Brownian Motion and the Riemann Zeta-Function

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My intention when I went to Oxford as an undergraduate was to be a physicist, but to do some mathematics first. In the first undergraduate year of mathematics, John Hammersley gave a course which included the quaternion proof of the Four-Squares Theorem, the 'elementary' proof of the Prime Number Theorem, and an introduction to Operational Research. It was strong meat for first-year undergraduates, and I'm sure that there was little which I understood fully. But it was marvellous material, conveyed with style and infectious excitement; and, more than anything, it persuaded me to stay with mathematics.

My intention when I was invited to contribute to this volume was to submit something on one or other of two concrete problems of interest to John Hammersley. But, unfortunately, while John had been able to convey enthusiasm to us students, it was of course impossible for him to grant us some of his creativity with hard problems; and, left therefore to my own devices, I have failed to make progress with either problem. Meanwhile ...

It seems that number-theorists have recently become interested in path-integral representations of the Riemann ζ-function. Such representations have for a long time been familiar to aficionados of Brownian excursion theory — I am sure that Kai Lai Chung and many others have known them as long as I have. C.M. Newman (1975) had explained that if it could be shown that a certain probability density function is 'ferromagnetic', then the Riemann Hypothesis would follow. The fact that this density function arises fairly naturally in the study of Brownian motion (of which more, I hope, in a later paper with Tim Mortimer) therefore has a certain entertainment value, though perhaps nothing more. Here — with thanks, apologies for not being brighter, and very best wishes, to John Hammersley — is a talk I gave recently to some of the number-theorists at Cambridge.

The theory of Brownian motion contains many remarkable identities. Many now have a complete explanation, though even in certain of these cases, there was a time when they were regarded as 'coincidences'. Amongst the identities for which a proper explanation remains to be found are some which are closely related to Riemann's ξ-function.

Contents

1. What is Brownian Motion?

For $t > 0$ and $x, y \in \mathbb{R}$, define

$$p_t(x,y) = (2\pi t)^{-1/2} \exp\{-(y-x)^2/2t\}. \tag{1.1}$$

Thus $p_t(x,\cdot)$ is the density of the *normal distribution* of mean x and variance t. The fact that p solves the *heat equation*:

$$D_t p = \tfrac{1}{2} D_{xx} p = \tfrac{1}{2} D_{yy} p \tag{1.2}$$

is best regarded as expressing the formula

$$P_t = \exp(t\tfrac{1}{2}\Delta), \text{ where } P_t f(x) := \int_{\mathbb{R}} p_t(x,y) f(y) dy, \tag{1.3}$$

which is made precise by Hille-Yosida theory. Let $\mathcal{C}$ be the smallest σ-algebra on *'path-space'* $C[0,\infty)$ such that, for each $t > 0$, the evaluation map $w \mapsto w(t)$ on $C[0,\infty)$ is $\mathcal{C}$ measurable.

Wiener's Theorem. *For $x \in \mathbb{R}$, there exists a unique measure W^x on $(C[0,\infty),\mathcal{C})$ such that for $n \in \mathbb{N}$, for $0 < t_1 < t_2 < \ldots < t_n$ and for $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathbb{R})$,*

$$W^x(\{w \in C[0,\infty) : w_{t_i} \in A_i \ (1 \le i \le n)\}) = \int_{x_1 \in A_1} \cdots \int_{x_n \in A_n} \prod_{i=1}^{n} \{p_{t_i - t_{i-1}}(x_{i-1}, x_i)dx_i\} \quad (1.4)$$

where $t_0 = 0, x_0 = x$.

The probability measure W^x is called *Wiener measure corresponding to starting position x.*

Suppose that we have a set-up $(\Omega, \mathcal{F}, \mathbb{P}^x, B)$ where Ω is a set, $\mathcal{F}$ is a σ-algebra on Ω, each $\mathbb{P}^x$ is a probability measure on $(\Omega, \mathcal{F})$, and

$$B : \Omega \to C[0,\infty), \qquad B^{-1} : \mathcal{C} \to \mathcal{F},$$
$$\omega \mapsto (t \mapsto B_t(\omega)).$$

Then B is called a *Brownian motion* if $\mathbb{P}^x \circ B^{-1} = W^x$ $(x \in \mathbb{R})$ on $\mathcal{C}$.

Canonical Brownian motion is the set-up:

$$(\Omega, \mathcal{F}, \mathbb{P}^x, B) = (C[0,\infty), \mathcal{C}, W^x, \mathrm{id}).$$

Properties (1.3) and (1.4) say: *Brownian motion is Markovian with transition density function p and with generator $\frac{1}{2}\Delta$.*

Expectation. If $Z : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable, we define

$$\mathbb{E}^x Z = \int_\Omega Z(w)\mathbb{P}^x(dw).$$

Example. For Borel function f on $\mathbb{R}$, $\mathbb{E}^x f(B_t) = \int f(y)\mathbb{P}^x(B_t \in dy) = P_t f(x)$.

2. Cauchy's Proof of the Functional Equation

Let Γ be the circle $\Gamma = \mathbb{R}/\mathbb{Z}\sqrt{2\pi}$, and let π (no confusion possible!) be the projection $\pi : \mathbb{R} \to \Gamma$. If B is $BM(\mathbb{R})$ (a Brownian motion on $\mathbb{R}$), then $B^\Gamma := \pi \circ B$ is a $BM(\Gamma)$, Markovian with transition density function

$$p_t^\Gamma(x,y) = \sum_{\{z : \pi z = y\}} p_t(x,z) \quad (2.1)$$

and generator $\frac{1}{2}\Delta^\Gamma$. Now, $\frac{1}{2}\Delta^{\mathcal{G}}$ has
normalized eigenfunctions: $(2\pi)^{-1/4}e^{in\sqrt{2\pi}\theta}$ $(n \in \mathbb{Z})$,
corresponding eigenvalues: $-n^2\pi$.
Hence $P_t^\Gamma := \exp(t\frac{1}{2}\Delta^\Gamma)$ has eigenvalues $e^{-n^2\pi t}$, so that

$$\mathrm{Trace}\left(P_t^\Gamma\right) = \theta(t) := \sum_{n\in\mathbb{Z}} e^{-n^2\pi t}.$$

But, using (2.1) and the obvious fact that $\mathrm{Trace}(P_t^\Gamma) = \sqrt{2\pi}p_t^\Gamma(0,0)$, we see that

$$\mathrm{Trace}(P_t^\Gamma) = \sqrt{2\pi}\sum_{n\in\mathbb{Z}}(2\pi t)^{-1/2}\exp(-n^2\cdot 2\pi/2t) = t^{-1/2}\theta(t^{-1}).$$

So, we have

$$\theta(t) = t^{-1/2}\theta(t^{-1}). \qquad \text{(Jacobi)}$$

As everyone knows, $\zeta(z) = \sum_{n\in\mathbb{N}} n^{-z}$ extended analytically from $\{\mathcal{R}z > 1\}$ to $\mathbb{C}\backslash\{1\}$. It was already known to Riemann that Jacobi's functional equation for θ implies the functional equation

$$\xi(z) = \xi(1-z)$$

for ξ (or ζ), where ξ is the entire function:

$$\xi(z) = \tfrac{1}{2}z(z-1)\pi^{-z/2}\Gamma(\tfrac{1}{2}z)\zeta(z).$$

The Riemann Hypothesis says: *if* $\xi(z) = 0$, *then* $\mathcal{R}z = \frac{1}{2}$.

3. Brownian Bridges and Bessel Bridges

(a) The 1-dimensional case. Intuitively, Brownian bridge with values in $\mathbb{R}$, $BB(\mathbb{R})$, is $BM(\mathbb{R})$ with time-parameter set [0,1] conditioned to be at 0 at times 0 and 1. Rigorously, there is unique measure $W^{0,0}$ on $C[0,1]$ with obvious σ-algebra $\mathcal{C}[0,1]$ $(= \mathcal{B}(C[0,1])!)$ such that for every $h \in C_b(C[0,1])$,

$$\int_{C[0,1]} h(w)W^{0,0}(dw) = \lim_{\epsilon\downarrow 0}\frac{\int_{C[0,\infty)\cap\{w:|w(1)|<\epsilon\}} h\left(w|_{[0,1]}\right)W^0(dw)}{W^0\{|w(1)|<\epsilon\}}.$$

A set-up $(\Omega,\mathcal{F},\mathbb{P},BB)$, where $BB:\Omega\to C[0,1]$ etc, is called a $BB(\mathbb{R})$ if

$$\mathbb{P}\circ BB^{-1} = W^{0,0} \text{ on } \mathcal{C}[0,1].$$

Example. Suppose that $(\Omega,\mathcal{F},\mathbb{P}^x,B)$ is a $BM(\mathbb{R})$, and that we set

$$BB_t(\omega) := \begin{cases} tB\left(\frac{1-t}{t},\omega\right) & \text{if } 0<t<1,\\ 0 & \text{if } t\in\{0,1\}.\end{cases}$$

Then $(\Omega,\mathcal{F},\mathbb{P}^0,BB)$ is a $BB(\mathbb{R})$.

(3.1) THEOREM. *Let BB be a $BB(\mathbb{R})$, and set*

$$R(\omega) := \sqrt{\frac{2}{\pi}}\left(\sup_{t\leq 1} BB_t(\omega) - \inf_{t\leq 1} BB_t(\omega)\right).$$

Then, for all z in $\mathbb{C}$,

$$\xi(z) = \tfrac{1}{2}\mathbb{E}(R^z) = \tfrac{1}{2}\int_\Omega R(\omega)^z \mathbb{P}(d\omega).$$

This, or some equivalent, has been known for some time. It appears in a fine paper by Biane and Yor (1987). I would like to say something about how the result relates to *'interchanging space and time'*, and also to the following *Fourier expansion of* $BB(\mathbb{R})$.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ carry independent random variables $G_1, G_2, \ldots$ each normally distributed with mean 0 and variance 1. Define

$$BB_t(\omega) := \sum_{n\geq 1} \frac{G_n(\omega)}{n\pi}\sqrt{2}\sin(n\pi t), \quad 0 \leq t \leq 1.$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, BB)$ is a $BB(\mathbb{R})$. Note that Parseval says:

$$\int_0^1 BB_t(\omega)^2 dt = \sum \frac{G_n(\omega)^2}{n^2\pi^2}. \tag{3.2}$$

(c) Brownian motion in $\mathbb{R}^n$, $BM(\mathbb{R}^n)$. We build $BM(\mathbb{R}^n)$ by making the component processes *independent* $BM(\mathbb{R})$ processes. Since

$$C([0,\infty);\mathbb{R}^n) = \prod_{i=1}^n C([0,\infty);\mathbb{R}) \quad \text{canonically,}$$

we can define, for $\mathbf{x} \in \mathbb{R}^n$,

$$\left(\mathbf{W}^{\mathbf{x}} \text{ on } (C[0,\infty);\mathbb{R}^n)\right) = \prod_{i=1}^n \left(W^{x_i} \text{ on } C([0,\infty);\mathbb{R})\right). \tag{3.3}$$

A $BM(\mathbb{R}^n)$ is a set-up $(\Omega, \mathcal{F}, \mathbb{P}^{\mathbf{x}}\colon \mathbf{x} \in \mathbb{R}^n, \mathbf{B})$, $\mathbf{B}\colon \Omega \to C([0,\infty);\mathbb{R}^n)$ such that $\mathbb{P}^{\mathbf{x}} \circ \mathbf{B}^{-1} = \mathbf{W}^{\mathbf{x}}$. A $BM(\mathbb{R}^n)$ is Markovian with generator

$$\tfrac{1}{2}\Delta = \sum \tfrac{1}{2}\frac{\partial^2}{\partial x_i^2} = \tfrac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{n-1}{2r}\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\tfrac{1}{2}\Delta^{S^{n-1}}. \tag{3.4}$$

The first formula for Δ only reiterates the product measure structure (3.3).

(d) Bessel process $BES(n)$ on $[0,\infty)$. Invariance of the family $\mathbf{W}^{\cdot}$ under $\mathrm{O}(n)$ implies that the radial part $r = |\mathbf{B}|$ of a $BM(\mathbb{R}^n)$ $\mathbf{B}$ is Markovian with generator $\frac{1}{2}\frac{d^2}{dr^2} + \frac{n-1}{2r}\frac{d}{dr}$; and we say that r is $BES(n)$, Bessel process on $[0,\infty)$ associated with dimension n. The second formula for $\frac{1}{2}\Delta$ at (3.4) means:

$$dr = d\beta + \frac{n-1}{2r}dt, \qquad \beta \text{ a } BM(\mathbb{R}),$$

$$\frac{B_t}{r_t} = BM^{S^{n-1}}\left(\int_0^t r_s^{-2} ds\right).$$

Making the first of these precise: if $(\Omega, \mathcal{F}, \mathbb{P}^{\mathbf{x}}, \mathbf{B})$ is a $BM(\mathbb{R}^n)$, then, for $\mathbf{x} \neq \mathbf{0}$, $\mathbb{P}^{\mathbf{x}} \circ \beta^{-1} = W^{|x|}$, where

$$\beta_t(\omega) := |\mathbf{B}_t(\omega)| - \int_0^t \frac{n-1}{2|\mathbf{B}_s(\omega)|} ds.$$

(e) Brownian bridge in $\mathbb{R}^n$, $BB(\mathbb{R}^n)$. If $(\Omega, \mathcal{F}, \mathbb{P}^x, \mathbf{B})$ is a $BM(\mathbb{R}^n)$, and $\mathbf{BB}_t(\omega) := t\mathbf{B}\left(\frac{1-t}{t}\right)$, then $\mathbb{P}^{\mathbf{0}} \circ \mathbf{BB}^{-1} = \mathbf{W}^{\mathbf{0},\mathbf{0}} := \prod_{i=1}^n W^{0,0}$. So, we say that $(\Omega, \mathcal{F}, \mathbb{P}^0, \mathbf{B}B)$ is a $BB(\mathbb{R}^n)$.

(f) $BES(n)BR$ with values in $[0,\infty)$. If $(\Omega, \mathcal{F}, \mathbb{P}, r)$ is a $BES(n)$ starting at 0, then $(\Omega, \mathcal{F}, \mathbb{P}, tr\left(\frac{1-t}{t}\right))$ is a $BES(n)BR$.

Pythagoras says: if $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, BB^i)$ is a $BB(\mathbb{R})$ $(1 \leq i \leq n)$ and we set $(\Omega, \mathcal{F}, \mathbb{P}) = \prod_{i=1}^n (\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$, then $(\Omega, \mathcal{F}, \mathbb{P}, \hat{r})$ is a $BES(n)BR$, where

$$\hat{r}_t(\omega) := \left(\sum_{i=1}^n BB_t^i(\omega)^2\right)^{1/2}.$$

4. The Excursion Picture of Reflecting Brownian Motion

Consider a reservoir which can hold any volume of water from $-\infty$ up to 0. Suppose input to reservoir is a $BM(\mathbb{R})$ process B starting at 0. This is represented by upper curve. Then

$L_t =$ overflow by time $t = \sup_{s \leq t} B_s$.

Actual volume at time t
$= Y_t = B_t - L_t$.

$(X_t) = (-Y_t)$ is reflecting BM
(RBM) on $[0,\infty)$, with the
same law as $(|B_t|)$.

New picture

X: RBM, L local time at 0 for X (as above).

Define
$\gamma_\tau := \inf\{t : L_t > \tau\}$

For circled excursion of X away from 0,
$L_u = L_v = \tau$ (say),
$\gamma_{(\tau-)=u}, \gamma_{(\tau)=v}$.

Path in circle:

U is space of excursion paths

is excursion at local time τ:
$e_\tau(t) = X(t + \gamma_{\tau-})$, $0 \leq t \leq \gamma_\tau - \gamma_\tau$.

Itô's Theorem. *The points $(\tau, e_\tau(t))$ in $[0, \infty) \times U$ are the points of a Poisson point process. (N.B. We have point (τ, e_τ) if and only if $\gamma_\tau > \gamma_{\tau-}$.) Numbers falling in disjoint regions of $[0, \infty) \times U$ are* ***independent*** *variables. There exists a sigma-finite measure n on U such that for (measurable) $\Gamma \subseteq [0, \infty) \times U$, the number N_Γ of points in Γ has Poisson distribution parameter $\lambda(\Gamma)$, that is,*

$$\mathbb{P}(N_\Gamma = k) = e^{\lambda(\Gamma)} \lambda(\Gamma)^k / k! \ \textit{where}\ \lambda = \text{Lebesgue} \times n.$$

The measure n on the space of excursion paths is called the ***Itô excursion law****. Given n, we can build X from its excursions.*

5. The Itô Excursion Law: Bessel Descriptions

Picture of an excursion

M: maximum (at time V_1).
$V = V_1 + V_2$: lifetime.

Description I.

I(a): $n(V \in dv) = \frac{1}{\sqrt{2\pi v^3}} dv$ (Lévy);

I(b): *under* n, *conditional on* V, $\{V^{-1/2} e(tV) : 0 \leq t \leq 1\}$ *is a* $BES(3)BR$ (Williams (1970) — after Lévy, Itô and McKean);

Description II.

II(a): $n(M \in dm) = m^{-2} dm$ (Lévy);

II(b): *under* n, *conditional on* M, $\{e_t : t \leq V_1\}$ *and* $\{e_{V-t} : t \leq V_2\}$ *are independent* $BES(3)$ *processes started at 0 and run until they hit* M (Williams (1970)).

Now consider the pictures (which relate to I(b) with $V = 1$ and II(b) with $M = 1$ — both 'scaled' versions):

Biane and Yor (1987) explain (see Appendix for a more direct proof) the initially-surprising fact that agreement of Descriptions I and II implies:

$$\tfrac{1}{2}\mathbb{E}\left\{ \left(\sqrt{\frac{2}{\pi}} R \right)^{z} \right\} = \tfrac{1}{2}\mathbb{E}\left\{ \left(\frac{\pi T}{2} \right)^{\frac{1}{2} - \frac{1}{2}z} \right\}. \tag{5.1}$$

It has been known for a long time (and will now be proved) that the right-hand side of (5.1) equals $\xi(z)$. The fact that the left-hand side of (5.1) equals $\xi(z)$ is (because of a result of Vervaat (1979)) equivalent to Theorem 3.1.

A calculation. Consider the following picture:

H is the hitting time of position 1 for $BES(3)$

Recall that $BES(3)$ has generator radial $\left(\tfrac{1}{2}\Delta^{\mathbb{R}^3}\right) = \frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}$.

Standard theory says that $u(x) = \mathbb{E}^x e^{-\lambda H}$ satisfies

$$\tfrac{1}{2}u'' + x^{-1}u' = \lambda u \text{ on } (0,1),\ u(1) = 1,$$
$$u \text{ bounded near } 0.$$

Hence $u(x) = \frac{\sinh \gamma x}{\gamma x} \cdot \frac{\gamma}{\sinh \gamma}$, $\gamma = (2\lambda)^{1/2}$.

It follows that if $T = T_1 + T_2$, where T_1 and T_2 have the same distribution as H under $\mathbb{P}^0$, then

$$\mathbb{E}e^{-\lambda T} = \left(\frac{\gamma}{\sinh \gamma}\right)^2.$$

For $\mathcal{R}z > 0$, we have

$$\mathbb{E}\int_{\lambda=0}^{\infty} e^{-\lambda T}\lambda^{z-1}d\lambda = \mathbb{E}\int_{u=0}^{\infty} e^{-u}\left(\frac{u}{T}\right)^{z-1}\frac{du}{T} = \Gamma(z)\mathbb{E}(T^{-z}),$$

so that

$$\begin{aligned}\Gamma(z)\mathbb{E}(T^{-z}) &= \int_0^\infty \left(\frac{\gamma}{\sinh \gamma}\right)^2 \lambda^{z-1}d\lambda = \int_0^\infty \frac{8\lambda}{(e^\gamma - e^{-\gamma})^2}\lambda^{z-1}d\lambda \\ &= 8\int_0^\infty e^{-2\gamma}\sum_{n=0}^\infty (n+1)e^{-2n\gamma}\lambda^z d\lambda \\ &= 8\sum_{n=1}^\infty \int_0^\infty ne^{-2n\sqrt{2\lambda}}\lambda^z d\lambda = 8\sum_{n=1}^\infty \int_0^\infty ne^{-u}\left(\frac{u^2}{8n^2}\right)^z \cdot \frac{u}{4n^2}du \\ &= 2\sum_{n=1}^\infty n^{-(1+2z)}\Gamma(2z+2)8^{-z} = 2\Gamma(2z+2)\zeta(1+2z)8^{-z}.\end{aligned}$$

Using duplication formula for Γ, we find that, initially for $\mathcal{R}z > 1$,

$$\tfrac{1}{2}\mathbb{E}\left\{\left(\frac{\pi T}{2}\right)^{\frac{1}{2}-\frac{1}{2}z}\right\} = \xi(z) \quad \left(= \tfrac{1}{2}\mathbb{E}\left\{\left(\frac{\pi T}{2}\right)^{\frac{1}{2}z}\right\} \begin{array}{l}\text{if we assume}\\ \text{functional}\\ \text{equation}\end{array}\right).$$

6. Integrated Local Time

Hints that interchanging time and space might be relevant have already been given. One of the standard ways of achieving such an interchange is via the celebrated Ray-Knight theorem.

Again consider the picture

but now insist that the $BES(3)r$ starts at 0.

Theorem (Ray, Knight). *For $\Lambda \in \mathcal{B}[0,1]$,*

$$\text{measure}\{t < H\colon r(t) \in \Lambda\} = \int_\Lambda \hat{r}_2(x)^2 dx,$$

where $\hat{r}_2$ is a $BES(2)BR$.

Hence

$$H = \int_0^1 \hat{r}_2(x)^2 dx = \sum_n \frac{G_{1,n}^2 + G_{2,n}^2}{n^2\pi^2}, \quad G\text{'s independent } N(0,1),$$

using Pythagoras to tell us that a $BES(2)BR^2$ is the sum of the squares of two independent $BB(\mathbb{R})$ processes and also using the Parseval result (3.2).

Now the sum of the squares of two independent $N(0,1)$ variables is exponential with mean 2, so

$$\mathbb{E}\exp\left(-\lambda\frac{G_{1,n}^2 + G_{2,n}^2}{n^2\pi^2}\right) = \frac{1}{1 + \lambda\cdot\frac{2}{n^2\pi^2}}.$$

Hence

$$\mathbb{E}e^{-\lambda H} = \prod_{n=1}^\infty \frac{1}{1 + \frac{\gamma^2}{n^2\pi^2}} = \frac{\gamma}{\sinh\gamma}, \ \gamma = (2\lambda)^{1/2},$$

giving another explanation for the $\gamma/\sinh\gamma$ term and hence of ξ.

7. Ferromagnetism and the Lee-Yang Theorem

I end with a result from another branch of probability theory which it would be fascinating to combine with results of earlier sections.

By an *isolated ferromagnetic spin-$\frac{1}{2}$ system on N sites* is meant the following set up:

ρ is measure $\{\frac{1}{2}, \frac{1}{2}\}^N$ on $\{-1,1\}^N$,
$\beta \geq 0$ (β is inverse temperature), $J_{ij} \geq 0 (i < j)$ (interaction),
for $\mathbf{x} \in \{-1,1\}^N$, $H(\mathbf{x}) = -\sum\sum_{i<j} J_{ij}x_ix_j$ (Hamiltonian),
ν is a probability measure on $\{-1,1\}^N$ (Gibbs measure) with

$$\frac{d\nu}{d\rho} = \exp(-\beta H)/Z,$$

where $Z = \int \exp(-\beta H)d\rho$ (partition function).

Define spins X_i $(1 \leq i \leq n)$ via $X_i(\mathbf{x}) = x_i$. Call a variable Y *special mean-zero ferromagnetic* if for some non-negative numbers λ_i $(1 \leq i \leq N)$,

$$Y = \sum_i \lambda_i X_i.$$

Now call a random variable Y *mean-zero ferromagnetic* if there exists a sequence $Y^{(n)}$ of special mean-zero ferromagnetic random variables such that

(i) $Y^{(n)} \Rightarrow Y$, that is, $\mathbb{E}h(Y^{(n)}) \to \mathbb{E}h(Y)\ \forall h \in C_b(\mathbb{R})$,

(ii) $\mathbb{E}((Y^{(n)})^2 \to \mathbb{E}(Y^2)$.

Examples.

(i) If $Y \sim N(0,\sigma^2)$, then Y is mean-zero ferromagnetic, and $\mathbb{E}e^{zY} = e^{\sigma^2 z^2/2}$.

(ii) If $Y \sim U[-\gamma,\gamma]$, then Y is mean-zero ferromagnetic, and $\mathbb{E}e^{zY} = \frac{\sinh \gamma z}{\gamma z} = \prod_{n=1}^{\infty}\left(1+\frac{\gamma^2 z^2}{n^2\pi^2}\right)$.

THEOREM (LEE-YANG-NEWMAN). *If Y is mean-zero ferromagnetic, then*

$$\mathbb{E}e^{zY} = e^{bz^2}\prod_j\left(1+\frac{z^2}{\alpha_j^2}\right),\ \textit{all } \alpha_j \textit{ real}.$$

Appendix. Proof of (5.1)

Notation

True Itô excursion: Lifetime V, maximum M.

'Scaled' excursion of duration $1 = BES(3)BR$: R maximum height.

Excursion of height 1; T duration, so that $T = T_1 + T_2$, etc, as before.

Descriptions I and II show that *under the Itô excursion law n*,

$$(M,V) \sim (M, M^2T) \sim (RV^{1/2}, V).$$

Biane and Yor remind us (and some of *us* needed reminding!) that, because n has infinite total mass, we can**NOT** conclude that $T^{-1/2} \sim R$.

Correct analysis using $n(M \in dx) = x^{-2}dx$, $n(V \in dv) = (2\pi v^3)^{-1/2}dv$: for a test function g

$$\begin{aligned}
n(g(M,V)) &= \int_x\int_y \frac{1}{x^2}g(x,y)\mathbb{P}(x^2T \in dy)\,dx \\
&= \int_x\int_y g(x,y)f_T\left(\frac{y}{x^2}\right)\frac{1}{x^2}\cdot\frac{1}{x^2}\,dxdy \\
&= \int_x\int_y \frac{1}{\sqrt{2\pi y^3}}g(x,y)\mathbb{P}(y^{1/2}R \in dx)\,dy \\
&= \int_x\int_y g(x,y)f_R\left(\frac{x}{y^{1/2}}\right)\cdot\frac{1}{y^{1/2}}\cdot\frac{1}{\sqrt{2\pi y^3}}\,dxdy,
\end{aligned}$$

whence, on taking $r = xy^{-1/2}$,

$$f_R(r) = \sqrt{2\pi}\frac{1}{r^4} f_T\left(\frac{1}{r^2}\right).$$

(This is trivially equivalent to (5.1).)

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All relevant background (except that for §7) is in Rogers and Williams (1987).

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