## 1. Bernoulli shifts

Let $m$ be a probability measure on $\mathbb{R}$. W may construct a (product) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there exists a sequence of independent random variables $(Y_n : n \in \mathbb{N})$, each having distribution $m$. Consider now the infinite product space

$$E = \mathbb{R}^{\mathbb{N}} = \{x = (x_n : n \in \mathbb{N}) : x_n \in \mathbb{R} \text{ for all } n\}$$

and the $\sigma$-algebra $\mathcal{E}$ on $E$ generated by the coordinate maps $X_n(x) = x_n$,

$$\mathcal{E} = \sigma(X_n : n \in \mathbb{N}).$$

Note that $\mathcal{E}$ is also generated by the $\pi$-system

$$\mathcal{R} = \left\{ \prod_{n \in \mathbb{N}} A_n : A_n \in \mathcal{B} \text{ for all } n,\ A_n = \mathbb{R} \text{ for sufficiently large } n \right\}.$$

Define the function $Y : \Omega \to E$ by $Y(\omega) = (Y_n(\omega) : n \in \mathbb{N})$. It is easily checked that $Y$ is measurable, and the image measure $\mu = \mathbb{P} \circ Y^{-1}$ satisfies,

$$\mu(A) = \prod_{n \in \mathbb{N}} m(A_n) \quad \text{for} \quad A = \prod_{n \in \mathbb{N}} A_n \in \mathcal{R}.$$

By the uniqueness-of-extension theorem, $\mu$ is the unique measure on $\mathcal{E}$ having this property. Note that, under the probability measure $\mu$, the coordinate maps $(X_n : n \in \mathbb{N})$ are themselves a sequence of independent random variables with law $m$. The probability space $(E, \mathcal{E}, \mu)$ is called the *canonical model* for such sequences.

Define the *shift map* $\theta : E \to E$ by

$$\theta(x) = (x_2, x_3, \dots) \quad \text{where } x = (x_1, x_2, \dots).$$

**Theorem 1.0.1.** *The shift map is an ergodic measure-preserving transformation.*

*Proof.* The details of showing that $\theta$ is measurable and measure-preserving are left as an exercise. To see that $\theta$ is ergodic, we recall the definition of the tail $\sigma$-fields,

$$\mathcal{T}_n = \sigma(X_m : m \geq n+1), \quad \mathcal{T} = \bigcap_n \mathcal{T}_n.$$

For $A = \prod_{k \in \mathbb{N}} A_k \in \mathcal{R}$ we have

$$\theta^{-n}(A) = \{X_{n+k} \in A_k \text{ for all } k\} \in \mathcal{T}_n.$$

Since $\mathcal{T}_n$ is a $\sigma$-field, we have that $\theta^{-n}(A) \in \mathcal{T}_n$ for all $A \in \mathcal{E}$. If $A$ lies in the invariant $\sigma$-field $\mathcal{E}_\theta$, then $A = \theta^{-n}(A) \in \mathcal{T}_n$ for all $n$, whence $A \in \bigcap_n \mathcal{T}_n = \mathcal{T}$ and $\mathcal{E}_\theta \subseteq \mathcal{T}$. By the Kolmogorov zero-one law, $\mathcal{T}$ is trivial in the sense that every member has probability either 0 or 1, and it follows that $\mathcal{E}$ is trivial also. $\square$