Probability and Measure 4

1. Let $\mathcal{R}$ be a family of random variables on the space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\sup_{X \in \mathcal{R}} \mathbb{E}\big(|X| I_{\{|X|>K\}}\big) \to 0 \qquad \text{as } K \to \infty.$$

Show that $\mathcal{R}$ is uniformly integrable.

2. Let μ_1, μ_2 be finite measures on $(\mathbb{R}, \mathcal{B})$ such that $\mu_1(g) = \mu_2(g)$ for all bounded continuous $g : \mathbb{R} \to \mathbb{R}$. Show that $\mu_1 = \mu_2$.

3. Show that the Fourier transform of a finite Borel measure on $\mathbb{R}$ is bounded and continuous.

4. Let X be a random variable and let $a, b \in \mathbb{R}$, Show that $Y = aX + b$ has characteristic function $\phi_Y(t) = e^{itb}\phi_X(at)$.

5. Show that there do not exist independent identically distributed random variables X, Y such that $X - Y$ has the uniform distribution on the interval $[-1, 1]$.

6. (a) Show that the characteristic function of the Cauchy density function, $f(x) = 1/\{\pi(1 + x^2)\}$, $x \in \mathbb{R}$, is

$$\phi(t) = e^{-|t|}, \quad t \in \mathbb{R}.$$

(b) Let $X_1, X_2, \ldots$ be independent random variables with the Cauchy distribution. Find the distribution of $n^{-1}(X_1 + X_2 + \cdots + X_n)$, and comment on your conclusion in the light of the law of large numbers and the central limit theorem.

7. Let μ be a finite Borel measure on $\mathbb{R}$, and suppose that its Fourier transform $\widehat{\mu}$ is Lebesgue integrable. Show that μ has a continuous density function f, in that

$$\mu(A) = \int_A f(x)\, dx, \quad A \in \mathcal{B}.$$

8. Let μ be a finite Borel measure on $\mathbb{R}$, and suppose that $\int_{\mathbb{R}} |x|^k \,\mu(dx) < \infty$ for some $k \in \{1, 2, \ldots\}$. Show that the Fourier transform $\widehat{\mu}$ has continuous kth derivative $\widehat{\mu}^{(k)}$, which at 0 is given by

$$\widehat{\mu}^{(k)}(0) = i^k \int_{\mathbb{R}} x^k \,\mu(dx).$$

9. Let $X_1, X_2, \ldots, X_n$ be jointly Gaussian with

$$\mathbb{E}(X_i) = \mu_i, \quad \operatorname{cov}(X_i, X_j) = \Sigma_{i,j}.$$

(a) Suppose the matrix $\Sigma = (\Sigma_{i,j})$ is invertible, and set $Y = \Sigma^{-\frac{1}{2}}(X - \mu)$. [Recall that, for a non-negative definite matrix V there exists a 'square-root' matrix W, written $V^{\frac{1}{2}}$, such that $W^2 = V$.] Show that $Y_1, Y_2, \ldots, Y_n$ are independent $N(0,1)$ random variables.

(b) Show that we can write X_2 in the form $X_2 = aX_1 + Z$ where Z is independent of X_1, and determine the distribution of Z.

10. Let $X_1, X_2, \ldots$ be independent $N(0,1)$ random varables. Show that the vectors

$$\left(\overline{X}, \sum_{k=1}^{n}(X_k - \overline{X})^2\right), \qquad \left(\frac{X_n}{\sqrt{n}}, \sum_{k=1}^{n-1} X_k^2\right)$$

have the same distribution, where $\overline{X} = n^{-1}(X_1 + X_2 + \cdots + X_n)$.

11. **Entropy.** The interval $[0,1]$ is partitioned into n disjoint sub-intervals with lengths $p_1, p_2, \ldots, p_n$, and the *entropy* of this partition is defined to be $h = -\sum_{i=1}^{n} p_i \log p_i$. Let $X_1, X_2, \ldots$ be independent random variables having the uniform distribution on $[0,1]$, and let $Z_m(i)$ be the number of the $X_1, X_2, \ldots, X_m$ which lie in the ith interval of the partition above. Show that

$$R_m = \prod_{i=1}^{n} p_i^{Z_m(i)}$$

satisfies $m^{-1} \log R_m \to -h$ a.s. as $m \to \infty$.

12. Prove that

$$\sum_{\substack{k: \\ |k-n| \le x\sqrt{n}}} \frac{n^k}{k!} e^{-n} \to \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du \qquad \text{as } n \to \infty.$$

13. Suppose $(\Omega, \mathcal{F}, \mu)$ is a finite measure space.

(a) Let $f : \Omega \to \mathbb{R}$ be integrable and $\theta : \Omega \to \Omega$ be measure-preserving. Show that $f \circ \theta$ is integrable with $\mu(f \circ \theta) = \mu(f)$.

(b) Let $\theta : \Omega \to \Omega$ be ergodic and $f : \Omega \to \mathbb{R}$ be invariant. Show that f is constant almost everywhere.

14. Let $\Omega = [0,1)$ with Lebesgue measure μ. Let $\alpha \in \Omega$, and consider the mapping $\tau(x) = x + \alpha$, with addition modulo 1.

(a) Show that τ is not ergodic when α is rational.

(b) Show that τ is ergodic when α is irrational. You may find it useful to prove first that the set $\{n\alpha : n \ge 1\}$ is dense in Ω.

(c) For each α and any integrable f, determine the value of the limit function

$$g = \lim_{n\to\infty} n^{-1}(f + f \circ \tau + \cdots + f \circ \tau^{n-1}).$$

15. **(Continuation)** Show that $\rho(x) = 2x$ (modulo 1) defines a measure-preserving and ergodic mapping on $[0,1)$. Determine the limit function g given in the previous example with τ replaced by ρ.

16. Let $X_1, X_2, \ldots$ be a sequence of random variables such that, for each $k \ge 1$, the vector $(X_{n+1}, X_{n+2}, \ldots, X_{n+k})$ has a distribution which does not depend on the value of n. (Such a sequence is called *strongly stationary*.) Assume that $X_1 \in L^p$ where $1 \le p < \infty$. Show that $n^{-1}\sum_{k=1}^{n} X_k$ converges a.s. and in L^p as $n \to \infty$.