Probability and Measure, Hand-out 4
Fourier transforms: uniqueness and inversion

A finite Borel measure is determined uniquely by its Fourier transform. There is a formula for the measure in terms of its transform, and this is quite simple in the integrable case.

Let X be a random variable taking values in $\mathbb{R}^n$. Its characteristic function ϕ_X is the Fourier transform of its law μ_X, which is to say that

$$\phi_X(u) = \mathbb{E}(e^{i\langle u,X\rangle}), \qquad u \in \mathbb{R}^n.$$

The 'heat kernel' function is given by

$$p(t,x,y) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{1}{2}|y-x|^2/t}, \qquad t > 0,\ x,y \in \mathbb{R}^n.$$

Recall the identity

$$e^{-\frac{1}{2}w^2} = \int_{\mathbb{R}} e^{iuw} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du.$$

Write $w = (x-y)/\sqrt{t}$ and make a change of variable to find when $n = 1$ that

$$p(t,x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iux} e^{-\frac{1}{2}u^2 t} e^{-iuy}\, du,$$

whence, for $n \geq 1$,

$$p(t,x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle u,x\rangle} e^{-\frac{1}{2}|u|^2 t} e^{-i\langle u,y\rangle}\, du.$$

Theorem. Let X be a random variable taking values in $\mathbb{R}^n$. The law μ_X of X is uniquely determined by knowledge of its characteristic function ϕ_X. Moreover, if ϕ_X is integrable, then X has a density function f_X where

$$f_X(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi_X(u) e^{-i\langle u,z\rangle}\, du, \qquad z \in \mathbb{R}^n.$$

Proof. Let Y be a standard Gaussian random variable in $\mathbb{R}^n$, independent of X, and let $g : \mathbb{R}^n \to \mathbb{R}$ be bounded and measurable. Then, for $t > 0$,

$$\begin{aligned}
\mathbb{E}(g(X + Y\sqrt{t})) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(x + y\sqrt{t}) \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}|y|^2} \, dy \, \mu_X(dx) \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p(t, x, z) g(z) \, dz \, \mu_X(dx) \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle u,x\rangle} e^{-\frac{1}{2}|u|^2 t} e^{-i\langle u,z\rangle} \, du \, \mu_X(dx) \right) g(z) \, dz \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi_X(u) e^{-\frac{1}{2}|u|^2 t} e^{-i\langle u,z\rangle} \, du \right) g(z) \, dz,
\end{aligned}$$

where we have used Fubini's theorem and the change of variables $z = x + y\sqrt{t}$.

By this formula, ϕ_X determines $\mathbb{E}(g(X + Y\sqrt{t}))$ for any given g. For any bounded continuous function g, we have by bounded convergence that

$$\mathbb{E}(g(X + Y\sqrt{t})) \to \mathbb{E}(g(X)) \qquad \text{as } t \downarrow 0,$$

whence ϕ_X determines $\mathbb{E}(g(X))$ for any given g. It follows that ϕ_X determines μ_X.

If ϕ_X is integrable and if g is continuous with compact support, then $|\phi_X(u)| \cdot |g(z)| \in L^1(\mathbb{R}^{2n}, \mathcal{B}, \lambda)$. By dominated convergence,

$$\begin{aligned}
\mathbb{E}(g(X + Y\sqrt{t})) &= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi_X(u) e^{-\frac{1}{2}|u|^2 t} e^{-i\langle u,z\rangle} \, du \right) g(z) \, dz \\
&\to \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi_X(u) e^{-i\langle u,z\rangle} \, du \right) g(z) \, dz,
\end{aligned}$$

as $t \downarrow 0$. Hence X has the claimed density function.