Critical Points, Large-Dimensionality Expansions, and the Ising Spin Glass

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Abstract

Critical points for the percolation process, for statistical-mechanical models of ferromagnets, and for self-avoiding and self-interacting walks are briefly discussed. The construction of expansions for such critical points in powers of $1/d$, where d is the dimensionality of the underlying hypercubic lattices, is reviewed. Corresponding expansions for the transition points, $T_c(d)$, of Ising model spin glasses with arbitrary symmetric distributions of couplings are derived to order $1/d^3$; for the $\pm J$ model results correct to fifth order are obtained. Numerical results are presented for $d = 3, 4, \ldots, 8$; the lower critical dimensionality appears to be about $d_< = 2.5$.

1. Introduction: Walks and Ferromagnets

Hammersley's pioneering work in formulating and analysing the *bond percolation problem* on a lattice (Broadbent and Hammersley 1957; Hammersley 1957a,b) laid a foundation for the systematic study of the statistics and statistical physics of random media. Our main aim here is to report some recent results concerning phase transitions in random media, specifically, for spin glasses; but, to set the topic in context, we first review the background in a little detail.

In the simplest percolation problem the nearest-neighbour bonds of a uniform space lattice, $\mathcal{L}$, are occupied (or present) with probability p and vacant (or absent) with probability $1 - p$. We will mainly focus on the d-dimensional hypercubic lattices $\mathcal{L}_d$ $(\equiv \mathbb{Z}^d)$ of coordination number

$$q \equiv \sigma + 1 = 2d. \tag{1.1}$$

Occupied bonds which are connected via common sites form clusters; above a *percolation threshold*, $p_c(\mathcal{L})$, an infinite cluster of connected bonds stretches across the lattice with probability one.

A relation between percolation and other statistical problems on lattices was established by Hammersley already in his earliest studies, via the general lower bound

$$p_c(\mathcal{L}) \geq 1/\mu(\mathcal{L}) \tag{1.2}$$

in which $\mu(\mathcal{L})$, the self-avoiding walk limit, was also first defined precisely by Hammersley. Specifically, if $c_m(\mathcal{L})$ is the number of distinct *self-avoiding* (i.e. nonself-intersecting) random walks of m nearest-neighbour steps starting at the origin of $\mathcal{L}$, then

$$\mu(\mathcal{L}) = \lim_{m\to\infty} |c_m(\mathcal{L})|^{1/m}, \tag{1.3}$$

where the existence of the limit follows by a subadditive argument (Hammersley and Morton 1954; Hammersley 1957a). Self-avoiding walks on lattices form natural, somewhat crude but nonetheless informative models of polymer molecules. In particular, the self-avoidance requirement represents the crucial *excluded volume* constraint which is the main theoretical obstacle to be faced in studying the statistical mechanics of polymeric systems.

Subsequently, Fisher and Sykes (1959) pointed out that there was a close parallel between the behavior of self-avoiding walks on a lattice $\mathcal{L}$ and the statistical mechanics of an Ising model of a ferromagnet on the same lattice[1]. An Ising ferromagnet is specified by its Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j)} s_i s_j, \tag{1.4}$$

in which $J > 0$ represents the strength of the coupling between the spin variables, $s_i, s_j, \ldots$ at lattice sites i and j while the sum runs over all nearest-neighbour pairs of spins, i.e., over the lattice bonds. In the standard Ising model each spin takes just two values, $s_i = \pm 1$ (all i). The basic control parameter is the temperature, T, which enters only in the dimensionless combination

$$K = J/k_{\mathrm{B}}T, \tag{1.5}$$

where k_{B} is Boltzmann's constant. The spin-spin correlation functions are defined, as usual, by the thermodynamic expectation values

$$\langle s_i s_k \rangle = \mathrm{Tr}_s\{s_i s_k \exp(-\mathcal{H}/k_{\mathrm{B}}T)\}/\mathrm{Tr}_s\{\exp(-\mathcal{H}/k_{\mathrm{B}}T)\} \tag{1.6}$$

[1] See also Fisher 1966.

in which the trace operation here corresponds to summing over all the values $\{s_i = \pm 1\}$. In terms of the correlation functions the (reduced) susceptibility is given by

$$\chi(T) = \sum_k \langle s_0 s_k \rangle, \tag{1.7}$$

the sum running over all sites k in $\mathcal{L}$.

An Ising ferromagnet in $d \geq 2$ dimensions displays a phase transition at a *critical point*, $T_c(\mathcal{L})$. Above T_c the correlation functions decay to zero exponentially fast with r_{ik}, the distance between sites, and the sum in (1.7) is absolutely convergent. As $T \to T_c+$, however, the susceptibility, $\chi(T)$, diverges strongly to ∞. Below T_c the system displays spontaneous magnetization and long-range order — the correlations do not decay.

An analogy with self-avoiding walks is obtained by considering the generating function

$$C(z;\mathcal{L}) = \sum_{m=0}^{\infty} c_m(\mathcal{L}) z^m. \tag{1.8}$$

One finds that $C(z;\mathcal{L})$ is similar in many ways to χ; in particular, $C(z)$ diverges strongly as $z \to z_c-$ where the critical point is simply

$$z_c(\mathcal{L}) = 1/\mu(\mathcal{L}). \tag{1.9}$$

More concretely one can establish the bound

$$\chi(T;\mathcal{L}) \leq C(\tanh K;\mathcal{L}); \tag{1.10}$$

see Fisher (1967). From this one immediately obtains a bound for the critical point analogous to (1.2), namely,

$$\tanh(J/k_{\mathrm{B}}T_c) \geq 1/\mu(\mathcal{L}) \tag{1.11}$$

(Fisher and Sykes 1959; Fisher 1967).

Ising models have been generalized in various ways important for the study of critical phenomena. In the first instance one has *rigid* or *fixed-length n-vector models* in which the simple Ising spins, s_i, are replaced by n-component vectors, $\vec{s_i}$, of magnitude which is most conveniently taken as $|\vec{s_i}| = \sqrt{n}$ (Stanley 1968, 1969). The coupling term in (1.4) is replaced by $\vec{s_i} \cdot \vec{s_j}$ and the trace operation in (1.6) becomes a product of integrations over the orientations of each $\vec{s_i}$.

A further generalization in this direction, crucial for renormalization group ϵ-expansion theory (Wilson and Fisher 1972), is to regard the spins, $\vec{s_i}$, as continuously variable in magnitude. In this *continuous* or *soft spin* *n-vector model* the trace operation becomes a product of integrals of the form

$$\int_{-\infty}^{\infty} ds_i^{(1)} \cdots \int_{-\infty}^{\infty} ds_i^{(n)} e^{-w(s_i^2)}, \tag{1.12}$$

in which the spin weighting function, $e^{-w(s^2)}$, decays rapidly as $s^2 \to \infty$; the form $w(s^2) = \frac{1}{2}s^2 + us^4$ with $u > 0$ is often considered.

Now it transpires, as first proposed by de Gennes (1972), that the connection between self-avoiding walks and magnetic models is much closer than originally suspected. Indeed, if one formally takes the zero-component limit, $n \to 0$, the susceptibility $\chi_n(t; \mathcal{L})$ for the rigid spin n-vector model becomes identical to the self-avoiding walk generating function, $C(z; \mathcal{L})$, with $z \propto K$. (See also Bowers and McKerrell 1973; Jasnow and Fisher 1976.)

If one studies the limit $n \to 0$ for the *continuous spin* models with weighting factor $e^{-w(s^2)}$, one obtains *self-interacting random walks*: self-intersections are now allowed but each site of the lattice which is visited r (> 0) times by the walk carries a Boltzmann factor or statistical weight given by

$$f_r = \frac{I(2r)}{(r-1)!\, I(2)} \left(\frac{e_0}{2I(2)} \right)^{r-1}, \tag{1.13}$$

in which

$$I(l) = \int_0^{\infty} e^{-w(s^2)} s^{l-1} ds, \tag{1.14}$$

while $e_0 = \exp[-w(0)]$. (Gerber and Fisher 1975; Jasnow and Fisher 1976.) Note that $f_1 \equiv 1$ always holds. When $e_0 = 0$, which is the case for rigid spins, one has $f_r = 0$ for all $r \geq 2$ so that the standard self-avoiding walk is recaptured.

Finally, we remark that the limits $n \to \infty$ and $n \to -2$ also have a special significance in that they correspond to exactly soluble models: this point is expanded in the next section.

Now all the model ferromagnets discussed above pertain, like the self-avoiding walks, to the spatially homogeneous, uniform, nonrandom medium. Considerable interest centers, however, on the study of phase transitions *in random media*. Following the example of the bond percolation problem, the simplest models to consider are *random-bond Ising models* in which, in place of (1.4), the Hamiltonian is

$$\mathcal{H} = -\sum_{(i,j)} J_{ij} s_i s_j, \tag{1.15}$$

where the interactions, J_{ij}, are independent, identically distributed random variables drawn from a specified distribution with a well defined mean and variance,

$$\bar{J} \equiv [J_{ij}]_J \quad \text{and} \quad \triangle J^2 \equiv [(J_{ij} - \bar{J})^2]_J. \tag{1.16}$$

Here and below, $[\cdot]_J$ denotes an expectation over the distribution of coupling constants. The simplest distribution is that of the so called $\pm J$ Ising model in which each bond takes the value $-J$ with probability p and $+J$ (>0) with probability $1-p$. A Gaussian distribution of couplings is also frequently considered.

Such random Ising models have a long history. If the mean, $\bar{J}$, is positive and the width, $\triangle J$, of the distribution is relatively small, one obtains a disorded or impure ferromagnet. The critical temperature, T_c, depends on the distribution of the couplings but the susceptibility still diverges strongly as $T \to T_c+$ and spontaneous magnetization arises below T_c, as it does for the uniform system. (However, the values of the critical exponents, γ, β, etc., describing the nature of the singularities at the critical point will, in general, change.)

On the other hand, if $\bar{J}$ is small enough relative to $\triangle J$, ferromagnetism is completely suppressed. The resulting, highly disordered system represents a *spin glass*. Real systems of this sort, made, for example, by alloying ferromagnetic metals with non-magnetic metals, show freezing phenomena into disordered states reminiscent of the behavior of ordinary glasses. A central question is whether or not this freezing behavior in a spin glass reflects the presence of a true, equilibrium phase transition of some sort: see the reviews by Binder and Young (1986) and Fisher, Grinstein and Khurana (1988). For this purpose it probably suffices to focus, as we will, on a *symmetric spin glass* for which $\bar{J}$ (along with all other odd moments of the coupling distribution) vanishes identically; the only parameter is then the width, $\triangle J$, or the reduced width, $\triangle J/k_{\mathrm{B}}T$.

If there is a transition to a low-temperature spin-glass state — as is now generally believed for systems of dimensionality $d=3$ or greater — many further questions arise. An obvious issue is the value of the transition temperature $T_c(\mathcal{L})$; we will address this specifically in Section 4 et seq.

2. Critical Points for Large Dimensionality

Obtaining explicit expressions for critical points — percolation thresholds, self-avoiding walk limits, or transition temperatures for Ising models — is, in general, a hard task. Exact results are available only in special cases. Thus for the ($d=2$)-dimensional square lattice the bond percolation threshold is $p_c = \frac{1}{2}$ (Harris 1960, Kesten 1980) and the standard Ising model critical point is given by $\tanh(J/k_{\mathrm{B}}T_c) = \sqrt{2}-1$ (Onsager 1944). For no other hypercubic lattices are the precise answers known.

For infinite Cayley trees of uniform coordination number q, or Bethe lattices (Domb 1960a), more detailed analytic progress can be made. The branching ratio on such pseudo-lattice structures is $\sigma = q-1$ and critical points are invariably closely related to σ. For example, for percolation on a Bethe lattice one has $p_c = 1/\sigma$ (Fisher and Essam 1961). Self-avoiding walks are obviously described by $\mu = \sigma$ while for Ising models one finds

$$\tanh(J/k_{\mathrm{B}}T_c) = 1/\sigma \tag{2.1}$$

(Domb 1960a). Note that the inequalities (1.2) and (1.11) became equalities in these cases. Indeed, the departures from these simple results which are found for real lattices are directly related to the existence of closed self-avoiding paths or cycles.

This last assertion can be seen in more quantitative terms in the formal expansions for critical points in inverse powers in the dimensionality, d, constructed by Fisher and Gaunt (1964) for the Ising model and self-avoiding walk problem on hypercubical lattices, $\mathcal{L}_d$. They obtained

$$\frac{T_c^{(d)}}{T_c^0} = 1 - \frac{1}{q} - 1\tfrac{1}{3}\frac{1}{q^2} - 4\tfrac{1}{3}\frac{1}{q^3} - 21\tfrac{34}{45}\frac{1}{q^4} - 133\tfrac{14}{15}\frac{1}{q^5} - \cdots, \tag{2.2}$$

with $T_c^0 = qJ/k_{\mathrm{B}}$ and, as before, $q = 2d = \sigma + 1$, and

$$\frac{\mu(d)}{q} = 1 - \frac{1}{q} - \frac{1}{q^2} - \frac{3}{q^3} - \frac{16}{q^4} - \frac{102}{q^5} - \cdots. \tag{2.3}$$

In terms of σ this last result simplifies to

$$\mu(d) = \sigma\left(1 - \frac{1}{\sigma^2} - \frac{2}{\sigma^3} - \frac{11}{\sigma^4} - \frac{62}{\sigma^5} - \cdots\right), \tag{2.4}$$

which shows that the corrections to the Bethe lattice value for μ are only of order $\sigma^{-2} \sim d^{-2}$; the same is true for the Ising model critical points. The detailed analysis sketched later shows that this correction reflects the fact that through each site on a hypercubic lattice pass precisely $2d(d-1)$ distinct squares constructed of nearest-neighbour bonds (Fisher and Gaunt 1964), as is readily checked.

The question of the convergence of these $1/d$ expansions will be addressed below. It is worth noting here, however, that the first three terms on the righthand sides of (2.2) and (2.3) provide rigorous upper bounds on $T_c(d)$ and $\mu(d)$ correct to order $1/d^3$ when $d \to \infty$; see Fisher (1967).

The Ising model result (2.2) was extended by Gerber and Fisher (1974) to the fixed-length n-vector model yielding

$$\begin{aligned}\frac{T_c(n,d)}{T_c^0} = 1 &- \frac{1}{q} - \frac{1}{q^2}\left(1 + \frac{n}{n+2}\right) - \frac{1}{q^3}\left(3 + \frac{4n}{n+2}\right)\\ &- \frac{1}{q^4}\left(16 + \frac{(21n+32)n}{(n+2)^2} - \frac{2n^2}{(n+2)(n+4)}\right)\\ &- \frac{1}{q^5}\left(102 + \frac{(129n^2+422n+340)n}{(n+2)^3} - \frac{16n^2}{(n+2)(n+4)}\right)\\ &- \cdots.\end{aligned} \tag{2.5}$$

For $n = 1$, which corresponds to the simple Ising case, this reduces correctly to (2.2). In addition, on setting $n = 0$, it evidently reproduces the self-avoiding walk result (2.3)!

Another limit is also of interest here: specifically one may take $n \to \infty$ to obtain

$$\frac{T_c(\infty, d)}{T_c^0} = 1 - \frac{1}{q} - \frac{2}{q^2} - \frac{7}{q^3} - \frac{35}{q^4} - \frac{215}{q^5} - \cdots . \tag{2.6}$$

Now, as first demonstrated by Stanley (1968b), the limit $n \to \infty$ in the n-vector model yields the *spherical model* devised by Berlin and Kac (see Joyce 1972). This model is exactly soluble in a wide variety of cases. For hypercubic lattices, the critical points, $T_c(d)$, are given by d-fold integrals over the basic lattice generating function which can be reduced to a single integral involving the dimensionality, d, only through a factor $[I_0(x)]^d$, where $I_0(x)$ is the Bessel function of zero order and pure-imaginary argument. Using this fact, Gerber and Fisher (1974) showed that $T_c(d)$ extends naturally into a function of d which is analytic on or near the real axis for $2 < d < \infty$. Furthermore, the inverse dimensionality expansion (2.6) can be checked (and extended indefinitely). The analysis also establishes that the $1/d$ expansion is (for $n = \infty$) asymptotic rather than convergent and suggests that truncation after the term of order $1/d^{l^\star}$ with $l^\star \simeq 1.62d$ is optimal numerically[2].

The analytic nature of $T_c(n, d)$ and the asymptotic character of the $1/d$ expansion have not been established for general n but it seems likely that both are, in fact, true. Certainly the expansion produces good numerical results even down to $d = 3$ and for $n = 0$ and 1 (see Gerber and Fisher 1974).

The dimensionality expansion can also be carried through for the general, continuous-spin n-vector model (Gerber and Fisher 1975). The expansion coefficients now depend on the reduced noninteracting-spin moments

$$M_{2k}(n) = m_{2k}(n)/m_2^k(n) \equiv \langle|\vec{s}|^{2k}\rangle_0 / \langle|\vec{s}|^2\rangle_0^k, \tag{2.7}$$

defined in terms of the spin-weighting function via

$$m_{2k}(n) = \int_0^\infty s^{2k+n-1} e^{-w(s^2)} ds \bigg/ \int_0^\infty s^{n-1} e^{-w(s^2)} ds. \tag{2.8}$$

[2] Abe (1976) has proposed a modification of the $1/d$ expansion which is actually convergent in the spherical model limit.

To order $1/d^2$ the calculations yield

$$\begin{aligned}\frac{T_c(n,d)}{T_c^0(n)} &= 1 - \frac{1}{q}\left[1 - \tfrac{1}{2}n(M_4 - 1)\right] \\ &\quad - \frac{1}{q^2}\left[2\frac{n+1}{n+2} - \tfrac{1}{2}\frac{n^2}{n+2}(M_4^2 - 1) - \tfrac{1}{8}n^2(M_6 - 2M_4^2 + M_4)\right] \\ &\quad - \cdots, \qquad \text{with} \quad T_c^0(n) = qJm_2(n)/k_{\mathrm{B}}. \end{aligned} \tag{2.9}$$

The term of order $1/d^3$ was also found by Gerber and Fisher (1975) but is not quoted here because of its length. It is interesting to note that the moments M_{2k}, of the spin-weighting factor enters for the first time only in order $1/d^{k-1}$.

Yet another exactly soluble magnetic model is now accessible, namely, the so called Gaussian model characterized by the spin weighting function $w(s^2) = w_0 s^2$. This model was originally studied by Kac and Berlin in connection with the spherical model (Joyce 1972). The spin moments for the Gaussian model are

$$M_{2k}(n) = (n+2)(n+4)\ldots(n+2k-2)/n^{k-1}, \tag{2.10}$$

for $k \geq 2$. On substituting this form into the terms of order q^{-1}, q^{-2}, and q^{-3} in (2.9) one finds that all the coefficients vanish identically! This is, in fact, in agreement with the exact Gaussian model result $T_c(n,d) = T_c^0(n)$; clearly the series converges absolutely in this special case!

It has, furthermore, been demonstrated that if the n-vector model is continued to $n = -2$, the Gaussian model is again recaptured (Balian and Toulouse 1973; Fisher 1973) irrespective of the form of $w(s^2)$. This conclusion, which is contingent on $w(s^2)$ remaining bounded as $s^2 \to 0$, can also be checked in (2.9) by analytically continuing the integrals in (2.8) to $n < 0$. One finds $M_{2k}(-2) = 0$ for $k \geq 2$ (see Gerber and Fisher 1975); this value agrees with the Gaussian form (2.10).

Lastly, it is worth quoting the limit for self-interacting walks which follows from (2.9) with (1.13). One obtains

$$\begin{aligned}\frac{M\{f_r\}}{q} &= 1 - (1-f_2)q^{-1} - (1-f_3)q^{-2} \\ &\quad - (3 + f_2 - 4f_2^2 - 2f_2f_3 + 3f_3^2 - f_4)q^{-3} + \cdots, \end{aligned} \tag{2.11}$$

which, of course, reproduces (2.3) when the weight factors, f_r, for r-fold intersections vanish. Conversely, when $f_r = 1$ for all r the series reduces simply to $\mu = q$.

3. Expanding in Inverse Dimensionality

The results summarized above for uniform, nonrandom lattices rest on an analysis of the high-temperature series expansions for the susceptibilities, $\chi_n(T;\mathcal{L}_d)$, for the magnetic systems in question (Fisher and Gaunt 1964). In terms of $K = J/k_\mathrm{B}T$ one has

$$\chi_n(T;\mathcal{L}_d) = \sum_{l=0}^{\infty} a_l(n,d)K^l. \tag{3.1}$$

In practice one observes that all the coefficients a_l are nonnegative. Then the nearest singularity of $\chi_n(T)$ lies on the real positive K axis at a K_c which locates the physical critical point, $T_c(d)$. Consequently one can write

$$\ln[k_\mathrm{B}T_c(d)/J] = \limsup_{l\to\infty} l^{-1}\ln a_l(n,d). \tag{3.2}$$

The aim is then to calculate $a_l(n,d)$ to leading orders in d for all sufficiently large l.

Now, rather generally, the susceptibility expansion coefficients can be written in graphical form as

$$a_l(n,d) = \sum_{G_l}(G_l,\mathcal{L}_d)A(n;G_l), \tag{3.3}$$

in which G_l represents a graph or multigraph of l lines (following the graph-theoretical terminology set out by Essam and Fisher 1970). The dominant graph in all cases is a chain, C_l, of l lines and $l+1$ vertices. The statistical weights, $A(n;G_l)$, depend on the nature of the interaction, the spin-dimensionality, n, and on the graph G_l but *not* on the lattice $\mathcal{L}_d$. Often the theory is arranged so that $A(n;G_l)$ vanishes if G_l is not a connected graph. In more favorable cases the weights may also vanish if G_l is not multiply connected; the sum in (2.14) may then be restricted to star graphs which is advantageous since they are much less multitudinous.

The lattice-dependence of $a_l(n,d)$ and, hence, the dependence on dimensionality, is isolated in the embedding constants or lattice constants, $(G_l,\mathcal{L}_d)$, which represent the number of ways of embedding the graph, G_l, in the lattice $\mathcal{L}_d$ *per site*. (The rules of embedding may depend on the details of the analysis but the so called *weak lattice constants* — see Essam and Fisher — are usually most convenient.) As stressed by Fisher and Gaunt (1964) the lattice constant, $(G_l,\mathcal{L}_d)$, for a graph of l lines is, on reflexion, easily seen to be a polynomial in d or, equivalently, in q or σ, of degree at most l. Thus for the chain, C_l, the lattice constant is $(2d)^l[1+O(d^{-1})]$; the square has a lattice constant $\frac{1}{2}d(d-1)$; in leading order the lattice constant for a double bond with one or two tails (of all possible lengths totalling $l-2$ lines) varies as $\frac{1}{2}(l-1)(2d)^{l-1}$; and so on.

Finally, then, the coefficient $a_l(n,d)$ is itself a polynomial in d of order l, the coefficient of d^{l-k} being a polynomial in l of order at most k. This structure enables one to remove a factor d^l in (3.2) leading to an overall additive term $\ln l$. Then one may expand the logarithm formally in powers of d^{-1} and take the limit $l \to \infty$ term by term. The desired large-dimensionality expansion or, rather its logarithm, results.

This method adapts readily to other problems when a susceptibility-like function can be identified. Thus Gaunt, Sykes and Ruskin (1976) and Gaunt and Ruskin (1978) considered site and bond percolation problems, respectively. Working with $S(p)$, the mean cluster size function for $p < p_c$, which diverges strongly as $p \to p_c$, they obtained expansions for the percolation threshold. Specifically Gaunt and Ruskin (1978) found

$$p_c(d) = \sigma^{-1}\bigl(1 + 2\tfrac{1}{2}\sigma^{-2} + 7\tfrac{1}{2}\sigma^{-3} + 57\sigma^{-4} + \cdots\bigr), \tag{3.4}$$

for bond percolation while, for site percolation one has

$$p_c(d) = \sigma^{-1}\bigl(1 + 1\tfrac{1}{2}\sigma^{-1} + 3\tfrac{3}{4}\sigma^{-2} + 20\tfrac{3}{4}\sigma^{-3} + \cdots\bigr), \tag{3.5}$$

(Gaunt, Sykes and Ruskin 1976). It is interesting that site percolation has a leading correction of order σ^{-1} whereas for bond percolation, as for self-avoiding walks, see equation (2.4), this term vanishes.

It should also be mentioned that Harris (1982) has reviewed the use of $1/d$ expansions in a more general context and has developed a method which, in principle, can cast a variety of problems into amenable form. With his methods, he obtained explicit results for the critical fugacity for lattice animals and for the mobility edge in localization theory.

Here we ask the question: "How can similar results be obtained for spin glasses and what do they tell us?" In the following sections we answer these questions. [A brief announcement of our results has been published (Singh and Fisher 1988).]

4. Ising Spin-Glass Susceptibilities

We will study the Ising spin glass with Hamiltonian (1.15) and spins $s_i = \pm 1$ at each site i of $\mathcal{L}_d$. For simplicity we restrict attention to symmetric spin glasses for which

$$[(J_{ij}^{2k+1}]_J = 0 \quad (k = 1, 2, \ldots). \tag{4.1}$$

Such distributions embody the crucial 'frustration' induced by competing interactions which lies at the heart of the spin-glass problem. While retaining symmetry, it is of interest, however, to allow for general distributions of the J_{ij}. In particular one would like a theoretical basis for comparing the predictions of Monte Carlo simulations of spin glass behavior, notably by Bhatt and Young (1985) and Ogielski and Morgenstern (1985) (see also Ogielski 1985), which in some cases have used only a Gaussian distribution, with extensive series expansion studies by Singh and Chakravarty (1986, 1987a,b) which employed the $\pm J$ model.

The basic indicators of order are the spin-spin correlation functions, $\langle s_i s_k \rangle$; see (1.6). For a general Ising model these have a graphical expansion in terms of the auxiliary temperature variables

$$v_{ij} = \tanh(J_{ij}/k_{\mathrm{B}}T), \tag{4.2}$$

which may be written

$$\langle s_i s_k \rangle = Z^{-1} \sum_{G^2(i,k)} \prod_{(j,l)} v_{jl} \qquad (i \neq k), \tag{4.3}$$

where the partition function, Z, has a similar expansion, namely,

$$Z = 1 + \sum_{G^0} \prod_{(j,l)} v_{jl}. \tag{4.4}$$

In these expressions the sums run over all distinct weak embeddings of the graphs $G^2(i,k)$ and G^0 in the lattice $\mathcal{L}$, which is most conveniently regarded here as finite with N sites and periodic boundary conditions. The products run over all lattice bonds, (j,l), covered in the embedding of the graph. The graphs G^0 are generalized polygons: they have no repeated lines and at each site of the lattice an even number, 0, 2, 4, ..., of lines must meet. The prescription for the two-rooted graphs, $G^2(i,k)$ is the same except that an *odd* number of lines must meet at the sites i and k. Note that both Z and the numerator for $\langle s_i s_k \rangle$ are *linear* functions of each v_{jl}.

Having obtained the correlation functions, or any other property, for a given realization, $\{J_{ij}\}$, of the couplings of the spin glass, one must perform the average, $[\cdot]_J$, over the coupling distributions. Because of the linearity of Z and $\langle s_i s_k \rangle Z$ in the v_{jl}, each term in the full expansion of $\langle s_i s_k \rangle$ contains an odd power of at least one bond variable, v_{jl}. Consequently we have

$$[\langle s_i s_k \rangle]_J \equiv 0 \quad (\text{all } i \neq k). \tag{4.5}$$

This result will remain true in the thermodynamic limit, $N \to \infty$, in the disordered, high-temperature region above any transition. As a result, the standard susceptibility of a spin glass, as calculated by averaging the expression in (1.7), reduces simply to a constant, explicitly one has $\chi =$

$[\langle s_0^2 \rangle]_J = 1$. Evidently this susceptibility is totally independent of the spin-spin interactions and contains no information about any possible spin-glass transition!

The way around this difficulty is to consider, instead, the generalized susceptibilities

$$\chi_{(q,r)}(T) = N^{-1} \sum_i \sum_k [\langle s_i s_k \rangle^q]_J^r \tag{4.6}$$

(Singh and Chakravarty 1987a). Indeed, the case $q = 2$, $r = 1$ corresponds to the so called spin-glass susceptibility, $\chi_{\rm SG}$, introduced originally by Edwards and Anderson (1975). We will focus on this special susceptibility. [Of course, $\chi_{(q,r)}$ vanishes identically whenever q is odd; the next nontrivial case, $q = 2, r = 2$, yields extra information in studies based on the numerical extrapolation of high-temperature power series (Singh and Chakravarty 1987a).]

The graphical expansion for $\chi_{\rm SG}$ follows from (4.3) by squaring, dividing through using (4.4), and averaging over the bond distribution term by term. It is instructive to consider, first, a spin glass on a Bethe lattice. This problem is analytically tractable having been first studied by Japanese workers (Oguchi and Ueno 1976; Katsura, Fujiki and Inawashiro 1979: see also references in Chayes et al. 1986). It has been revisited more recently by Thouless (1986) and coworkers (Chayes, Chayes, Sethna and Thouless 1986), particularly to investigate behavior in an external field.

Now there are no closed polygons on a Bethe lattice so, by (4.4), one has $Z = 1$. Likewise, any two sites, i and k, are connected by a single chain of $l(i,k)$ bonds. Thus, after averaging, the only graphical contribution to $[\langle s_i s_k \rangle^2]_J$ comes from a chain, $C^2(i,k)$, of *doubled bonds* reaching from i to k. If we define the moments of the coupling distribution via

$$w_q(T) = [v_{ij}^{2q}]_J \equiv [\tanh^{2q}(J_{ij}/k_{\rm B}T)]_J, \tag{4.7}$$

we thus have

$$[\langle s_0 s_k \rangle^2]_J^0 = w_1^{l(0,k)}, \tag{4.8}$$

where the superscript zero indicates the Bethe lattice. To use this, we may formally take the thermodynamic limit in (4.6) by dropping the first summation and the factor N^{-1}; this yields the analogue of (1.7). To perform the remaining sum over the sites k, we note that there are just $q\sigma^{l-1}$ distinct self-avoiding paths of l steps leaving the origin, 0, of a Bethe lattice of coordination number q. Summing on l yields the explicit, high-temperature spin-glass susceptibility for a Bethe lattice, namely,

$$\chi_{\rm SG}^0 = [1 + w_1(T)]/[1 - \sigma w_1(T)]. \tag{4.9}$$

Evidently $\chi_{\rm SG}^0$ diverges at a critical point given by

$$w_{1c}^0 \equiv [\tanh^2(J_{ij}/k_{\rm B}T_c^0)]_J = 1/\sigma. \tag{4.10}$$

This is a natural analogue of the formula (2.1) for a ferromagnetic Bethe lattice. Note, incidentally, that for a $\pm J$ distribution one has $\triangle J = J\ (>0)$ and the moments become $w_q = \tanh^{2q}(J/k_{\mathrm{B}}T)$. Thus (4.10) reduces simply to $\tanh(J/k_{\mathrm{B}}T_c) = 1/\sqrt{\sigma}$. Comparing with (2.1) shows that the critical temperature of the spin glass is much lower than of the corresponding ferromagnet, in accord with the obvious effects of having negative antiferromagnetic bonds competing with positive, ferromagnetic couplings.

5. Expansion for a Hypercubic Spin Glass

We may anticipate that (4.10) will provide the leading large-d behavior for spin glasses on hypercubic lattices with $q = 2d$. To show this, we must allow for the lattice polygons. The first point then is that, $c_l(\mathcal{L}_d)$, the total number of self-avoiding paths or chains, C_l, of l steps leaving the origin, is no longer given by $q\sigma^{l-1}$. Rather this large-d form must be multiplied by a correction factor $\lambda(C_l)$ — the *reduced lattice constant* (Gerber and Fisher 1974). It is instructive to reproduce the calculation of this lattice constant in leading nontrivial order. The dominant correction to $c_l \simeq q\sigma^{l-1}$ comes from the closure of a square, P_4, of four bonds (or steps). This may occur at any one of $(l-4+1) = (l-3)$ positions along the chain. As mentioned in Section 2, through each point in the lattice there pass $2d(d-1) = \frac{1}{2}(\sigma+1)(\sigma-1)$ distinct squares. Each such square may be traced by a chain/walk in two possible senses. The remaining $l-4$ bonds of the chain may, in leading order be regarded as 'free' and so are associated with $q\sigma^{l-5}$ configurations. In total, therefore, one must subtract the term $(l-3)(\sigma^2-1)q\sigma^{l-5}$ from $q\sigma^{l-1}$. Finally, to leading order, the desired correction factor is thus

$$\lambda(C_l) = 1 - (l-3)\sigma^{-2} - \cdots . \tag{5.1}$$

A little reflection shows that allowing for hexagons, P_6, yields a correction of order σ^{-4}; however, a correction of lower order, σ^{-3}, arises from subtracting generic configurations, to be denoted $CP_4^{(1)}$, in which the chain or path overlaps one side of the square so yielding a doubled bond: see the graph labelled [c] in Fig. 1. Such configurations were not eliminated in the leading order calculation. This term and higher order ones up to order σ^{-5} were originally calculated by Fisher and Gaunt (1964). The resulting formula for $\lambda(C_l)$ and for other reduced lattice constants needed here have been listed correct to order σ^{-5} by Gerber and Fisher (1974).

To go further in the calculation of χ_{SG}, one must account for polygons which arise directly in the graphical expansion as products of bond factors v_{ij}. To this end let $\{G\}$ denote the sum of all products of the v_{ij} corresponding to the embeddings in $\mathcal{L}$ of all graphs isomorphic to G, as required in (4.3) and (4.4). Then the expansion of the correlation function may be

FIG. 1. Generic graphs of l lines which are needed in the calculation of the $1/d$ expansion for a spin glass. Except for $[u]$ and $[x]$, the labelling follows Gerber and Fisher (1974) who give the reduced lattice constants. Note the graphs $[c]$, $[j]$ and $[x]$ have the same skeleton graph of $l' = l-1, l-3$, and $l-5$ lines and thus have simply related lattice constants.

written

$$\langle s_0 s_k \rangle = \sum_l \frac{\{C_l\} + \{C_l, P_4\} + \{C_l, P_6\} + \{C_l, P_4, P_4\} + \cdots}{1 + \{P_4\} + \{P_6\} + \{P_4, P_4\} + \{P_8\} + \cdots}, \tag{5.2}$$

for $k \neq 0$. Here it is understood that each chain of l lines, C_l, is rooted at site 0 and terminates at site k. The polygons of m sides, P_m, may occur singly or, as indicated by a comma, as *disconnected* multiplets, which means, in the present case, that they have *no common bonds* although they may share one or more sites. In dividing out the denominator one obtains products of terms; most, however, cancel to leave only terms with repeated bonds. Thus one has

$$\{C_l, P_m\} - \{C_l\}\{P_m\} = -\sum_{r=1}^{m-1}\{C_l P_m^{(r)}\} - \sum\{C_l P_m^*\}. \tag{5.3}$$

Here, as above, $C_l P_m^{(r)}$ denotes a generic multigraph consisting of a polygon P_m and a chain, C_l, which overlaps it in all possible ways, on r bonds (which are thus doubled); such graphs cannot appear in the numerator of (5.2). The extra terms $\{C_l P_m^*\}$ include contributions from connected graphs in which the chain touches the P_m at an isolated vertex two or more times but does not overlap a bond of P_m; such graphs do arise in the numerator of (5.2) but they appear only once whereas the product in (5.3) generates them more often. Likewise, excess terms having both overlapping bonds and vertex contacts must be subtracted. Leading contributions to the expansion on a hypercubic lattice are then

$$\begin{aligned}\langle s_0 s_k\rangle = \sum_l \Big(&\{C_l\} - \{C_l P_4^{(1)}\} - (\{C_l P_4^{(2)}\} + \{C_l P_6^{(1)}\} + \{C_l P_6^{\dagger}\}) \\ &- (\{C_l P_4^{(3)}\} + \{C_l P_6^{(2)}\} + \{C_l P_8^{(1)}\} + \cdots) \\ &- (\{C_l P_6^{(3)}\} + \{C_l P_6^{(4)}\} + \{C_l P_6^{(5)}\} + \cdots) \\ &- (-\{C_l P_4^{2(1)}\} + \cdots) - \cdots\Big).\end{aligned} \tag{5.4}$$

The second term, involving $P_4^{(1)}$, generates a contribution of relative order σ^{-3} in the expansion for the critical point of an Ising *ferro*magnet; successive terms in parentheses likewise contribute to terms of order $\sigma^{-4}, \sigma^{-5}, \ldots$. The symbol $C_l P_6^{\dagger}$ in the third term denotes a chain which cuts a hexagon diametrically forming two squares sharing a common bond: see the graph $[i]$ in Fig. 1. This will not actually be needed to the order developed here. However, the generic graph $C_l P_4^{2(1)}$, in the last term displayed, will be needed: this is a chain of l single bonds that overlaps a square, P_4, of *doubled* bonds along one side: see graph $[x]$ in Fig. 1.

The first three terms presented in (5.4) actually suffice to generate the expansion (2.2) for a hypercubic Ising ferromagnet correct to order $1/q^4$. For the spin glass, however, it transpires that most of the further graphical terms exhibited are also needed even at order $1/q^3$. To see which matter, we square the expansion for $\langle s_0 s_k\rangle$ and perform the spin-glass average, $[\cdot]_J$. Prior to averaging, the expansion will contain multigraphs with bonds of all multiplicities; but, on averaging, any graph containing a bond of odd multiplicity makes a vanishing contribution. On the other hand, each double bond contributes a factor $w_1(T)$, each quadruple bond, a factor $w_2(T)$, and so on. Evidently the square of each term in (5.4) contributes directly so that, for example, $\{C_l P_4^{(1)}\}$ (see [c] in Fig. 1) appears with each bond doubled and weight $w_1^{l+2} w_2$, the factor w_2 arising from the doubling of the original doubled bond. However, cross terms also contribute. Thus the product $\{C_l\}\{C_l P_4^{2(1)}\}$ of the first and last terms displayed in (5.4) appears twice and yields two further terms of the same weight. The product $\{C_l P_4^{(1)}\}\{C_l\}$ yields a contribution with the same skeleton as $C_l P_{\underline{4}}^{(1)}$ but with each bond doubled and hence weight w_1^{l+3}. (In the *skeleton*, $\overline{G}$, of a multigraph G, all multibonds are collapsed to single lines.)

Overall, we obtain the spin-glass susceptibility in the form

$$\chi_{\mathrm{SG}}(T; d) \equiv \sum_k [\langle s_0 s_k \rangle^2]_J = 1 + \sum_{l=1} w_1^l b_l(d), \tag{5.5}$$

with coefficients given graphically by

$$\begin{aligned} b_l = [\overline{C}_l] - 4[\overline{C_{l-3} P_4^{(1)}}] - 2[\overline{C_{l-2} P_4^{(2)}}] - 2\sum_{r=1}^{5} [\overline{C_{l-6+r} P_6^{(r)}}] \\ + 3(w_2/w_1^2)[\overline{C_{l-4} P_4^{(1)}}] + \cdots, \end{aligned} \tag{5.6}$$

where the notation $[\overline{G}]$ now denotes the generic lattice constants with one end of the chain rooted at the origin. Note the factor (w_2/w_1^2) which comes from the quadruple bond which arises as explained above. Factors, (w_4/w_1^4), (w_3/w_1^3), etc. appear in higher order terms.

As mentioned, all but one of the required lattice constants in (5.6) have been computed by Gerber and Fisher (1974). It is clear that a given lattice constant depends only on the skeleton graph; however, the precise expressions for the reduced lattice constants, $\lambda(G)$, depend on the total number of lines. With this in mind, the spin-glass susceptibility coefficient may be written, adapting the notation of Gerber and Fisher (1974), as

$$\begin{aligned} b_l(d) = q\sigma^{l-1}\Big\{\lambda([a]_l) - 4\lambda([c]_{l+1}) - \lambda([d]_{l+2}) - 4\lambda([h]_{l+1}) - 4\lambda([n]_{l+2}) \\ - 4\lambda([u]_{l+3}) - 3(w_2/w_1^2)\lambda([c]_l) + \cdots\Big\}. \end{aligned} \tag{5.7}$$

The generic graph [u] corresponds to $C_l P_6^{(3)}$ (see Fig. 1). It was not considered by Gerber and Fisher but its lattice constant is the same, to leading order, as that for [n]. Finally, using the data for the $\lambda(G)$ yields

$$b_l(d) = q\sigma^{l-1}\left\{1 - \frac{7l - 11}{\sigma^2} - \frac{24l - \delta - 3(l-4)w_2/w_1^2}{\sigma^3} - \cdots\right\}, \tag{5.8}$$

where the integer δ is determined by terms of order l^0 in the lattice constants, which were not retained by Gerber and Fisher (1974); however, the value of δ proves immaterial here.

Now, following the procedure outlined in Section 3, we may finally compute the expansion for $(b_l)^{1/l}$ and take the limit $l \to \infty$ in order to identify the critical value, $(w_{1c})^{-1}$: this yields

$$\frac{1}{w_1(T_c)} = \sigma\left(1 - \frac{7}{\sigma^2} - \frac{24 - 3(w_2/w_1^2)}{\sigma^3} - \cdots\right), \tag{5.9}$$

which is the desired critical point expansion.

In the case of the ferromagnetic models we remarked, in partial justification of the last step, that the known expansion coefficients, a_l, for the susceptibility, χ, are observed to be positive; if true for all l, this means the limit of $(a_l)^{1/l}$ does correctly generate the physical singularity. For the spin-glass susceptibility, χ_{SG}, however, one finds *negative* coefficients, b_l, for $d = 2$ and 3 (Singh and Chakravarty 1986). The known coefficients (with $l \leq 15$) for $d = 4$ are all positive but have a strong alternation and might well become alternating in sign for a larger l. For low d, at least, it thus seems likely that the nearest singularity in the complex $1/T$ plane is *not* the physical singularity. In that case an exact computation of the limit of $[b_l(d)]^{1/l}$ at fixed d would not yield the critical point. We believe, nonetheless, that the procedure we have used will generate the correct asymptotic expansion for the spin-glass critical point.

The factor (w_2/w_1^2) on the right of (5.9) takes the value unity for the $\pm J$ distribution. More generally, however, it must depend on T_c: see equation (4.7). In that case the expansion is really implicit rather than explicit. Furthermore, by examining the higher order terms one sees that factors (w_2/w_1^2) and $(w_2/w_1^2)^2$ appear in order σ^{-4}. The reduced sixth moment of the bond distribution appears first, via a factor (w_3/w_1^3) only in order σ^{-6}. One may, however, generate an explicit expansion for the spin-glass critical temperature for a fixed bond-coupling distribution with reduced moments

$$\rho_q = [(J_{ij})^q]_J / [(J_{ij})^2]_J^{q/2}, \tag{5.10}$$

by expanding $w_1 = [\tanh^2(J_{ij}/k_{\mathrm{B}}T)]_J$ in powers of $1/T$, reverting the series and using (5.9). This yields

$$\frac{k_B^2 T_c^2}{\triangle J^2} = \sigma\Big[1 - \tfrac{2}{3}\rho_4\frac{1}{\sigma} - \big(7 + \tfrac{4}{9}\rho_4^2 - \tfrac{17}{45}\rho_6\big)\frac{1}{\sigma^2} - \big(24 - 3\rho_4 + \tfrac{62}{315}\rho_8 - \tfrac{34}{45}\rho_4\rho_6 + \tfrac{16}{27}\rho_4^3\big)\frac{1}{\sigma^3} - \cdots\Big]. \tag{5.11}$$

Before discussing the expansion in quantitative terms we present an alternative method of derivation which has enabled us to generate the series for the $\pm J$ models correct to order σ^{-5}.

6. The Inverse Susceptibility Expansion

Singh and Chakravarty (1986, 1987a) discovered that it was possible to expand the free energy and inverse spin-glass susceptibility, $1/\chi_{\mathrm{SG}}$, of an Ising spin glass on a general lattice in a way that only required star graphs (i.e., multiply connected graphs). The significant advantage of such an expansion, which entails a cluster algorithm to generate the appropriate weights, is that for a given number of lines there are far fewer star graphs than the more general graphs required in a direct calculation of χ_{SG}. As a result, for hypercubic lattices Singh and Chakravarty were able to calculate the expansion for the $\pm J$ model to orders w_1^{19}, w_1^{17}, and w_1^{15} for the square, simple cubic, and $(d = 4)$-dimensional hypercubic lattices, respectively. This greatly extended the pioneering work of Fisch and Harris (1977) whose series proved, unfortunately, too short for reliable numerical extrapolation.

Now, as explained in Section 3, the lattice constant for a given star graph is a finite polynomial in d (or q) with, as one easily sees, a vanishing constant term. The most 'open' star graph of l lines is a polygon. Since a polygon of $l = 2p$ lines can explore at most p different spatial dimensions its lattice constant is of order at most d^p (Fisher and Gaunt 1964). Thus each term in the expansion for $1/\chi_{\mathrm{SG}}$ can be written as a polynomial in d. Furthermore, since the star graphs of l lines enter no earlier than in the term of order w_1^l, the expansion to order w_1^{2p} entails no powers of d higher than d^p.

Given this information it is actually possible to obtain the polynomials representing the expansion coefficients of $1/\chi_{\mathrm{SG}}$ for a fixed bond distribution knowing only their numerical values for various dimensionalities. Specifically, from the numerical expansions in dimensions $d = 1$ [for which the Bethe lattice form (4.9) is exact], and $d = 2, 3,$ and 4 (Singh and Chakravarty 1986) one can compute the first nine polynomials. However, using the lattice constants of Fisher and Gaunt, which include all stars of 10 lines, one can go to order w_1^{10} and use the numerical results as a cross check. Writing $w_1 \equiv w$, the result for the $\pm J$ model is found to be

$$\begin{aligned}\chi_{\mathrm{SG}}^{-1}(T) = 1 &- qw + qw^2 - qw^3 + (7q^2 - 13q)w^4 \\ &- (30q^2 - 59q)w^5 + (44q^3 - 169q^2 + 163q)w^6 \\ &- (352q^3 - 1712q^2 + 2017q)w^7 \\ &+ (405q^4 - 3026q^3 + 8503q^2 - 8141q)w^8 \\ &- (3968\tfrac{2}{3}q^4 - 35266\tfrac{2}{3}q^3 + 107011\tfrac{1}{3}q^2 - 104704\tfrac{1}{3}q)w^9 \\ &+ (4712q^5 - 61157q^4 + 336356q^3 - 846314q^2 + 761069q)w^{10} \\ &- \cdots . \end{aligned} \tag{6.1}$$

Now the critical point is determined by the divergence of χ_{SG} or the vanishing of χ_{SG}^{-1}. If we set $qw = y$ in the result (6.1), we can write the critical equation $\chi_{\mathrm{SG}}^{-1}(y) = 0$ as

$$\begin{aligned} y = 1 &+ y^2 q^{-1} + (7y^4 - y^3)q^{-2} + (44y^6 - 30y^5 - 13y^4)q^{-3} \\ &+ (405y^8 - \cdots + 59y^5)q^{-4} + (4712y^{10} - \cdots + 163y^6)q^{-5} \\ &+ O(q^{-6}). \end{aligned} \tag{6.2}$$

This is readily solved by reversion which finally yields the expansion

$$\frac{1}{w_1(T_c)} = q\left(1 - \frac{1}{q} - \frac{7}{q^2} - \frac{28}{q^3} - \frac{219}{q^4} - \frac{1905\frac{1}{3}}{q^5} - \cdots\right). \tag{6.3}$$

Of course, this result for the $\pm J$ model can be checked to order $1/q^3$ against the original $1/\sigma$ expansion (5.9). However, the two further terms prove helpful in using the expansion numerically. For convenience we also quote

$$\frac{1}{\tanh\left(J/k_{\mathrm{B}}T_c\right)} = \sqrt{\sigma}\left(1 - \frac{3\frac{1}{2}}{\sigma^2} - \frac{10\frac{1}{2}}{\sigma^3} - \frac{91\frac{1}{8}}{\sigma^4} - \frac{699\frac{5}{12}}{\sigma^5} - \cdots\right). \tag{6.4}$$

d	$1/\sigma$ series w_c	order	High-T series w_c	Biassed Padé w_c
3	0.5036	4	0.48 $\pm$ 0.04	0.48 ($\pm$ 0.04)
4	0.2133	5	0.21 $\pm$ 0.01	0.21 ($\pm$ 0.01)
5	0.1322	5	0.139 $\pm$ 0.002	0.133 $\pm$ 0.003
6	0.1002	5	0.102 $\pm$ 0.002	0.1005 $\pm$ 0.0006
7	0.0818	5	0.083 $\pm$ 0.001	0.0819 $\pm$ 0.0002
8	0.0696	5	0.070 $\pm$ 0.001	0.06964 $\pm$ 0.00005

Table 1. Estimates of critical temperatures for the $\pm J$ Ising spin glass on a d-dimensional hypercubic lattice: values of $w_c = \tanh^2(J/k_{\mathrm{B}}T_c)$ are listed.

7. Spin-Glass Critical Temperatures

Having obtained the expansions (5.9), (6.3) and (6.4) for the critical points, $T_c(d)$, of Ising spin-glass models on hypercubic lattices, let us examine the numerical aspects. Note, first, that the ratios of successive coefficients in (6.3) are increasing rapidly; the pattern is somewhat erratic but suggests an approximately linear increase with order. Thus it seems likely, as in other

FIG. 2. Plots of $\sqrt{w_c} = \tanh(J/k_{\mathrm{B}}T_c)$ vs. $1/\sqrt{\sigma}$ with $\sigma = 2d - 1$ for the critical temperature of the $\pm J$ Ising spin-glass model on a d-dimensional hypercubic lattice. The numerals $1, \ldots, 5$ label the order of truncation of the $1/d$ expansion for $1/w_c$. The first-order truncation is exact for Bethe lattices of coordination number $q = \sigma + 1$. The curves marked 'Padé' represent approximants to the $1/\sigma$ expansion, of equation (6.4), biassed to reproduce the favored estimates for $d = 3$ and 4, marked by solid circles. (These estimates result from high-temperature series analysis.)

cases, that the series for $T_c(d)$ is no better than asymptotic as $d \to \infty$. For numerical estimation, truncation of the series close to the smallest term, which is roughly that of order $1/q^{d+1}$, thus seems reasonable. Table 1 shows values of $w_c = \tanh^2(J/k_{\mathrm{B}}T_c)$ for the $\pm J$ model calculated this way from (6.4) for dimensions $d = 3, 4, \ldots, 8$. Also shown are corresponding estimates for w_c based on the high-temperature series extrapolation analysis of Singh and Chakravarty (1986, 1987). The agreement is rather encouraging.

A graphic portrayal of the large d series is presented in Fig. 2 which plots $\sqrt{w_c} = \tanh(J/k_{\mathrm{B}}T_c)$ vs. $1/\sqrt{\sigma}$. In such a plot the values for a Bethe lattice of coordination number $q = \sigma + 1$ lie on a straight line. The partial sums to order $1/q^k$ of the series (6.3) are shown; the corresponding sums of the $1/\sigma^k$ series, (6.4), are quantitatively very similar.

In order to improve the numerical performance of the large-d expansion we have accepted the central values of the high-temperature series estimates for $d = 3$ and 4: see Table 1. Then one may generate Padé approximants to the (truncated) series (6.4) *biassed* to ensure that the preferred values are reproduced for $d = 3$ and 4. The four near-diagonal biassed approximants, [2/5], [3/4], [4/3], and [5/2] are displayed: they agree very closely for $d > 4$. Indeed we believe that these approximants are rather reliable for $d \gtrsim 3$: their predictions are listed in the last column of Table 1. The uncertainties assigned there take into consideration the uncertainties in the biassing points at $d = 3$ and 4.

An interesting theoretical point may be addressed using the Padé approximants for $T_c(d)$; this concerns the *lower critical* (or borderline) *dimensionality*, $d_<$, for Ising spin glasses. The lower critical dimensionality is defined for systems with Hamiltonians that belong to a given universality class, in the usual renormalization-group or critical-phenomena sense (see e.g. Fisher 1983), as the dimensionality below which the critical point, $T_c(d)$, vanishes. For Ising-like ferromagnets etc., (with $n = 1$) one has $d_< = 1$; however, for n-vector ferromagnets with $n \geq 2$ one finds $d_< = 2$. As regards Ising spin glasses, the Monte Carlo simulations and the high-T series analyses strongly suggest $2 < d_< < 3$ (Bhatt and Young 1985; Ogielski and Morgenstern 1985; Singh and Chakravarty 1986). If one notes that $w_c = 1$ implies $T_c = 0$, it is evident that the intersections of the plots of the biassed Padé approximants for $T_c(d)$ with the frame in Fig. 2 provide explicit estimates for $d_<$. It is reasonable to conclude

$$d_< \simeq 2.5_0 \pm 0.01_5. \tag{7.1}$$

However, if, as is not implausible, $T_c(d)$ departs from zero like $(d - d_<)^\omega$ with $\omega > 1$, this estimate could prove somewhat too high since such behavior cannot be accounted for in the high-d approximants. Nevertheless, the value $d_< \simeq 2.5$ agrees quite well with various approximate real-space renormalization-group calculations (see, e.g., Bray and Moore 1984).

Finally, it is of interest to gain some feel for the effect of the actual distribution of couplings on the spin-glass critical temperature (Singh and Fisher 1988). By (5.9) we can write the deviation of $T_c(d)$ from that for the $\pm J$ model in the form

$$\frac{w_c - w_c^{(\pm J)}}{w_c^{(\pm J)}} = \frac{3[1 - (w_2/w_1^2)]}{\sigma^3} + O\left(\frac{1}{\sigma^4}\right), \tag{7.2}$$

while to the same order one might replace w_2/w_1^2 by ρ_4: see (5.10). The Gaussian distribution of the J_{ij} is usually taken proportional to

$\exp(-\frac{1}{2}J_{ij}^2/J^2)$. From the value of $w_c \equiv w_1(T_c)$ for $d = 3$ in Table 1 we find $k_{\rm B}T_c^{(\pm J)}/J = 1.2 \pm 0.1$; for the Gaussian model, however, this value of $w_1(T_c)$ implies $k_{\rm B}T_c^{(G)}/J \simeq 0.79 \pm 0.09$. The correction to this implied by (7.2) increases $k_{\rm B}T_c^{(G)}/J$ by about 0.02 . Although small, the change does serve to bring the series-based estimate for $T_c^{(G)}$ somewhat closer to the central estimate of $k_{\rm B}T_c^{(G))}/J = 0.9 \pm 0.1$ obtained by Bhatt and Young (1988) in their Monte Carlo simulations. For $d = 4$ the value of w_c in Table 1 yields $k_{\rm B}T_c^{(\pm J)}/J = 2.02 \pm 0.06$; the corresponding, uncorrected Gaussian estimate is $k_{\rm B}T_c^{(G)}/J \simeq 1.74 \pm 0.07$. The result (7.2) again yields only a small increase, to about 1.76; however, that compares well with the Monte Carlo estimate $k_{\rm B}T_c^{(G)}/J \simeq 1.8$ (Bhatt and Young 1988).

In summary, the large-dimensionality expansions for the critical points of Ising spin glasses prove effective and informative. Of course, the remaining theoretical challenge, which seems likely to prove hard, is to provide some better basis for the expansions (5.9) and (6.3) than the heuristic calculations we have expounded here. Mathematical progress along such lines could add significant insight into the statistical mechanics of random systems, a subject in which Hammersley's pioneering contributions remain a striking landmark.

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References

Abe, R. (1976). A modified $1/d$ expansion for critical temperature of spherical model on hypercubic lattice. *Progress of Theoretical Physics* 56, 494–497.

Balian, R. and Toulouse, G. (1973). Critical exponents for transitions with $n = -1$ components of the order parameter. *Physical Review Letters* 30, 544–546.

Bhatt, R.N. and Young, A.P. (1985). Search for a transition in the three-dimensional $\pm J$ Ising spin glass. *Physical Review Letters* 54, 924–928.

——— (1988). Numerical studies of Ising spin glasses in two, three and four dimensions. *Physical Review B* 37, 5606–5614.

Binder, K. and Young, A.P. (1986). Spin glasses: experimental facts, theoretical concepts, and open questions. *Reviews of Modern Physics* 58, 801–976.

Bowers, R.G. and McKerrell, A. (1973). An exact relation between the classical n-vector model ferromagnet and the self-avoiding walk problem. *Journal of Physics C: Solid State Physics* 6, 2721–2732.

Bray, A.J. and Moore, M.A. (1984). Lower critical dimension of Ising spin glasses: a numerical study. *Journal of Physics C: Solid State Physics* 17, L463–L468.

Broadbent, S.R. and Hammersley, J.M. (1957). Percolation processes I. Crystals and mazes. *Proceedings of the Cambridge Philosophical Society* 53, 629–641.

Chayes, J.T., Chayes, L., Sethna, J.P., and Thouless, D.J. (1986). A mean field spin glass with short-range interactions. *Communications in Mathematical Physics* 106, 41–89.

Domb, C. (1960a). On the theory of cooperative phenomena in crystals. *Advances in Physics* 9, 149–361, Sects. 4.3 and 4.6.

——— (1960b). *Loc. cit.*, Sects. 3.4.1 and 3.6.2.

Edwards, S.F. and Anderson, P.W. (1975). Theory of spin glasses. *Journal of Physics F: Metal Physics* 5, 965–974.

Essam, J.W. and Fisher, M.E. (1970). Some basic definitions in graph theory. *Reviews of Modern Physics* 42, 271–288.

Fisch, R. and Harris, A.B. (1977). Series study of a spin-glass model in continuous dimensionality. *Physical Review Letters* 38, 785–787.

Fisher, D.S., Grinstein, G.M., and Khurana, A. (1988). Theory of random magnets. *Physics Today* 41 (12), 56–67.

Fisher, M.E. (1966). The shape of a self-avoiding walk or polymer chain. *Journal of Chemical Physics* 44, 616–622.

——— (1967). Critical temperatures of anisotropic Ising lattices II. Upper bounds. *Physical Review* 162, 480–485.

——— (1973). Classical n-component spin systems or fields with negative even integral n. *Physical Review Letters* 30, 679–681.

——— (1983). Scaling, universality, and renormalization group theory. In *Critical Phenomena*, ed. F.J.W. Hahne, Lecture Notes in Physics No. 186, 1–139, Springer-Verlag, Berlin.

Fisher, M.E. and Essam, J.W. (1961). Some cluster size and percolation problems. *Journal of Mathematical Physics* 2, 609–619.

Fisher, M.E. and Gaunt, D.S. (1964). Ising model and self-avoiding walks on hypercubical lattices. *Physical Review* 133, A224–A239.

Fisher, M.E. and Sykes, M.F. (1959). Excluded volume problem and the Ising model of ferromagnetism. *Physical Review* 114, 45–58.

Gaunt, D.S. and Ruskin, H. (1978). Bond percolation processes in d dimensions. *Journal of Physics A: Mathematical and General* 11, 1369–1380.

Gaunt, D.S., Sykes, M.F., and Ruskin, H. (1976). Percolation processes in d dimensions. *Journal of Physics A: Mathematical and General* 9, 1899–1911.

Gennes, P.G. de (1972). Exponents for the excluded volume problem as derived by the Wilson method. *Physics Letters A* 38, 339–340.

Gerber, P.R. and Fisher, M.E. (1974). Critical temperatures of classical n-vector models on hypercubic lattices. *Physical Review* 10, 4697–4703.

——— (1975). Critical temperatures of continuous spin models and the free energy of a polymer. *Journal of Chemical Physics* 63, 4941–4946.

Hammersley, J.M. (1957a). Percolation processes II. The connective constant. *Proceedings of the Cambridge Philosophical Society* 53, 642–645.

——— (1957b). Percolation processes: lower bounds for the critical probability. *Annals of Mathematical Statistics* 28, 790–795.

Hammersley, J.M. and Morton, K.W. (1954). Poor man's Monte Carlo. *Journal of the Royal Statistical Society B* 16, 23–38.

Harris, A.B. (1982). Renormalized $(1/\sigma)$ expansion for lattice animals and localization. *Physical Review B* 26, 337–366.

Harris, T.E. (1960). A lower bound for the critical probability in a certain percolation process. *Proceedings of the Cambridge Philosophical Society* 56, 13–20.

Jasnow, D. and Fisher, M.E. (1976). Self-interacting walks, random spin systems, and the zero-component limit. *Physical Review B* 13, 1112–1118.

Joyce, G.S. (1972). Critical properties of the spherical model. In *Phase Transitions and Critical Phenomena*, Vol. 2, ed. C. Domb and M.S. Green, Academic Press, New York, 375–442.

Katsura, S., Fujiki, S., and Inawashiro, S. (1979). Spin-glass phase in the site Ising model. *Journal of Physics C: Solid State Physics* 12, 2839–2846.

Kesten, H. (1980). The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Communications in Mathematical Physics* 74, 41–59.

Ogielski, A.T. (1985). Dynamics of three-dimensional Ising spin-glasses in thermal equilibrium. *Physical Review B* 32, 7384–7398.

Ogielski, A.T. and Morgenstern, I. (1985). Critical behavior of three-dimensional Ising spin-glass model. *Physical Review Letters* 54, 928–932.

Oguchi, T. and Ueno, Y. (1976). Statistical theory of the random ordered phase in quenched bond mixtures. *Journal of the Physical Society of Japan* 41, 1123–1128.

Onsager, L. (1944). Crystal statistics I. A two-dimensional model with an order-disorder transition. *Physical Review* 65, 117–149.

Singh, R.R.P. and Chakravarty, S. (1986). Critical behavior of an Ising spin-glass. *Physical Review Letters* 57, 245–248.

——— (1987a). Critical exponents for Ising spin-glasses through high temperature series analysis. *Journal of Applied Physics* 61, 4095–4096.

——— (1987b). High temperature series expansion for spin glasses. I: Derivation of the series. II: Analysis of the series. *Physical Review B* 36, 546–558 and 559–566.

Singh, R.R.P. and Fisher, M.E. (1988). Short-range Ising spin-glasses in general dimensions. *Journal of Applied Physics* 63, 3994–3996.

Stanley, H.E. (1968a). Dependence of critical properties on dimensionality of spins. *Physical Review Letters* 20, 589–592.

——— (1968b). Spherical model as the limit of infinite spin dimensionality. *Physical Review* 176, 718–722.

——— (1969). Exact solution for a linear chain of isotropically interacting classical spins of arbitrary dimensionality. *Physical Review* 179, 570–577.

Thouless, D.J. (1986). Spin-glass on a Bethe lattice. *Physical Review Letters* 56, 1082–1085.

Wilson, K.G. and Fisher, M.E. (1972). Critical exponents in 3.99 dimensions. *Physical Review Letters* 28, 240–243.

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