Directed Compact Percolation II: Nodal Points, Mass Distribution, and Scaling

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Abstract

Directed compact percolation is a limiting case of a cellular automaton model which also includes directed site and bond percolation. Existing results for the latter are reviewed and previous calculations for compact percolation are extended so that comparison of several critical exponents may be made. New results are obtained for the probability distribution of the number of nodal points and for the centre of mass and moments of inertia of compact percolation clusters. Also for this model scaling is verified for the cluster size distribution and for the longitudinal moments of the pair connectedness.

1. Introduction

A percolation process, as introduced by Broadbent and Hammersley (1957) in the first published work on percolation theory, is the passage of *fluid* through a *random medium*. The terms fluid and medium are capable of a very broad interpretation and as foreseen by Frisch and Hammersley (1963) such processes have been found to occur widely in all branches of science. The early history of the subject is described by Hammersley in an article in the book edited by Deutscher et al. (1983) where a collection of recent applications of percolation theory may also be found.

The medium was originally modelled by a random maze derived from a *crystal* of *atoms* between which local connections were made by *bonds*. The bonds could either be directed or undirected and each had independently probability p of being open to the passage of fluid. This model has become known as bond percolation. It was shown by Broadbent and Hammersley (1957) that under certain conditions there is a critical value of $p \in (0, 1)$, called the *critical probability* p_c, below which only finitely many atoms are wet from any source with probability 1. Above p_c there is positive probability $P(p)$, the percolation probability, that the set of atoms which are wet is unbounded and percolation is said to take place.

Percolation processes now form an important branch of critical phenomena theory. This connection was first noted by Domb (1959) who observed that if the crystal atoms are of two species, A and B, and species A occurs with probability p, then the expected size $S(p)$ of A-clusters will become infinite at a value of p which he identified with Hammersley's critical probability, except that the randomness was now in the atoms rather than the bonds. This type of percolation has become known as site percolation since the sites of a crystal lattice are randomly occupied by species of two types. Clusters can also be defined for bond percolation as maximal sets of atoms connected by open bonds. This yields an alternative definition of critical probability for bond percolation as the point at which the cluster size becomes infinite. (These definitions have recently been shown to coincide in cases of physical interest, see Kesten 1982 for references.) Subsequently Hammersley (1961) showed that the critical probability for bond percolation cannot exceed that for site percolation on the same crystal lattice.

Here we consider percolation on a square lattice with nearest neighbour bonds and all bonds which are parallel are oriented in the same direction. For analytical work on this oriented percolation model see Durrett (1984) where references to earlier work may also be found. As anticipated by Frisch and Hammersley (1963) it turns out to be useful to allow both bonds and sites to be random elements and we denote by p_b and p_s the probabilities that a given bond or site is open respectively. A site can only become wet if it can be reached by a path from the source of fluid, all elements of which are open (including the site itself). For bond percolation on this lattice it was shown by Hammersley (1959) that $\frac{1}{2} \le p_c \le 0.85$. These bounds have subsequently been improved but the exact value of p_c still remains unknown. For the general model the critical probability becomes a critical curve in the p_b–p_s plane.

The lattice sites will be labelled by cartesian co-ordinates t and y chosen in such a way that the two types of directed bond make angles $\pm 45°$ with the positive t-axis. A line source perpendicular to the t-axis will be considered which has m sites and the origin of co-ordinates is chosen as the centre of this line. For m even, the lattice sites are (t, y) where t and y are any pair of integers with odd sum and the source sites have co-ordinates $(0, y)$ with $y = \pm 1, \pm 3, \ldots, \pm(m-1)$. For m odd, the sum of t and y is even and the source sites have $y = 0, \pm 2, \pm 4, \ldots, \pm(m-1)$. Fluid arriving at site (t, y) may come from either of the sites $(t-1, y \pm 1)$. The probability p_1, that (t, y) is wet given that exactly one of these sites is wet, is $p_b p_s$ and the probability p_2, that (t, y) is wet given that both of them are wet, is $(2p_b - p_b^2)p_s$. We denote the cluster which is wet from this source by c_m.

As p_b and p_s range over $[0, 1]$ it can be seen that given p_1, p_2 must lie in the interval $[p_1, 2p_1 - p_1^2]$. Durrett and Schonmann (1987) have considered a generalised percolation process in which the probabilities of the bonds leading to (t, y) being open are not necessarily independent. In this model p_2 lies in the extended range $[p_1, 2p_1]$. Domany and Kinzel (1984) considered an even more general cluster growth model in which p_1 and p_2 both range over $[0, 1]$. They considered this to be a one-dimensional stochastic cellular automaton model in which each cell has two states (wet and dry) and t is the time variable and also noted that this automaton could be mapped (Verhagen 1976) onto a triangular lattice Ising model with three-spin interactions in every other triangle (in addition to the usual pair interactions). All of these models are examples of the stochastic growth model of Durrett and Schonmann (1987) who have obtained very general analytical results on the shape of the infinite cluster above p_c. Their work has recently been extended to $p \geq p_c$ and all space dimensions by Bezuidenhout and Grimmett (1989).

The special case $p_2 = 1$, $p_1 = p$ has the simplifying feature that a given site will always be wet whenever its two predecessors are wet irrespective of the value of p. The cluster c_m therefore cannot branch and hence is free from holes; for this reason we shall call it a compact percolation cluster. Finite compact clusters also have a unique terminal site. In this case a complete description of c_m can be obtained by specifying, for each t, the number of wet sites n_t in the tth column and the centre of mass of these sites. Domany and Kinzel (1984) noted that n_t could be thought of as the position of a one-dimensional random walker after t steps with transition probabilities p^2, $(1-p)^2$ and $2p(1-p)$ to positions $n_t + 1$, $n_t - 1$ and n_t respectively. Using this they deduced that $p_c = \frac{1}{2}$ and also determined the probability distribution of the cluster length by calculating the probability that the walker reaches the origin for the first time after t steps. In terms of the walk model the percolation probability is the probability that the walker never reaches the origin. The clusters under consideration are known by Delest and Viennot (1984) as parallelogram polyominoes and Domany and Kinzel's result can also be deduced from an enumeration formula given by these authors (see Essam 1989). Another source of combinatorial information on compact percolation clusters is the work of Huse et al. (1983) on the enumeration of domain walls in a chiral clock model.

In Section 2.1 we will define critical exponents and we shall see that the exponents for bond and site percolation are numerically equal but are quite different from those for compact percolation. There is a critical curve in the p_1–p_2 plane on which the critical points of all three models lie. Critical phenomena theory suggests that critical exponents will normally remain constant along such curves but that there may be special crossover points at which they may change discontinuously. We believe that compact percolation corresponds to such a point and here we investigate this model in detail as a preliminary to investigation of the crossover phenomenon. The special nature of this point is suggested by the fact that it is the point at which $p_b \to 0$ and $p_s \to \infty$ such that $p_b p_s \to \frac{1}{2}$. Also it is the point at which the clusters become qualitatively different in that branching and holes will occur for any value of $p_2 < 1$.

In a previous paper (Essam 1989) the work of Domany and Kinzel was extended to asymmetric compact percolation by which we mean the following. Denote the sites $(t-1, y+1)$ and $(t-1, y-1)$ by A and B respectively. If both A and B are wet then as before (t, y) is wet with probability 1. However if A is dry and B is wet then (t, y) is wet with probability p_u in which event the upper edge of the cluster moves upwards. Finally if A is wet and B is dry then (t, y) is wet with probability p_d the subscript denoting downward motion of the lower edge. Of course if A and B are both dry then (t, y) is certainly dry. Domany and Kinzel's results were for $p_u = p_d = p_1$ and their walk model may still be used by defining the transition probabilities from n_t to $n_t + 1$, $n_t - 1$ and n_t as $c = p_u p_d$, $d = (1-p_u)(1-p_d)$ and $1-c-d$ respectively. Percolation now occurs with positive probability when $c > d$ and the critical curve in the p_u–p_d plane is $p_u + p_d = 1$. A duality relation between $c < d$ and $c > d$ was shown to exist. The moment generating function for the cluster length distribution and the first two moments of the cluster size distribution were also obtained by solving recurrence relations.

In this paper we investigate the probability distribution of the number of nodal sites and bonds. We also rederive the previous results for the cluster size distribution by a method which enables the general moment to be considered and also the expected values of the centre of mass and moments of inertia to be calculated.

2. Definitions and Previous Results

2.1. Definitions

In numerical work on directed percolation using series expansion techniques (see Essam et al. 1988 for references) the mean size and moments of the mass distribution of the cluster c_m have been investigated in the case $m = 1$. These functions are conveniently defined in terms of the *pair connectedness* $C_m(t, y)$ which is the probability that the atom (t, y) is wet from a line source of m sites given that c_m is finite, thus the moments of the mass distribution of finite clusters about the y-axis are given by

$$\mu_k^{(t)}(p) = \sum_{t=0}^{\infty} \sum_{y=-y_{\max}(t)}^{y_{\max}(t)} t^k C_m(t, y) \tag{2.1}$$

with a similar definition of $\mu_k^{(y)}(p)$, the moments about the t-axis. The *mean cluster size* $S_m(p)$ is the expected number of sites which are wet and is given by the same expression with $k = 0$. In the case of compact percolation, p represents the pair of variables p_u, p_d. The position of the centre of mass of finite clusters is estimated by $(\langle t\rangle, \langle y\rangle)$ and the radii of gyration about the y and t axes by $\langle t^2\rangle^{1/2}$ and $\langle y^2\rangle^{1/2}$ respectively where

$$\langle t^k\rangle = \mu_k^{(t)}(p)/S_m(p) \quad \text{and} \quad \langle y^k\rangle = \mu_k^{(y)}(p)/S_m(p). \tag{2.2}$$

We shall picture the formation of the cluster, c_m, which is wet from a source of width m, as taking place in a number of growth stages in which at the tth stage column t is wet from column $t-1$. The probability that c_m has t growth stages (i.e. $t+1$ is the first dry column) will be denoted by $r_t(m)$. In terms of the random walk problem described in the introduction this is also the probability that a walker starting at position $m > 0$ reaches the origin for the first time on the $(t+1)$th step. In terms of $r_t(m)$ we may obtain $Q_m(p)$, the probability that c_m is finite, and hence the *percolation probability* $P_m(p) = 1 - Q_m(p)$:

$$Q_m(p) = \sum_{t=0}^{\infty} r_t(m). \tag{2.3}$$

The *length* L_m of c_m will be defined as the number of atoms in a path from the source to a terminal point and the *probability distribution of the length of finite clusters* is determined by $r_t(m)$ and has moments which are given by:

$$E(L_m^k) = \sum_{t=0}^{\infty}(t+1)^k r_t(m)/Q_m(p). \tag{2.4}$$

The probability $p_s(m)$, that c_m has s atoms (sites) will be known as the *cluster size distribution* and has the same normalisation factor as $r_t(m)$, i.e. $Q_m(p)$. The normalised kth moment of this distribution is

$$m_k(p) = \sum_{s=1}^{\infty} s^k p_s(m)/Q_m(p) \tag{2.5}$$

and in particular $m_1(p) = S_m(p)$ is the mean size.

For compact clusters the moments $m_k(p)$ may be related (see Section 4) to the transition probability $r_{mn}(t)$ that a cluster with source of width m has width n after t growth stages. In terms of walks, $r_{mn}(t)$ is the probability that the walker moves from position $m > 0$ to position n in t steps without visiting 0. Notice that

$$r_{m1}(t) = r_t(m)/d \tag{2.6}$$

and so the transition probability for $m = 1$ is determined by previous results (Essam 1989). An explicit formula for the following moment generating function of $r_{mn}(t)$ is given in Section 4:

$$R_{mn}(z) = \sum_{t=0}^{\infty} e^{-zt} r_{mn}(t). \tag{2.7}$$

A *nodal (articulation) point* of c_m in the compact case is an intermediate site of c_m through which all open paths from the source to the terminal point must pass. At a nodal point c_m has width one. In Section 3 we obtain the probability $g_a(p, m)$ that c_m has exactly a nodal points given that it is finite. We shall see that this is determined in terms of the transition probability $w_{mn}(t)$ which is defined in the same way as $r_{mn}(t)$ except that there must be no intermediate values of t at which the cluster width is one. In the walk analogy intermediate visits to the point distant 1 from the origin must also be avoided. It is convenient to take $w_{mn}(0) = 0$. We shall see that $w_{mn}(t)$ is simply related to $r_{mn}(t)$ by a duality argument. The moment generating function, $W_{mn}(z)$, of $w_{mn}(t)$ is defined in the same way as $R_{mn}(z)$.

For $p_2 < 1$ there is no unique terminal point and nodal points are defined relative to each site of the cluster. A *nodal point relative to the site* (t, y) of c_m is an intermediate site of c_m through which all open paths from the source to (t, y) must pass. An estimator of the number of nodal points on any open path from the source to a given lattice site averaged over all such sites is

$$\langle a \rangle = \sum_{t=0}^{\infty} \sum_{y=-y_{\max}(t)}^{y_{\max}(t)} a_m(t, y)/S_m(p) \tag{2.8}$$

where $a_m(t, y)$ is the expected number of nodal points on an open path to (t, y) (if there is no path the number of nodal points is zero).

Nodal bonds are defined similarly to nodal sites in terms of bonds through which all open paths must pass. The average of the expected number of nodal bonds over all sites will be denoted by $\langle b \rangle$.

Critical exponents are widely used in critical phenomena theory to characterise the divergence of functions as the critical point (critical probability in the present context) is approached. Thus for a typical function $F(p)$ of this kind we write

$$F(p) \cong A|1 - p/p_c|^{-\epsilon} \tag{2.9}$$

to denote that the ratio of $F(p)$ to the right-hand side approaches unity as $p \to p_c$. Here ϵ is the *critical exponent* and A is the *amplitude.* In the case of asymmetric compact percolation we shall find that u, defined by (4.13), is an appropriate variable rather than p and $u_c = 1$. In the symmetric case $p_u = p_d = p$, $1 - u$ may be replaced by $4(1 - p/p_c)$, where $p_c = \frac{1}{2}$, as p_c is approached so that only the amplitude changes in going from u to p. Notice also that in the symmetric case $d - c = 1 - 2p = 1 - p/p_c$. In a case where the exponent but not the amplitude has been determined we replace $\cong$ by $\sim$.

Again from critical phenomena theory we expect that the various distribution functions above will have *scaling forms* which imply that the ratio of moment k to moment $k-1$ has a critical exponent which is independent of k. We shall verify that this is the case for compact percolation and write

$$\begin{gathered}\langle s^k\rangle/\langle s^{k-1}\rangle \sim \sigma(p), \quad \langle t^k\rangle/\langle t^{k-1}\rangle \sim \xi_{\Uparrow}(p), \\ \langle y^{2k}\rangle/\langle y^{2k-2}\rangle \sim \xi_{\perp}(p)^2\end{gathered} \tag{2.10}$$

where in the latter case we have taken even moments since we shall consider only the symmetric case, for which the odd moments are zero. These functions are known respectively as the scaling size and the parallel and perpendicular connectedness lengths. The notation used for their critical exponents and those of other functions is given in the table below. The percolation probability exponent β describes its vanishing at p_c rather than divergence. We shall see that the ratio $E(L_m^k)/E(L_m^{k-1})$ has the same critical exponent as $\xi_{\Uparrow}(p)$.

2.2. Previous Results for Bond and Site Percolation

Recent numerical work on bond and site percolation (Essam et al. 1988) has given extremely accurate estimates of the critical probabilities for both directed square and triangular lattices. For example on the square lattice $p_c(\text{bond}) = 0.644701 \pm 0.000001$ and $p_c(\text{site}) = 0.705489 \pm 0.000004$. The critical exponent γ of $S_1(p)$ for all four percolation processes was estimated to be 2.278 ± 0.002. Assuming that γ is a 'simple' rational, the value $41/18$ was chosen and then biased estimates of the exponents $\nu_{\Uparrow}$ and $\nu_{\perp}$ of the connectedness lengths were obtained from the moments $\mu_1^{(t)}$, $\mu_2^{(t)}$, $\mu_2^{(y)}$. The existence of the scaling length $\xi_{\Uparrow}(p)$ was strongly supported by the results. A search for simple rational values of all exponents which were consistent with the data and also with scaling relations failed. A value of $\beta = 199/720$ predicted by the relation $\beta = \frac{1}{2}(\nu_{\Uparrow} + \nu_{\perp} - \gamma)$ was considered not to be simple but is supported by a direct numerical estimate (Baxter and Guttmann 1988). The rational values quoted for bond and site percolation in the table below reproduce the estimated exponents to three decimal places.

It has been shown by Coniglio (1982) that for bond percolation $\langle b\rangle = p\,d(\ln S_m(p))/dp$ so that this function diverges with exponent 1. A similar argument shows that for site percolation $\langle a\rangle = p\,d(\ln S_m(p))/dp - 1$ which therefore also has a simple pole at the critical point. On the other hand we shall show that for compact clusters $\langle a\rangle$ is finite on the critical line. This marked difference may be attributed to the ramified nature of non-compact clusters.

function	exponent		
	usual symbol	compact	bond and site
$P_m(p)$	β	1	199/720
$\langle L_m \rangle$		1	
$S_m(p)$	γ	2	41/18
$\xi_{\Uparrow}(p)$	$\nu_{\Uparrow}$	2	26/15
$\xi_{\perp}(p)$	$\nu_{\perp}$	1	79/72
$\sigma(p)$	Δ	3	
$\langle a \rangle$	ζ_a	$\langle a \rangle$ finite	1 (site)
$\langle b \rangle$	ζ_b	$\langle b \rangle$ finite	1 (bond)

2.3. Previous Results for Compact Clusters

The probability $r_m(t)$ that c_m has t growth stages satisfies (Essam 1989) the recurrence relation

$$r_t(m) = cr_{t-1}(m+1) + (1-c-d)r_{t-1}(m) + dr_{t-1}(m-1) \qquad (2.11)$$

with boundary conditions $r_0(1) = d$, together with $r_t(m) = 0$ for $m \geq t+2$ and $m = 0$. This leads to an explicit formula for the moment generating function of $r_t(m)$, namely

$$R_m(z) = e^z[\lambda(z)]^m. \qquad (2.12)$$

where $\lambda(z)$ is the root of

$$c\lambda^2 + (1-c-d-e^z)\lambda + d = 0 \qquad (2.13)$$

which tends to zero as $z \to \infty$, i.e.

$$\lambda(z) = \frac{1}{2c}\left\{c + d + e^z - 1 - \sqrt{(1-c-d-e^z)^2 - 4cd}\right\}. \qquad (2.14)$$

Using (2.6) and (2.7) the generating function $R_{m1}(z)$ is given by

$$R_{m1}(z) = R_m(z)/d = e^z\lambda(z)^m/d. \qquad (2.15)$$

Similar analysis in the case $c + d = 1$ may be found in Feller (1968).

The percolation probability

From (2.3) and (2.7) the probability that c_m is finite is equal to $R_m(0)$ and hence

$$Q_m(p) = \begin{cases} 1 & \text{for } c < d \\ (d/c)^m & \text{for } c \geq d, \end{cases} \tag{2.16}$$

and the percolation probability $P_m(p) = 1 - Q_m(p)$, just above the critical curve $c = d$, has asymptotic form

$$P_m(p) \cong \frac{m(c-d)}{c} = \frac{m(p_u + p_d - 1)}{p_u p_d} \tag{2.17}$$

and hence the critical exponent $\beta = 1$ for all m and all points on the curve as expected.

Moments of the cluster length distribution

The generating function for the cumulants of the distribution of L_m is $\ln[e^{-z} R_m(z)] = m \ln \lambda(z)$. Thus

$$E(L_m) = mG(0) \tag{2.18}$$

where

$$\begin{aligned} G(z) &= -(d/dz) \ln \lambda(z) \\ &= e^z[(1 - c - d - e^z)^2 - 4cd]^{-1/2}, \end{aligned} \tag{2.19}$$

which is a symmetric function of c and d which we will use later. Setting $z = 0$,

$$G(0) = \frac{1}{|d-c|} \tag{2.20}$$

which diverges on the critical line with exponent 1 which is therefore the critical exponent of the mean length. Further

$$\mathrm{Var}(L_m) = -mG'(0) = m[(c+d)G(0)^3 - G(0)] \tag{2.21}$$

which therefore has critical exponent 3. The kth order cumulant average of L_m is equal to $(-1)^{k-1} G^{(k-1)}(0)$ and it follows from (2.19) that near the critical line

$$G^{(k)}(0) \cong (-c)^k (2k)_k |d-c|^{-2k-1}. \tag{2.22}$$

The critical exponent of the kth moment of L_m is therefore $2k+1$, for all m and all points on the critical curve in agreement with Essam (1989) for the symmetric case. This implies the existence of a scaling length with critical exponent 2 which we shall see is the same as that for the radius of gyration about the y-axis, i.e. $\nu_{\Uparrow} = 2$.

Moments of the cluster size distribution

The first two moments of the cluster size distribution of compact clusters have also been determined previously (Essam 1989). In particular the mean cluster size is

$$S_m(p) = \tfrac{1}{2}m\left[\frac{c+d}{(d-c)^2} + \frac{m}{|d-c|}\right] \tag{2.23}$$

a result we will need in the next section. Thus the critical exponent $\gamma = 2$. We rederive these moments in Section 4 by a method which enables the critical exponent of the general moment to be obtained.

3. Nodal Points and Bonds in Compact Clusters

In this section we break new ground in the theory of compact percolation and determine the probability distributions of nodal points and bonds.

3.1. The Probability that c_m has no Nodal Points

The probability that no nodal points (bonds) occur in c_m is the probability of finding at least two paths from the source to the terminal point of c_m which have no intermediate site (bond) in common. We first consider clusters with t growth stages ($t \geq 1$) and note that

$$\mathrm{pr}(c_m \text{ has } t \text{ growth stages and no nodal vertices}) = w_{m1}(t)d \tag{3.1}$$

and we will calculate this quantity first by relating it to $r_{m1}(t)$. For $m = 1$ and $t \geq 2$

$$w_{11}(t) = cdr_{11}(t-2) \tag{3.2}$$

since any cluster contributing to $w_{11}(t)$ may be associated with the maximal cluster on the dual lattice which it contains (see Figure 1) and all clusters which contribute to $r_{11}(t-2)$ occur as such dual clusters. For $m > 1$ and $t \geq 1$ a similar bijection exists between clusters contributing to $w_{m1}(t)$ and $r_{m-1,1}(t-1)$ which gives

$$w_{m1}(t) = dr_{m-1,1}(t-1). \tag{3.3}$$

These results also have simple interpretations in terms of translating walks of $r_{m1}(t)$ one step to the right.

Summing (3.1) over all $t \geq 1$, remembering that $w_{mn}(0)$ is defined as zero, we obtain

$$\mathrm{pr}(c_m \text{ is finite, has at least one growth stage and no nodal points}) = W_{m1}(0)d \tag{3.4}$$

FIG. 1. The open circles are the vertices in a compact cluster contributing to $w_{11}(21)$, and the filled circles are the vertices in the maximal cluster on the dual lattice which it contains. The latter has length 20, size 32, source width $m = 1$, maximum width 3, 7 nodal points and 4 nodal bonds.

and

$$\begin{aligned} g_0(p) &= \mathrm{pr}(c_m \text{ has no nodal points} \mid c_m \text{ finite}) \\ &= \frac{W_{m1}(0)d + d\delta_{m,1}}{Q_m(p)}, \end{aligned} \tag{3.5}$$

where the term $d\delta_{m,1}$ is from the cluster having just one site which can occur only when $m = 1$ and is not included in $W_{11}(0)$. The above results give the following generating function relations

$$\begin{aligned} W_{11}(z) &= w_{11}(1)e^{-z} + cdR_{11}(z)e^{-2z} \\ &= [(1 - c - d) + c\lambda(z)]e^{-z}, \end{aligned} \tag{3.6}$$

$$W_{m1}(z) = \lambda(z)^{m-1} \quad \text{for } m > 1, \tag{3.7}$$

which for $c < d$ leads to

$$g_0(p) = \begin{cases} (2 - d)d & \text{for } m = 1 \\ d & \text{for } m > 1, \end{cases} \tag{3.8}$$

and for $c > d$, d is replaced by c.

3.2. The Number of Nodal Points and Bonds in c_m

For $a > 0$ the probability distribution of the number of nodal points in c_m is

$$\begin{aligned} g_a(p, m) &\equiv \mathrm{pr}(c_m \text{ has exactly } a \text{ nodal vertices given that it is finite}) \\ &= \frac{W_{m1}(0)W_{11}(0)^a d}{Q_m(p)} \end{aligned} \tag{3.9}$$

since a cluster with a nodal points may be constructed as the series combination of $a+1$ non-nodal subclusters which have width 1 at their initial and final stages, except for the first subcluster which is initially of width m. The W factors arise from the repeated convolution of $w_{m1}(t)$ with $w_{11}(t)$ and the final factor d is the probability that the cluster terminates after the last subcluster is complete. Substituting from (3.6) and (3.7) gives, for $c < d$

$$g_a(p,m) = \begin{cases} (1-d)^{a+1}d & \text{for } m=1, a>0 \\ (2-d)d & \text{for } m=1, a=0 \\ (1-d)^a d & \text{for } m>1, \end{cases} \tag{3.10}$$

and computing the moments gives

$$E(a) = \begin{cases} (1-d)^2/d & \text{for } m=1 \\ (1-d)/d & \text{for } m>1 \end{cases} \tag{3.11}$$

and

$$\mathrm{Var}(a) = \begin{cases} (1-d)^2/d^2 & \text{for } m=1 \\ (1-d)/d^2 & \text{for } m>1. \end{cases} \tag{3.12}$$

For $c > d$, d should be replaced by c. Notice that these quantities have only a cusp on the critical line $c = d$ whereas we saw in Section 2 that for site percolation $\langle a \rangle$ is infinite at p_c. For compact clusters we can show that the expected number of growth stages, T, between nodal points is infinite when $c = d$, thus

$$\begin{aligned} T &= -(d/dz)\ln[W_{11}(z)]|_{z=0} \\ &= \frac{d(1-d+c)}{(1-d)(d-c)} \quad \text{for } c<d \end{aligned} \tag{3.13}$$

and again c and d are interchanged for $c > d$. This is consistent with the divergence of the expected cluster length and in fact

$$E(L_m) = 1 + [E(a+1) - d]T. \tag{3.14}$$

The probability distribution for the number of nodal bonds b for clusters with a point source is for $b > 0$, $m = 1$ and $c < d$:

$$h_b(p,1) = (1-c-d)^b(1+c)^{b+1}d \tag{3.15}$$

with c and d interchanged for $c > d$.

3.3. Spatial Average of the Number of Nodal Vertices for $m = 1$

Let s_i, $i = 0, \ldots, a$, be the number of sites in the ith non-nodal subcluster in the series combination constituting c_1, not counting the initial site. The total number of sites in c_1 is therefore

$$s = 1 + \sum_{i=0}^{a} s_i. \tag{3.16}$$

For $i > 0$ let S_{11} be the expected value of s_i; this is independent of i by translational symmetry. The expected value of s for c_1 given that it has a articulation points is therefore $1+(a+1)S_{11}$ except when $a = 0$ and $s = 1$. The unconditional expectation of s which we have previously calculated and denoted by $S_1(p)$ is therefore:

$$S_1(p) = 1 + [E(a+1) - d]S_{11} \tag{3.17}$$

which determines S_{11} in terms of $S_1(p)$ which is given by (2.23) and using (3.11), for $c < d$:

$$S_{11} = \frac{d}{1-d}\left[\frac{d}{(d-c)^2} - 1\right]. \tag{3.18}$$

The un-normalised spatial average of the number of nodal points defined in Section 2.1 may be found by (i) counting, for each site of c_m, the number of nodal sites on a path from the origin to the chosen site, (ii) summing over all sites of c_m, (iii) averaging over all clusters. For a cluster with a articulation points the result of (ii) may be written in the form $s_1 + 2s_2 + 3s_3 + \cdots + as_a$ which has conditional expectation value $\frac{1}{2}a(a+1)S_{11}$ and hence

$$\begin{aligned}\langle a\rangle &= E[a(a+1)]\frac{S_{11}}{2S_1(p)}\\ &= \frac{1-d}{d}\left[1 - \frac{1}{S_1(p)}\right].\end{aligned} \tag{3.19}$$

Again c replaces d for $c > d$ and $\langle a\rangle$ is finite and continuous on the critical curve but using (2.23) the approach is quadratic in contrast to $E(a)$ for which the approach to the critical curve was linear with a discontinuous first derivative.

4. The Width Distribution, Cluster Size, Centre of Mass, and Moments of Inertia

In this section we calculate the quantities which were defined in Section 2.1 and verify the existence of the scaling size $\sigma(p)$, with exponent $\Delta = 3$, and the scaling length $\xi_{\Uparrow}(p)$, with exponent $\nu_{\Uparrow} = 2$.

4.1. The Cluster-Width Distribution and Moments of the Cluster-Size Distribution

We now turn to the moments of the cluster-size distribution. The size s of the cluster c_m is the number of vertices it contains and if n_t is the number of vertices which are added at stage t of the growth then

$$s = \sum_{t=0}^{\infty} n_t. \tag{4.1}$$

We first obtain formal expressions for the moments in terms of the following cumulative distribution which gives the cluster width after t growth stages:

$$\begin{aligned} \rho_{mn}(t) &= \mathrm{pr}(c_m \text{ grows for at least } t \text{ stages and has width } n \text{ after stage } t \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{given that it is finite}) \\ &= r_{mn}(t) Q_n / Q_m \end{aligned} \tag{4.2}$$

where $r_{mn}(t)$ and $Q_m(p)$ are defined in Section 2.1 and we have suppressed the argument of the latter. The following moments of $\rho_{mn}(t)$ will be important in all subsequent calculations.

$$M_k = \sum_{t=0}^{\infty} \sum_{n=1}^{\infty} n^k \rho_{mn}(t). \tag{4.3}$$

It follows, using (2.5) and (4.1), that

$$E(L_m) = M_0 \quad \text{and} \quad S_m(p) = M_1. \tag{4.4}$$

The second moment $m_2(p)$ of the size distribution is the expected value of

$$s^2 = \sum_{t=0}^{\infty} \sum_{t'=0}^{\infty} n_t n_{t'}. \tag{4.5}$$

Separating the diagonal and off-diagonal terms

$$m_2(p) = 2U_m^{(2)} - M_2 \tag{4.6}$$

where

$$U_m^{(k+1)} = \sum_{t=0}^{\infty} \sum_{n=1}^{\infty} n \rho_{mn}(t) U_n^{(k)} \tag{4.7}$$

and $U_m^{(1)} = S_m(p)$. We shall see that $U_m^{(2)}$ is expressible in terms of the M_k as far as M_3.

Carrying out the t-summation in (4.3), using (2.7), we see that the M_k are the x-derivatives at $x = 0$ of the generating function:

$$M(x) = \sum_{n=1}^{\infty} e^{-nx} R_{mn}(0) Q_n / Q_m. \tag{4.8}$$

It may be shown using the reflection principle (Feller 1968) that the moment generating function $R_{mn}(z)$ is given by

$$R_{mn}(z) = G(z) H_{mn}(\lambda(z)) \tag{4.9}$$

where $G(z)$ is given by (2.19) and

$$H_{mn}(\lambda) = \begin{cases} \lambda^{m-n} - \lambda^m \lambda^{\star n} & \text{for } n \leq m \\ \lambda^{\star n-m} - \lambda^m \lambda^{\star n} & \text{for } n \geq m \end{cases} \tag{4.10}$$

where $\lambda^\star(z)$ is the reciprocal of the second root of (2.13),

$$\lambda^\star(z) = (c/d)\lambda(z). \tag{4.11}$$

A lengthy but straightforward calculation using (4.8) and (4.9) yields

$$M(x) = \frac{G(0)(1-u)(1-e^{-mx})}{(e^x - u)(1 - e^{-x})} \tag{4.12}$$

where

$$u = \lambda(0)\lambda^\star(0) = \begin{cases} c/d & \text{for } c < d \\ d/c & \text{for } c > d \end{cases} \tag{4.13}$$

and has the value 1 on the critical curve. The critical exponents associated with the M_k may be deduced by expanding $M(x)$ in the form

$$M(x) = G(0) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} v_k (e^x - 1)^j (1-u)^{-j} \frac{x^k}{k!} \tag{4.14}$$

where $v_0 = m$ and for $k > 0$

$$v_k = \sum_{n=1}^{m-1} n^k \tag{4.15}$$

which is a polynomial in m of degree $k + 1$. Collecting together powers of x in (4.14) we find

$$M_0 = mG(0) = E(L_m), \tag{4.16}$$

where $G(0)$ is given explicitly by (2.18) and so M_0 has critical exponent 1.

$$S_m(p) = M_1 = M_0\left[\frac{1}{1-u} + \tfrac{1}{2}(m-1)\right], \tag{4.17}$$

which thus has critical exponent $\gamma = 2$ and rederives (2.23).

$$M_2 = M_0\left[\frac{1+u}{(1-u)^2} + \frac{m-1}{1-u} + \tfrac{1}{6}(m-1)(2m-1)\right], \tag{4.18}$$

which has critical exponent 3 and

$$M_3 = M_0\left[\frac{1+4u+u^2}{(1-u)^3} + \frac{3(m-1)(1+u)}{2(1-u)^2} + \frac{(m-1)(2m-1)}{2(1-u)} + \tfrac{1}{4}m(m-1)^2\right] \tag{4.19}$$

which has critical exponent 4. In general, from (4.14), M_k/M_0 is the product of a polynomial in u and a factor $(1-u)^{-k}$ and hence M_k has critical exponent $k+1$.

Using (4.6), (4.7), (4.17) and (4.3) we see that the second moment of the cluster-size distribution is given by

$$m_2(p) = G(0)\left[M_3 + M_2\frac{1+u}{1-u}\right] - M_2, \tag{4.20}$$

and hence $m_2(p)$ has critical exponent 5 which comes from the first term since the second term, which came from the diagonal contributions in (4.5), has the smaller exponent 3. We have checked that for $m = 1$ this result correctly reproduces equation (4.19) of Essam (1989).

In general we can show the critical exponent of $m_k(p)$ is that of $U_m^{(k)}$. Also we can prove by induction that the terms in $U_m^{(k)}$ with dominant critical exponent are of the form

$$\frac{G(0)^{k-1}M_i}{(1-u)^{2k-i-1}} \quad \text{with } i = 2, 3, \ldots, 2k-1 \tag{4.21}$$

and since M_i has exponent $i+1$ it follows that $U_m^{(k)}$ has exponent $3k-1$. The existence of $\sigma(p)$, the scaling size with critical exponent $\Delta = 3$ is thus established. The induction uses the fact that M_i is a combination of terms with singularities of the form

$$\frac{G(0)\nu_{ij}}{(1-u)^{i-j}} \quad \text{with } j = 0, \ldots, i, \tag{4.22}$$

where, from (4.14), ν_{ij} is a linear combination of the ν_k in (4.15), with $k \le j$. It also uses the fact that, since ν_{ij} is a polynomial in m of degree $j+1$, when $U_m^{(k)}$ is substituted in the right hand side of (4.7) a term involving ν_{ij} produces an M_{j+2}.

4.2. The Centre of Mass and Moments of Inertia

The first moment of the mass distribution of c_m about the y-axis may be written

$$\mu_{1t}(p) = \sum_{t=0}^{\infty}\sum_{n=1}^{\infty} nt\rho_{mn}(t). \tag{4.23}$$

In calculating the moments about the t-axis we shall assume, for simplicity, the symmetric case $p_u = p_d = p$ so that the first moment is zero. It may be shown that the second moment is then given by

$$\mu_{2y}(p) = 2p(1-p)\sum_{t=0}^{\infty}\sum_{n=1}^{\infty} n(t+1)\rho_{mn}(t) + \tfrac{1}{3}[M_3 - M_1] \tag{4.24}$$

where the first term arises from the second moment of the centre of mass of column t about the t-axis and the second is the second moment of column t about its centre of mass. It is therefore possible to express this moment in terms of the M_k and $\mu_{1t}(p)$ which we now calculate:

$$\mu_{2y}(p) = 2p(1-p)[\mu_{1t}(p) + M_1] + \tfrac{1}{3}[M_3 - M_1]. \tag{4.25}$$

Now using (4.9)

$$\begin{aligned}\mu_{1t}(p) &= \sum_{n=1}^{\infty} n\left[-\frac{dR_{mn}(z)}{dz}\right]_{z=0}\frac{Q_n}{Q_m}\\ &= \left[-\frac{d}{dz}\ln G(z)\right]_{z=0} S_m(p) + \left[-\frac{d}{dz}\ln\lambda(z)\right]_{z=0} G(0)A_m\end{aligned} \tag{4.26}$$

where

$$A_m = \sum_{n=1}^{\infty} n\lambda\frac{dH_{mn}(\lambda)}{d\lambda}\frac{Q_n}{Q_m}. \tag{4.28}$$

After some lengthy calculation we find

$$A_m = \tfrac{1}{6}m(m^2-1) \tag{4.28}$$

and

$$\mu_{1t}(p) = [(c+d)G(0)^2 - 1]S_m(p) + \tfrac{1}{6}m(m^2-1)G(0)^2 \tag{4.29}$$

which has critical exponent 4 and it follows that the mean length $\langle t \rangle = \mu_{1t}(p)/S_m(p)$ has critical exponent 2. Assuming the existence of the scaling length $\xi_{\Uparrow}(p)$ this gives $\nu_{\Uparrow} = 2$. This assumption will be verified later in the case $m = 1$.

Combining this result with (4.25) gives

$$\mu_{2y}(p) = 2p(1-p)(c+d)G(0)^2 S_m(p) + \tfrac{1}{6}m(m^2-1)G(0)^2 - \tfrac{1}{3}[M_3 - M_1] \tag{4.30}$$

which has critical exponent 4 and therefore the mean square displacement $\langle y^2 \rangle = \mu_{2y}(p)/S_m(p)$ of the cluster mass from the t-axis has critical exponent 2. Assuming the existence of the scaling length $\xi_{\perp}(p)$ gives $\nu_{\perp} = 1$. This agrees with the value given by Domany and Kinzel (1984) which they deduced from a result of Verhagen (1976) for a triangular lattice Ising model with three spin interactions in alternate triangles. Our derivation is direct and uses the same definition as is used in series expansion calculations (2.2).

In the case $m = 1$ we have the explicit formula

$$R_{1n}(z) = \frac{1}{c} e^z \lambda^\star(z)^n \tag{4.31}$$

and using this we may show that the kth moment of $t+1$ may be written

$$\begin{aligned} \langle (t+1)^k \rangle &= \sum_{n=1}^{\infty} n \left[\frac{d^k(e^{-z}R_{1n}(z))}{d(-z)^k} \right]_{z=0} \frac{Q_n}{Q_1 S_1} \\ &= \sum_{n=1}^{\infty} n\alpha_k(n) R_{1n}(0) \frac{Q_n}{Q_1 S_1} \end{aligned} \tag{4.32}$$

where, with $C_k = n(-1)^{k-1}G^{(k-1)}(0)$,

$$\begin{gathered} \alpha_1(n) = C_1, \quad \alpha_2(n) = C_2 + C_1^2, \\ \alpha_3(n) = C_3 + 3C_1C_2 + C_1^3, \\ \alpha_4(n) = C_4 + 3C_1C_3 + 3C_2^2 + 6C_1^2C_2 + C_1^4, \end{gathered} \tag{4.33}$$

and in general the subscripts are the possible partitions of k and the coefficients are the number of partitions of each type. Carrying out the n-summation using (4.8) we find

$$\langle (t+1) \rangle = G(0)M_2/S_1 \tag{4.34}$$

$$\langle (t+1)^2 \rangle = [-G'(0)M_2 + G(0)^2 M_3]/S_1 \tag{4.35}$$

$$\langle (t+1)^3 \rangle = [G''(0)M_2 - 3G'(0)G(0)M_3 + G(0)^3 M_4]/S_1 \tag{4.36}$$

and so on. Equation (4.34) may be seen to agree with (4.29) when $m = 1$ by using (4.18) and $\langle t+1 \rangle = \langle t \rangle + 1$. Using our previous results that M_k has exponent $k+1$ and that $G^{(k)}(0)$ has exponent $2k+1$ it can be seen that each of the terms has the same critical exponent and hence that $\langle t^k \rangle$ has critical exponent $2k$ which verifies the existence of the scaling length $\xi_{\Uparrow}(p)$ having critical exponent $\nu_{\Uparrow} = 2$ the same as found from the cluster length distribution in Section 2.3.

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