Probability and Measure 2

1. (For 'revision') Let $(x_n : n \geq 1)$ be a sequence of reals, and define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{m\geq n} x_m, \quad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{m\geq n} x_m.$$

Show that $x_n \to x$ ($\in \mathbb{R}$) if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$.

2. (For 'review') Check over the proofs of extension and uniqueness in Handout I.

3. Let μ_F be the Lebesgue–Stieltjes measure associated with the distribution function F. Show that F is continuous at x if and only if $\mu_F(\{x\}) = 0$. What is the corresponding condition on the so-called 'joint distribution function' F when working in $\mathbb{R}^d$ with $d \geq 2$?

4. Let F_n, $n \geq 1$, be a sequence of distribution functions such that the limit $F(x) = \lim_{n\to\infty} F_n(x)$ exists for all $x \in \mathbb{R}$. Show that F need not be a distribution function.

5. (**'Skorohod representation theorem'**) Let F, F_n $(n \geq 1)$ be distribution functions such that $F(x) = \lim_{n\to\infty} F_n(x)$ for all $x \in \mathbb{R}$ at which F is continuous. Show that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with random variables $X, X_n : \Omega \to \mathbb{R}$ such that:
(i) X has distribution function F,
(ii) X_n has distribution function F_n,
(iii) $X_n \to X$ almost surely as $n \to \infty$.

6. Let $i, j \geq 1$ and let $f : \mathbb{R}^i \to \mathbb{R}^j$ be continuous. Show that f is measurable. [You may work with either the Borel or Lebesgue σ-fields, but should be specific about your choice.]

7. Let $(\Omega, \mathcal{F})$ be a measurable pair, and let $f : \Omega \to \mathbb{R}^d$ be written in the form $f(\omega) = (f_1(\omega), f_2(\omega), \ldots, f_d(\omega))$. Working with the Borel σ-fields of $\mathbb{R}^d$ and $\mathbb{R}$, show that f is measurable if and only if each $f_i : \Omega \to \mathbb{R}$ is measurable.

8. (Continuation) Let $(\Omega, \mathcal{F})$ be a measurable pair, and let f, g be (Borel) measurable functions from Ω to $\mathbb{R}$. Show that $f + g$, fg, $\max\{f, g\}$, $\min\{f, g\}$ are measurable functions.

9. (Continuation) Show further that, if f_n, $n \geq 1$ are measurable functions, then so are
(i) $\inf_n f_n$, $\sup_n f_n$,
(ii) $\liminf_n f_n$, $\limsup_n f_n$,
(iii) the set L of all $\omega \in \Omega$ at which the limit $f(\omega) = \lim_n f_n(\omega)$ exists,
(iv) the function

$$g(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Note. If the above infima and/or suprema etc are infinite, it will be necessary to extend the range to the extended real line $[-\infty, \infty]$ with the extended σ-field.

10. Let $f : \mathbb{R} \to \mathbb{R}^*$.

(a) Show that $\mu(f) = 0$ if $f = 0$ μ-almost everywhere (μ-a.e., that is, $\mu\{\omega : f(\omega) \neq 0\} = 0$).

(b) If $f \geq 0$ μ-a.e. and $\mu\{\omega : f(\omega) > 0\} > 0$, show that $\mu(f) > 0$.

(c) If $|\mu(f)| < \infty$ show that $|f| < \infty$ μ-a.e.

(d) If $f = g$ μ-a.e., show that $\mu(f) = \mu(g)$ whenever one of these integrals is defined.

11. Consider an alternative definition of the integral. Let $f : \mathbb{R} \to [0, \infty)$ be Borel-measurable, and define

$$\mu'(f, \mathcal{A}) = \sum_{i=1}^{n} \left\{ \sup_{\omega \in A_i} f(\omega) \right\} \mu(A_i), \qquad \mu'(f) = \inf_{\mathcal{A}} \mu'(f, \mathcal{A}).$$

[Here, $\mathcal{A}$ denotes a finite partition of $\mathbb{R}$ into Borel sets.] Show that $\mu'(f) = \infty$ if

$$\begin{aligned} &\text{either (i)} \quad \mu\{\omega : f(\omega) > 0\} = \infty, \\ &\text{or (ii)} \quad \mu\{\omega : f(\omega) > x\} > 0 \quad \text{for all } x. \end{aligned}$$

Would you regard this as a satisfactory definition?

12. Let X be a non-negative integer-valued random variable. Show that

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$$

Deduce that, if $\mathbb{E}(X) = \infty$, and $X, X_1, X_2, \ldots$ is a sequence of iid random variables, then $\limsup_{n\to\infty}(X_n/n) \geq 1$ a.s., and indeed

$$\limsup_{n\to\infty}(X_n/n) = \infty \quad \text{a.s.}$$

13. (Continuation) Suppose that $Y_1, Y_2, \ldots$ is a sequence of iid random variables with $\mathbb{E}|Y_1| = \infty$. Show that $\limsup_{n\to\infty}(|Y_n|/n) = \infty$ a.s., and indeed

$$\limsup_{n\to\infty} \frac{1}{n}|Y_1 + Y_2 + \cdots + Y_n| = \infty \quad \text{a.s.}$$

14. Let $\alpha \in (0, \infty)$ and $p \in [1, \infty)$, and let $f_\alpha(x) = x^{-\alpha}$ for $x > 0$. Show that

$$\begin{aligned} f_\alpha \in L^p((0, 1], dx) \quad &\Leftrightarrow \quad \alpha p < 1, \\ f_\alpha \in L^p([1, \infty), dx) \quad &\Leftrightarrow \quad \alpha p > 1. \end{aligned}$$

15. Show that, as $n \to \infty$,

$$\text{(a)} \quad \int_0^\infty \frac{\sin e^x}{1 + nx^2}\, dx \to 0, \qquad \text{(b)} \quad \int_0^1 \frac{n \cos x}{1 + n^2 x^{3/2}}\, dx \to 0.$$

16. Show that the function $f(x) = x^{-1}\sin x$ is not Lebesgue integrable over the interval $[1, \infty)$. By using Fubini's theorem and the identity $x^{-1} = \int_0^\infty e^{-xt}\, dt$, $x > 0$, (or otherwise; you may use integration by parts) prove that

$$\lim_{y\to\infty} \int_0^y \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$