal{O}_\lambda$.

Let for any $\lambda \in \mathbb{R}^1$

$$G_{N,\lambda}^n = \left.\frac{d^n H_N(z)}{dz^n}\right|_{z=\lambda}. \tag{4.7}$$

For $\lambda = 0$ these quantities are called the semi-invariants (or cumulants) of order n of the variables (4.1). It follows from (4.5) and well known properties of holomorphic functions that the limits exist

$$G_\lambda^n = \lim_{N\to\infty} \frac{1}{N} G_{N,\lambda}^n = \left.\frac{d^n H(z)}{dz^n}\right|_{z=\lambda}. \tag{4.8}$$

We shall omit the index λ in the notation $G_{N,\lambda}^n$ when $\lambda = 0$ and let

$$E_N = G_N^1, \quad E = G^1, \quad D_N = G_N^2, \quad D = G^2. \tag{4.9}$$

The quantities E_N and D_N are, of course, the mean value and the variance of S_N. It is easy to see that $G_{N,\lambda}^2 \ge 0$ for all $\lambda \in \mathbb{R}^1$. So the functions $H_N(\lambda)$ and $H(\lambda)$ are concave functions of $\lambda \in \mathbb{R}^1$. We suppose additionally that $H_N(\lambda)$ and $H(\lambda)$ are strictly concave. (For the functions $H_N(\lambda)$ this property holds under the condition that the probability distribution $P_N(b)$ is not concentrated in a single point; for the limiting function $H(\lambda)$ it is an essential new restriction.) This assumption implies in particular that the limiting variance D is positive. Our hypothesis about the asymptotic behavior of the functions $N^{-1}H_N(\lambda)$ and $H(\lambda)$ at $\lambda = \pm\infty$ is that their derivatives $N^{-1}G^1_{N,\lambda}$ and G^1_λ go to ± 1 when $\lambda \to \pm\infty$.

We define *the action functions*

$$I_N(b) = \sup_{\lambda \in \mathbb{R}^1} (\lambda b - H_N(\lambda)), \quad I(\hat{b}) = \sup_{\lambda \in \mathbb{R}^1} (\lambda \hat{b} - H(\lambda)). \tag{4.10}$$

Under the strict concavity hypothesis introduced above, there exist unique values $\lambda_N(b)$, $b \in (-N, N)$, and $\lambda(\hat{b})$, $\hat{b} \in (-1, 1)$, such that

$$b = G^1_{N,\lambda} = \left.\frac{dH_N(\lambda)}{d\lambda}\right|_{\lambda=\lambda_N(b)}, \quad \hat{b} = \left.\frac{dH(\lambda)}{d\lambda}\right|_{\lambda=\lambda(\hat{b})} \tag{4.11}$$

and so

$$I_N(b) = \lambda_N(b)b - H_N(\lambda_N(b)), \quad I(\hat{b}) = \lambda(\hat{b})\hat{b} - H(\lambda(\hat{b})), \tag{4.12}$$

and

$$I(\hat{b}) = \lim_{N\to\infty} \frac{1}{N} I_N(N\hat{b}). \tag{4.13}$$

So the implicit function theorem implies that $I_N(b)$ and $I(\hat{b})$ can be extended to holomorphic functions in a neighborhood of $\mathbb{R}^1$. It is easy to see that

$$I_N(E_N) = I(E) = 0, \quad \left.\frac{dI_N(b)}{db}\right|_{b=E_N} = \left.\frac{dI(\hat{b})}{d\hat{b}}\right|_{\hat{b}=E} = 0. \tag{4.14}$$

So the Taylor expansions of $I_N(b)$ and $I(\hat{b})$ at E_N, E have the following form:

$$I_N(b) = \frac{(b-E_N)^2}{2D_N} + \sum_{j=3}^{\infty} \frac{K^j_N}{j!} \left(\frac{(b-E_N)}{D_N}\right)^j, \tag{4.15}$$

$$I(\hat{b}) = \frac{(\hat{b}-E)^2}{2D} + \sum_{j=3}^{\infty} \frac{K^j}{j!} \left(\frac{(\hat{b}-E)}{D}\right)^j, \tag{4.16}$$

where the limits satisfy

$$\lim_{N\to\infty} \frac{1}{N} K^j_N = K^j. \tag{4.17}$$

It is easy to see that

$$K^j_N = D_N Q_j \left(\frac{G^3_N}{D_N}, \dots, \frac{G^j_N}{D_N}\right), \quad j = 3, 4, \dots, \tag{4.18}$$

where Q_j is a polynomial of $j-2$ variables. In particular,

$$K^3_N = -G^3_N, \quad K^4_N = -G^4_N + 3(G^3_N)^2 (D_N)^{-1}, \tag{4.19}$$

and so on. Similar relations hold between the limiting coefficients K^j and the limiting semi-invariants G^j.

Observe that it follows from (4.11)–(4.12) that the derivatives satisfy

$$\frac{d^2 I_N(b)}{db^2} = \left(\frac{dH_N(\lambda_N(b))}{d\lambda}\right)^{-1}, \quad \frac{d^2 I(\hat{b})}{d\hat{b}^2} = \left(\frac{dH(\lambda(\hat{b}))}{d\lambda}\right)^{-1}. \tag{4.20}$$

Because we have supposed that the functions $H_N(\lambda)$ and $H(\lambda)$ are strictly concave, the relation (4.20) implies that the same holds for the functions $I_N(b)$ and $I(\hat{b})$.

We say that the 'classical large deviation principle in the strong form' holds for the sequence S_N if for any sequence $b_N, N = 1, 2, 3, \ldots$, satisfying the condition

$$b_N - N \equiv 0 \,(\text{mod } 2), \tag{4.21}$$

and

$$\limsup_{N\to\infty} \frac{|b_N|}{N} < 1 \tag{4.22}$$

the following relation holds:

$$P_N(b_N) = 2\left(2\pi G^2_{N,\lambda_N(b_N)}\right)^{-1/2} \exp\{-I_N(b_N)\}(1 + o_N(1)), \tag{4.23}$$

where $o_N(1) \to 0$ as $N \to \infty$. It follows from (4.23) and (4.12) that if the sequence b_N satisfies the condition (4.21) and the condition

$$\lim_{N\to\infty} \frac{b_N}{N} = \hat{b}, \tag{4.24}$$

where $|\hat{b}| < 1$ then

$$\lim_{N\to\infty} \frac{1}{N} \ln P_N(b_N) = -I(\hat{b}). \tag{4.25}$$

This is a familiar form of the 'large deviation principle' as it appears usually in the literature. Of course, it is weaker than (4.23).

On the other hand, we can consider the case when

$$\lim_{N\to\infty} \frac{|b_N - E_N|}{N} = 0, \tag{4.26}$$

which is called the 'moderate deviations' case. Then $\lambda_N(b_N) \to 0$ as $N \to \infty$, and so the classical large deviation principle in the strong form implies that

$$P_N(b_N) = 2(2\pi D_N)^{-1/2} \exp\{-I_N(b_N)\}(1 + o_N(1)). \tag{4.27}$$

If an even stronger condition holds, viz. for some $k = 2, 3, \ldots$

$$\lim_{N\to\infty} \frac{|b_N - E_N|}{N^{k/k+1}} = 0, \tag{4.28}$$

then

$$P_N(b_N) = q_N(b_N) \exp\left\{ -\sum_{j=3}^{k} \frac{K_N^j}{(j)!} \left(\frac{b_N - E_N}{D_N} \right)^j \right\} (1 + o_N(1)), \tag{4.29}$$

where

$$q_N(b_N) = 2(2\pi D_N)^{-1/2} \exp\left\{ -\frac{1}{2} \frac{(b_N - E_N)^2}{D_N} \right\} \tag{4.30}$$

is the usual normal approximation for the probabilities $P_N(b_N)$. In the case $k = 2$ ('small deviation case') it implies that

$$P_N(b_N) = q_N(b_N)(1 + o_N(1)) \tag{4.31}$$

if

$$\lim_{N \to \infty} \frac{|b_N - E_N|}{N^{2/3}} = 0, \tag{4.32}$$

and this is a natural region of the validity of the local central limit theorem. The equations (4.25) and (4.15), and the positivity of the limiting variance D imply a normal approximation in the domain of moderate deviations (4.26)

$$\ln P_N(b_N) = \ln q_N(b_N)(1 + o_N(1)) = -\frac{1}{2} \frac{(b_N - E_N)^2}{D_N} (1 + o_N(1)). \tag{4.33}$$

In the domain of large deviations (4.24) with $b \neq 0$ we can reduce (4.27) further to

$$\ln(-\ln P_N(b_N)) = \ln(-\ln q_N(b_N))(1 + o_N(1)) = \ln N(1 + o_N(1)). \tag{4.34}$$

So the normal approximation predicts correctly the order of the exponential asymptotics of the large deviation probabilities.

5. Classical Behavior of Probabilities of Deviations for the Ising Model

We fix a sequence of finite volumes $V_1 \subset V_2 \cdots \subset \mathbb{Z}^\nu$, such that $\bigcup_N V_N = \mathbb{Z}^\nu$ and the following Van Hove condition holds:

$$\lim_{N \to \infty} \frac{|\partial V_N|}{|V_N|} = 0. \tag{5.1}$$

For $\sigma = (\sigma_t, t \in V_N) \in \Omega_{V_N}$ we consider

$$S_N = S_N(\sigma) = \sum_{t \in V_N} \sigma_t \tag{5.2}$$

and we will study a large deviation behavior of this random variable. We will suppose that

$$b_N - |V_N| \equiv 0 \,(\mathrm{mod}\ 2), \quad \limsup_{N \to \infty} \frac{|b_N|}{|V_N|} < 1. \tag{5.3}$$

The following probability distributions for $\sigma \in \Omega_{V_N}$ in (5.2) will be considered: the Ising model distribution in V_N with +-boundary conditions $\bar{\sigma}^+ = (\sigma_t \equiv +1)$. We let

$$P_N^+(b_N) = P_{V_N}^{\beta,+}(\sigma : S_N(\sigma) = b_N). \tag{5.4}$$

Similar notation will be used for other boundary conditions also.

Theorem 5.1. *Suppose that the Van Hove condition (5.1) and the conditions (5.3) are fulfilled. There exists a value $\beta_0 > 0$ such that for the Ising model with the inverse temperature $\beta \le \beta_0$ the classical large deviation principle in the strong form (see Section 4) holds together with its implications (4.25), (4.27), (4.29), (4.31), (4.33), (4 38). This limit action function is a strictly concave holomorphic function of $\hat{b}$.*

6. The Behavior of Deviations in the Phase Transition Regime

As we have mentioned already, the behavior of the large deviations in the phase transition regime is not classical. In particular it is known ([I]) that the function $H(z)$ is not holomorphic at $z = 0$ if β is large enough, contrary to the functions $H_N(z)$, which are holomorphic in some neighborhoods around zero, even they shrink to zero as $N \to \infty$. The same holds for the functions $I(z)$ and $I_N(z)$. Still, we shall use the coefficients K_N^j defined by the semi-invariants G_N^j by the relations (4.18)–(4.19) even in the phase transition region.

The statements of the present section are using a notion of a *contractible volume*. By such we call any finite volume $V \subset \mathbb{Z}^\nu$ such that the complement V^c is connected. We will sometimes use a following additional restriction for a sequence of finite volumes V_N which will be called a *thickness condition*: there exists a constant $L < \infty$ such that for any N and any integer $K \le |V_N|$ there exists a contractible subvolume $V_N' \subset V_N$ such that $|V_N'| = K$ and

$$|\partial V_N'| \le L |V_N'|^{\nu - 1/\nu}. \tag{6.1}$$

This restriction is a weak one. For example if V_N is a sequence of parallelepipeds $\{t = (t_1, \dots, t_\nu) : 0 \le t_i \le l_i, i = 1, \dots, \nu\}$ then the thickness condition (and also the Van Hove condition and the contractibility condition) are fulfilled if all the ratios l_i/l_j, $i, j = 1, \dots, \nu$ are uniformly bounded.

Theorem 6.1. *Suppose that a sequence of contractible volumes V_N and a sequence of integers b_N are given such that the Van Hove condition (5.1) and the conditions (5.3) are fulfilled.*

Consider the $+$-boundary conditions. Let $D > 0$ and

$$|V_N| \tanh D_\beta \ge b_N \ge E_N^+, \tag{6.2}$$

where

$$E_N^+ = \langle S_N \rangle_{V_N}^{+,\beta} \tag{6.3}$$

is the mean value of S_N with respect to the considered Gibbs distribution (cf. the notations (4.9) and (2.6)). Then for the random variable S_N defined by (5.3) the classical large deviation principle in the strong form holds, i.e., the relation (4.23) holds, together with (4.25), (4.27), (4.29), (4.31), (4.33), (4.34) (which in the situation of Section 4 were its corollaries) provided $\beta \ge \beta_1(D)$. The limit action function

$I(\hat{b})$ (see (4.13)) defined for the corresponding Gibbs distribution is a holomorphic strictly concave function of $\hat{b}$ in the interval

$$\hat{b} > m(\beta). \tag{6.4}$$

In the opposite case

$$b_N - E_N^+ < 0 \tag{6.5}$$

for any integer k, such that

$$k \leq \nu \tag{6.6}$$

and for values b_N, satisfying

$$\lim_{N\to\infty} \frac{|b_N - E_N^+|}{|V_N|^{k/k+1}} = 0, \tag{6.7}$$

the equality (4.29) and its implication (4.31) hold provided $\beta \geq \beta_2$ with β_2 large enough.

Suppose now additionally that the thickness condition (6.1) is fulfilled. Then in the complementary region

$$\limsup_{N\to\infty} \frac{b_N - E_N^+}{|V_N|^{\nu/\nu+1}} < 0, \tag{6.8}$$

if also for some constant $Q > 0$

$$b_N \geq -E_N^+ - Q|V_N|^{(2\nu-1)/2\nu}, \tag{6.9}$$

there exist constants $\underline{c} > 0$, $\overline{c} < \infty$ depending on Q, a sequence of volumes V_N and the value of inverse temperature β_2, such that for all N and $\beta \geq \beta_2$

$$\underline{c}\big(E_N^+ - b_N\big)^{(\nu-1)/\nu} \leq -\ln P_N^+(b_N) \leq \overline{c}\big(E_N^+ - b_N\big)^{(\nu-1)/\nu}. \tag{6.10}$$

Finally, if

$$\limsup_{N\to\infty} \frac{b_N + E_N^+}{|V_N|^{(2\nu-1)/2\nu}} = -\infty \tag{6.11}$$

and

$$\limsup_{N\to\infty} \frac{b_n}{|V_N|} > -\tanh D_\beta \tag{6.12}$$

then for $\beta \geq \beta_1(D)$

$$\lim_{N\to\infty} \frac{-\ln P_N^+(b_N)}{I_N(-b_N)} = 1, \tag{6.13}$$

where $I_N(b)$ is the action function defined by the relation (4.12) for the probability distribution in question.

Similar statements hold for the case of the limit pure $+$-state. The limit action functions for the $+$-boundary conditions and for the limit pure $+$-state coincide and do not depend on a choice of the sequences of volumes V_N (cf. (5.5)).

To compare the classical and non-classical behavior of the deviations described above we suppose that for some $1 \geq \kappa \geq \frac{1}{2}$ and some $a \neq 0$

$$\lim_{N\to\infty} \frac{b_N - E_N^+}{|V_N|^\kappa} = a. \tag{6.14}$$

Then it follows from the previous theorem that

$$-\ln P_N^{+,0}(b_N) \asymp \begin{cases} |V_N|^{2\kappa-1}, & \text{if } a>0,\ \kappa<1 \text{ or } 1-m(\beta)>a>0,\ \kappa=1 \\ & \text{or if } a<0,\ \kappa<\nu/(\nu+1), \\ |V_N|^{\kappa(\nu-1)/\nu}, & \text{if } a<0,\ \nu/(\nu+1)<\kappa<1, \\ & \text{or } -2m(\beta)\le a<0,\ \kappa=1, \\ |V_N|, & \text{if } -1-m(\beta)<a<-2m(\beta),\ \kappa=1, \end{cases} \tag{6.15}$$

where the relation $\asymp$ means that the ratio of the corresponding quantities is uniformly in N bounded from both sides. This relation illustrates a difference between the asymptotic behavior of large deviations in the region of the phase transitions versus region without it. Another interesting picture arises if we suppose that for some $0<\kappa\le 1$, $a>0$

$$\lim_{N\to\infty} \frac{b_N+E_N^+}{|V_N|^\kappa} = a. \tag{6.16}$$

Then using the statements (6.10) and (6.13) and a normal approximation for the action function $I_N^0(b)$ for $b=-b_N\approx E_N^+$ (cf. (4.33)) we see that

$$-\ln P_N^+(b_N) \asymp \begin{cases} |V_N|^{(\nu-1))/\nu}, & \text{if } \kappa\le(2\nu-1)/2\nu \\ |V_N|^{2\kappa-1}, & \text{if } \kappa\ge(2\nu-1)/2\nu. \end{cases} \tag{6.17}$$

Finally, it follows from (6.13) that if

$$-1<\lim_{N\to\infty}\frac{b_N}{|V_N|}=\hat{b}<-m(\beta) \tag{6.18}$$

then we know a more exact asymptotic and can state that

$$\lim_{N\to\infty}\frac{-\ln P_N^+(b_N)}{|V_N|}=I_N(-\hat{b}). \tag{6.19}$$

7. Large Deviations and the Shape of the Droplets

The relation (6.10) gives much rougher information compared with the precise knowledge of the large deviation behavior one gets in the classical regime. So it is natural to try to get at least the value of a constant c which describes the true asymptotic in (6.10):

$$\lim_{N\to\infty}\frac{-\ln P_N^+(b_N)}{\left(E_N^+-b_N\right)^{(\nu-1)/\nu}}=c. \tag{7.1}$$

It turns out that this is a difficult question, and the answer to it is closely connected with an important physical theory which describes the typical shape of a droplet of one phase floating in the opposite one. In the recent book [DKS] the question of the typical shape was rigorously studied in the simplest non-trivial situation of the two-dimensional Ising model with periodic boundary conditions, and this is about the only case when we can prove the existence of the limit (7.1) and exhibit a construction which produce the value of this constant c. For this we have to repeat some notions discussed in details in the book [DKS].

Let

$$W_N = \{t = (t^1, t^2) \in \mathbb{Z}^2 : -N \le t^i \le N, i = 1, 2\} \tag{7.2}$$

be a square. For any direction $n \in \mathbb{S}^1$ (where $\mathbb{S}^1 \subset \mathbb{R}^2$ is the unit circle centered at the origin) we introduce the boundary conditions $\bar{\sigma}^n$ such that

$$\bar{\sigma}^n_t = \begin{cases} +1 & \text{if } (t, n) > 0, \\ -1 & \text{if } (t, n) \le 0. \end{cases} \tag{7.3}$$

Suppose that the inverse temperature β is large enough. We introduce the surface tension in the direction n as the limit

$$\tau_\beta(n) = -\lim_{N\to\infty} \frac{1}{\beta d(N, n)} \ln \frac{Z_{W_N}(\beta, \bar{\sigma}^n)}{Z_{W_N}(\beta, \bar{\sigma}^+)} \tag{7.4}$$

where $\bar{\sigma}^+$ are $+$–boundary conditions (see (2.5)) and $d(N, n)$ is the length of the segment

$$\{t \in \mathbb{R}^2 : (t, n) = 0, t^1 \in [-N, N]\}. \tag{7.5}$$

The existence of the limit (7.4) for the case of large enough inverse temperatures was proven in [DKS].

Let γ be a closed non-self-intersecting smooth curve in $\mathbb{R}^2$. Let

$$\mathcal{W}_\beta(\gamma) = \int_\gamma \tau_\beta(n_s)\, ds, \tag{7.6}$$

where $n_s \in \mathbb{S}^1$ is the normal to the curve γ at the point s and ds is the differential of its length. Let

$$F_\beta = \inf \mathcal{W}_\beta(\gamma) \tag{7.7}$$

where the infimum is taken over all curves γ such that the area enclosed is equal to one.

Theorem 7.1. *There exists a value $\beta_1 < \infty$ such that for the two-dimensional Ising model with inverse temperature $\beta > \beta_1$ and for any sequence of non-negative integers b_N such that the conditions (5.4) hold and the limit*

$$\liminf_{N\to\infty} \frac{b_N - E^{\mathrm{per},\beta,+}_{\mathbf{T}_N}}{|\mathbf{T}_N|^{2/3}} < 0, \tag{7.8}$$

where $E^{\mathrm{per},\beta,+}_{\mathbf{T}_N}$ is the conditional mean value $\langle S_{\mathbf{T}_N} | S_{\mathbf{T}_N} \ge 0\rangle^{\mathrm{per},\beta}_{\mathbf{T}_N}$, the following equality holds:

$$\lim_{N\to\infty} \frac{-\ln P^{\mathrm{per},\beta}_{\mathbf{T}_N}(b_N)}{w_N(b_N)} = 1, \tag{7.9}$$

where

$$w_N(b_N) = \begin{cases} \beta F_\beta \left(\dfrac{E^{\mathrm{per},\beta,+}_{\mathbf{T}_N} - b_N}{2m(\beta)} \right)^{1/2} & \textit{if } F_\beta \left(\dfrac{E^{\mathrm{per},\beta,+}_{\mathbf{T}_N} - b_N}{2m(\beta)} \right)^{1/2} \le 2\beta N \tau_\beta(n_0), \\ 2\beta N \tau_\beta(n_0) & \textit{otherwise,} \end{cases} \tag{7.10}$$

and $n_0 = (1, 0)$ *is the unit lattice vector.*

The case of positive b_N can be reduced to the case of negative ones by the help of the symmetry.

8. References and Generalizations

8.1. In the exposition of the classical theory of deviations in Section 4 we followed essentially the pioneering papers of Khinchin [Kh], Smirnov [Sm], Cramér [Cr], who laid the foundations of the theory (see also books by Ibragimov–Linnik [IL], Petrov [P], Saulis–Statulevichius [SS]). Instead of the local variant of the theory used in Section 4, its integral variant is discussed more often. The local approach was used in papers [Ri], [CS1], [CS2] and in the book [B]. In the integral theory the asymptotic of sums

$$P_N = \sum_{b \in B_N} P_N(b), \tag{8.1}$$

where $B_N \subset \mathbb{R}^1$ is a sequence of subsets, is studied. Since the main estimate (4.23) is true for any sequence b_N, we can estimate the sum (8.1) termwise uniformly in N, so an estimate for the sum can be obtained as an easy implication of the local theory. The possibility to obtain the results both about large and moderate deviations as an implication of a unique estimate (4.23), discussed in Section 4, might be a methodological innovation. The elegant relation (4.25) became a starting point of deep generalizations to the cases of dependent variables, of variables with values in general functional spaces, to a study of empirical distributions and so on (see the books [DeSt], [DZ], [V], [VF]). The main idea is to prove a general result which shows that the classical large deviations principle in the strong form follows easily from some analyticity properties of the logarithmic generating function. This result is influenced by similar results of Saulis–Statulevichius [SS].

8.2. Theorem 5 is a simple implication of the general result mentioned above. The analyticity condition which one has to check was intensively studied in the literature on rigorous results in statistical physics (see, for example, [DS1], [MM]) and so this theorem can be extended to a wide class of situations. On the other hand, it is natural to expect that Theorem 5 remains true for all β in the domain $\beta_0 < \beta < \beta^{\nu}_{\mathrm{crit}}$, but at present there is no known methods to check the corresponding analyticity properties. Some result of the similar type were obtained by Čepulenas [Č].

Recently several authors have developed a general theory of large deviations of empirical distribution of Gibbs fields (see [B], [BD], [Co], [FO1], [G1], [G2], [GZ], [O]). This theory implies, in particular, the existence of the limit (4.25) and can be applied to a wide class of Gibbs fields including the phase transition region. This does not contradict the special behavior of large deviations in the regime of phase transition described in Section 6: the general theory does not exclude a possibility of the vanishing of the action function $I(b)$ on some interval, and it is a typical situation in the case of phase transitions of the first order. In such situation the equality (4.25) does not reveal the true asymptotic behavior of the probabilities.

8.3. The main estimate (6.10) for the case of the deviations proportional to the volume was proved also in the recent papers of Föllmer and Ort [FO2] and Schonmann [Sch]. They obtained for this case explicit estimates of constants $\underline{c}$ and $\overline{c}$ in (6.10). Their papers are based on methods different from the one used in the present paper. Without an explicit formulation such results were essentially contained already in the old paper of Minlos and Sinai [MS]. The results of our paper about moderate deviations seem to be new. The results of Sections 3, 6 and 7 can be extended to a wide class of Gibbs fields undergoing phase transitions, for which it is possible to have a full control of the probabilistic properties of contour systems (such as in Pirogov–Sinai situation, see [S]). Again, it is natural to expect that the theorems of Section 6 are valid for all values of $\beta > \beta^{\nu}_{\mathrm{crit}}$, but there is no methods to prove it.

8.4. Results of Section 3 of the book [DKS] about the shape of the droplet. Its proof is an extension of the proof in [DKS]. Since we wanted to use directly the constructions of [DKS] we restrict ourselves in Section 3 to the case of two-dimensional Ising models with periodic boundary conditions. Generalizations to a wider class of two-dimensional Gibbs models and other boundary conditions seems to be possible. A study of the three-dimensional case meets very serious mathematical difficulties (see discussions in Section 1 of [DKS]).

8.5. The book [DKS] contains some of the results of the present paper for the two-dimensional case, which were necessary to obtain the main result of [DKS]. Some other results were formulated in the papers [DS2], [Sh].

References

[BD] Bolthausen, E. and Deuschel, J.-D. Critical large deviations for Gaussian fields in the phase transition regime; I. *Annals of Probability* (to appear); II (to appear).

[BI] Borgs, C. and Imbrie, J. (1989). A unified approach to phase diagrams in field theory and statistical mechanics. *Communications in Mathematical Physics* **123**, 305–328.

[B] Borovkov, A. A. (1986). *Probability Theory.* Nauka, Moscow.

[Br] Bryc, W. (1992). On the large deviation principle for stationary weakly dependent random fields. *Annals of Probability* **20**, 1004–1030.

[Č] Čepulenas, S. (1985). Probabilities of large deviations for random fields. *Litov. Mat. Sbornik* **5**, 164–176; *Lith. Math. J.* (1985) **25**, 381–390.

[CS1] Chaganty, N. R. and Sethuraman, J. (1985). Large deviations local limit theorems for arbitrary sequences of random variables. *Annals of Probability* **13**, 97–114.

[CS2] Chaganty, N. R. and Sethuraman, J. (1987). Limit theorems in the area of large deviations for some dependent random variables. *Annals of Probability* **15**, 628–645.

[Co] Comets, F. (1989). Large deviation estimates for a conditional probability distribution. Applications to random interacting Gibbs measures. *Probability Theory and Related Fields* **80**, 407–432.

[Cr] Cramér, H. (1938). Sur un nouveau théoreme-limite de la théorie des probabilités. *Actualités Scientifiques et Industrielles* **736**, 5–23.

[DZ] Dembo, A. and Zeitoni, O. (1992). *Large Deviations Techniques and Applications.* Jones and Bartlett Publishers, Boston.

[DeSt] Deuschel, J.-D. and Stroock, D. (1989). *Large Deviations*, vol. 137. Academic Press.

[DKS] Dobrushin, R. L., Kotecký, R., and Shlosman, S. (1992). *The Wulff Construction: A Global Shape from Local Interactions*. Translations of Mathematical Monographs, vol. 104. AMS, Providence, Rhode Island.

[DN] Dobrushin, R. L. and Nakhapetyan, B. S. (1974). Strong convexity of the pressure for lattice systems of classical statistical physics. *Teoret. Mat. Fiz.* **20**, 223–234.

[DS1] Dobrushin, R. L. and Shlosman, S. B. (1987). Completely analytic interactions. Constructive description. *Journal of Statistical Physics* **46**, 983–1014.

[DS2] Dobrushin, R. L. and Shlosman, S. B. (1992). Large deviation behavior of statistical mechanics models in the multiphase regime. *Proceedings of the Xth Congress on Mathematical Physics, Leipzig 1991* (K. Schmudgen, ed.), Springer-Verlag, Berlin.

[DT] Dobrushin, R. L. and Tirozzi, B. (1977). The central limit theorem and the problem of the equivalence of ensembles. *Communications in Mathematical Physics* **54**, 173–192.

[FO1] Föllmer, H. and Orey, S. (1988). Large deviations for the empirical field of a Gibbs measure. *Annals of Probability* **16**, 961–977.

[FO2] Föllmer, H. and Ort, M. (1988). Large deviations and surface entropy for Markov fields. *Astérisque* **157–158**, 173–190.

[GKK] Gawedski, K., Kotecký, R., and Kupiainen, A. (1987). Coarse-graining approach to the first-oder phase transitions. *Journal of Statistical Physics* **47**, 701–724.

[G1] Georgii, H.-O. (1993). Large deviations and maximum entropy principle for interacting random fields on $\mathbb{Z}^d$. *Annals of Probability* (to appear).

[G2] Georgii, H.-O. (1994). Large deviations and the equivalence of ensembles for Gibbsian particle systems with superstable interactions. *Annals of Probability* (to appear).

[GZ] Georgii, H.-O. and Zessin, H. (1993). Large deviations and the maximum entropy principle for marked point random fields. *Probability Theory and Related Fields* **96**, 177–204.

[IL] Ibragimov, I. A. and Linnik, Yu. I. (1977). *Independent and Stationary Sequences of Random Variables*. Walters–Noordhoff, Groningen.

[I] Isakov, S. N. (1984). Nonanalytic features of the first order phase transition in the Ising model. *Communications in Mathematical Physics* **95**, 427–443.

[Kh] Khinchin, A. I. (1929). Über einen neuen grenzwertsatz der wahrscheinlichkeitsrechnung. *Mathematische Annalen* **101**, 745–752.

[KP] Kotecký, R. and Preiss, D. (1986). Cluster expansion for abstract polymer models. *Communications in Mathematical Physics* **103**, 491–498.

[MM] Malyshev, V. A. and Minlos, R. A. (1991). *Gibbs States; Cluster Expansions*. Kluwer, Dordrecht.

[MS] Minlos, R. A. and Sinai, Ya. G. The phenomenon of 'phase separation' at low temperatures in some lattice models of a gas; I (1967). *Matem. Sbornik* **73**, 375–448; English translation *Math. USSR Sbornik* (1967) **2**, 335–395; II (1968). *Tr. Moskow Mat. Obshch.* **19**, 113–178; English translation *Trans. Moscow Math. Soc.* (1968) **19**, 121–196.

[O] Olla, S. (1988). Large deviations for Gibbs random fields. *Probability Theory and Related Fields* **77**, 343–357.

[P] Petrov, V. V. (1975). *Sums of Independent Random Variables*. Springer-Verlag, Berlin.

[Ri] Richter, W. (1957). Local limit theorem for large deviations. *Teor. Probab. Appl.* **2**, 206–219.

[R] Ruelle, D. (1978). *Thermodynamic Formalism*. Addison-Wesley, Reading.

[SS] Saulis, L. and Statulevičius, V. A. (1991). *Limit Theorems for Large Deviations*. Kluwer, Dordrecht.

[Sch] Schonmann, R. (1987). Second order large deviation estimates for ferrromagnetic systems in the phase coexistence region. *Communications in Mathematical Physics* **112**, 409–422.

[Sh] Shlosman, S. B. (1989). The droplet in the tube: A case of phase transition in the canonical ensemble. *Communications in Mathematical Physics* **125**, 81–90.

[S] Sinai, Ya. G. (1982). *Theory of Phase Transitions: Rigorous Results*. Pergamon Press.

[Sm] Smirnoff, N. (1933). Uber wahrscheinlichkeiten grosser abweichungen. *Rec. Soc. Math. Moscou* **40**, 441–455.

[V] Varadhan, S. R. S. (1984). *Large Deviations and Applications*. Society of Industrial and Applied Mathematics, Philadelphia.

[VF] Ventzel, A. D. and Freidlin, M. I. (1984). *Random Perturbations of Dynamical Systems.* Springer, Berlin.

[Za] Zahradnik, M. (1984). An alternative version of Pirogov–Sinai theory. *Communications in Mathematical Physics* **93**, 559–581.