Percolation: Questions 2

1. Let ϕ be a random-cluster measure on a finite graph $G = (V, E)$ with parameters p and q. Let A be an increasing event, and prove that
$$\frac{d}{dp}\phi(A) = \frac{1}{p(1-p)}\Big\{\phi(N1_A) - \phi(N)\phi(A)\Big\}$$
where N is the number of open edges of a configuration ω and 1_A is the indicator function of the event A.

2. (continuation) Show that ϕ satisfies the FKG inequality if $q \geq 1$, but does not generally have this property when $q < 1$.

3. Show that the critical point $p_c(q)$ of the random-cluster model satisfies
$$p_c(1) \leq p_c(q) \leq \frac{qp_c(1)}{1+(q-1)p_c(1)}, \quad q \geq 1.$$
You may appeal to the comparison inequalities.

4. Consider the following random fractal. Start with the unit cube C_0 in d dimensions, where $d \geq 1$. Divide C_0 into M^d subcubes of side-length M^{-1}. Now, pick some subset of this set of subcubes according to some given probability measure ψ, and call these subcubes *open*. Now iterate the process on the open subcubes only, using the same measure ψ at each stage. Iterate repeatedly, to obtain a limit set $C = \lim_{n\to\infty} C_n$ as before. Propose a formula for the Hausdorff dimension of C, and justify this formula as was done in lectures in a special case.

5. Present a sketch proof that, for bond percolation on $\mathbb{L}^2$ with $p = \frac{1}{2}$, there exists a.s. an open circuit with the origin in its interior.

6. The following random 'dynamical system' has acquired a certain notoriety. At each vertex of $\mathbb{L}^2$ we place a mirror. These mirrors are plane and double-backed, and are placed such that light incident along an axis is reflected along a perpendicular axis. Each mirror has two possible configurations, labelled in the natural way NW and NE, and we stipulate that, for each vertex x, the mirror at x is equally likely to be NW as NE (and that mirrors at different vertices have independent orientations). We light a candle at the origin, and let C be the set of illuminated vertices. Show that $P(|C| < \infty) = 1$. (*Hint*. Use the result of Question 5, on a suitable copy of $\mathbb{L}^2$ inclined at 45° to the horizontal.)

7. **Ising model**. Let $G = (V, E)$ be a finite graph, and let π be the probability measure on $\Sigma = \{-1, +1\}^V$ given by
$$\pi(\sigma) \propto \exp\left(\beta \sum_{e=\langle i,j\rangle} \sigma_i\sigma_j\right), \quad \sigma \in \Sigma,$$
where $\beta > 0$. Thinking of Σ as a partially ordered set (where $\sigma \leq \sigma'$ iff $\sigma_i \leq \sigma'_i$ for all i), show that π satisfies the FKG inequality.

8. Show that random-cluster measures do not generally satisfy the BK inequality if $q > 1$.

9. (Research problem, likely to be hard if true) Prove that random-cluster measures satisfy the BK inequality if $q < 1$.

10. Read Section 12.3 of the notes, and check you understand the calculations for paragraphs (a)–(c), and the proof of Theorem 12.8.